Secular Stagnation, Rational Bubbles, and Fiscal Policy

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December 29, 2014

Abstract

As is well known, rational bubbles can be sustained in a dynamically inefficient economy, where the return to capital $r$ is below the growth rate $g$. Empirical evidence has shown that modern economies tend to be on a efficient path. But the recent decline in $r$ suggests that the economy might be temporarily inefficient. This paper shows that rational bubbles can be sustained when $r < g$ for some episodes. The economy is shown to be more efficient with than without bubbly assets, though it does not allow the implementation of first best. A proper fiscal policy can increase both financial stability and welfare. Contrary to common wisdom, trade in bubbly assets implements intergenerational transfers, while fiscal policy is needed to implement intragenerational transfers.

The author thanks Jeffrey Campbell, Axel Gottfries, Jasper Lukkezen, Larry Summers, and Jaume Ventura for comments on previous versions.

1 Introduction

“There is increasing concern that we may be in an era of secular stagnation in which there is insufficient investment demand to absorb all the financial savings done by households and corporations, even with interest rates so low as to risk financial bubbles.”

Lawrence Summers, Boston Globe, April 11, 2014

Real interest rates have come down steadily over the past thirty years. This phenomenon has been dubbed secular stagnation by Lawrence Summers (2013), see Richard Baldwin and Coen Teulings (2014) for an overview of this debate. High precautionary saving in China and ageing have increased the supply of savings. Lower growth and a shift of economic activity towards IT with a low demand for capital might have reduced investment demand. These factors can explain a worldwide decline in real interest rates, see e.g. IMF (2014, Chapter 3). This decline raises two types of worries, about the effectiveness of monetary
policy and about the emergence of bubbles. This paper is about the latter. Are bubbles a threat to economic stability? We address this question in a Walrasian world, where Say’s Law always holds, where expectations are rational, and where all markets clear and are perfectly competitive. We assume that monetary authorities are able to avoid the zero lower bound for the nominal interest rate by some divine touch. The only missing market is that for intergenerational transfers. In this setting, bubbles will be shown to partially fill the gap of this missing market.

Jean Tirole’s (1985) celebrated paper on the feasibility of rational bubbles is the point of departure. He considers an overlapping generation model similar to Olivier Blanchard (1985). Tirole shows that rational bubbles can be sustained only when the real rate of interest \( r \) is smaller than the real growth rate \( g \). The intuition is that if subsequent generations of youngsters save a fixed share of their income for future consumption, they can either invest it to obtain a return \( r \) or buy a bubbly asset to be sold later to the next generation. Since savings grow at a rate \( g \), the next generation will have a fraction \( g \) more to spend on this asset. When its supply is fixed, the price of the bubbly asset will therefore go up at a rate \( g \). Since \( r < g \), buying bubbly assets is the more profitable strategy. The condition \( r < g \) is Peter Diamond’s (1965) condition for dynamic inefficiency. In fact, \( r < g \) is also the Aaron condition for Pay-As-You-Go (PAYG) pension systems to be more efficient than funded systems. Bubbles are a substitute for PAYG pensions: either the young pay the old by buying their bubbly asset providing the old the means for consumption, or the government taxes the young to pay pensions to the old. Though widely different from a financial point of view, both institutions yield the same physical outcome.

Tirole’s analysis considers a non-stochastic, steady state economy, where \( r - g \) is constant over time. Hence, an economy is either permanently efficient (\( r > g \)), or permanently inefficient (\( r < g \)). It cannot jump back and forth between both states. The contribution of this paper is to consider what happens if \( r < g \) during some episodes, while \( r > g \) during others. Jumping back and forth between \( r > g \) and \( r < g \) poses its own problems. Switching from a funded to a PAYG pension system (such as required for dynamic efficiency when \( r < g \)) is a piece of cake, since the stock of saving becomes available for current consumption. Switching back from PAYG to a funded system is hard as a cohort has to give up consumption to rebuild the capital stock. We show that bubbly assets provide a useful instrument in this context. Suppose that \( r < g \) temporarily. Then, the young do not want to invest their savings. Instead, they are willing to buy the bubbly asset. Since many buy the bubbly asset, its price is high. Hence, the elderly, who hold the bubbly asset at the beginning of the period, receive a windfall profit. They use this windfall for extra consumption. A reverse mechanism occurs when investment opportunities are high: the price of the bubbly asset will be low and hence the elderly incur a loss and will have to reduce their consumption. The variation in the price of the bubbly asset is therefore an instrument for shifting resources between consumption and investment depending on the return on physical investment. These shifts between investment and consumption imply shifts of consumption between generations.
One would expect that such shifts can only be implemented when enforced by the government. Trade in bubbly assets is shown to be a substitute for transfers enforced by the government, though an imperfect substitute. A bubbly equilibrium is therefore more efficient than the naive market equilibrium where all assets are priced according to the net present value of the stream of expected future dividends.

Though bubbly assets are shown to further efficiency, policy makers might be rightly afraid to let them inflate too much. Bubbly assets are risky, since their value relies on a social convention that a particular asset serves as the store of value in bubbly episodes. If one generation decides to coordinate its beliefs on another asset, the previous generation incurs a capital loss as its purchases of bubbly assets do not pay off. Bubbles are therefore a threat to financial stability. Hence, policy makers might seek to constrain the total value of the stock of bubbly assets. Moreover, though bubbly assets entail an efficiency gain compared to the naive market equilibrium, they are not first best. Hence, the legitimate question is whether there are simple institutions that can improve on the bubbly equilibrium, preferably simultaneously reducing the total value of the bubbly assets. A simple fiscal policy rule that commits the government to issue public debt turns out to achieve both goals. We enter a strange world where bubbles implement intergenerational transfers without enforcement by the government, but where the government’s fiscal policy is indispensable for implementing intragenerational transfers. Since a large share of lifetime wealth is consumed when young, intragenerational transfers of consumption between the two stages of the life cycle are a major source of flexibility when the capacity to store resources is temporarily low due to an investment slump. Fiscal policy can therefore improve welfare.

Research done after Tirole (1985) has cast doubt on the practical relevance of his results. Andrew Abel, Gregory Mankiw, Lawrence Summers, and Richard Zeckhauser (1987) showed that for the economy to be dynamically inefficient the capital sector must be a net sink: investment should exceed dividends. They showed that this condition is violated empirically by a wide margin. However, Francois Geerolf (2013) has shown recently that when adjusting the criterion for the existence of rents and for a number of other factors the economy might have been in a dynamic inefficient state quite frequently. This paper shows that when the economy flips back and forth between $r > g$ and $r < g$, the capital sector has positive outlays on average. Others have argued that the condition of dynamic inefficiency does not correspond to what we usually associate with bubbly episodes. These episodes are characterized by high investment driven by waves of optimism, not by low investment, as in Tirole’s model, see Alberto Martin and Jaume Ventura (2012) for a discussion. They consider a world with distorted financial markets where only bubbles allow to transfer wealth from inefficient to efficient investors or where bubbles provide the collateral needed to support these transfers (see their 2014), see also Ricardo Caballero, Emmanuel Farhi and Mohamad Hammour (2006). However, not all bubbly episodes seem to support waves of high investment, see e.g. the analysis of the hike oil prices just before the demise of Lehman Brothers in 2008 by Caballero, Farhi, and

The set up of the paper is as follows. In Section 2 we consider the simplest economy that exhibits our mechanism. The young save a fixed amount, which can either be stored in productive investment or be spent on buying bubbly assets. The decision how to store resources is the only margin of adjustment in this simple economy. Section 3 extends this model by allowing for intertemporal substitution in consumption within a cohort: saving is no longer fixed. This introduces a second margin of adjustment, between consumption today and consumption tomorrow. In Section 2 and 3, we assume risk neutrality. This assumption is relaxed in Section 4. Risk aversion introduces some new mechanisms. However, the main conclusions from Section 3 survive unchanged. In Section 5 we ask ourselves where bubbly assets come from. Bubbles are often viewed as some remote possibility. Our point of departure is exactly the opposite: bubbly assets are the natural state of the economy. Section 6 concludes. During an episode of low returns to investment, policy makers face a trade off. Either they let the real interest rate fall sufficiently into negative territory to allow the capital market to clear. That leads to bubbles in asset prices. Or they let sovereign debt increase to provide an alternative store of value.

2 The basic model

2.1 Core assumptions for all three models

The three models considered in this paper share a number of common assumptions. We consider an economy that is populated by overlapping generations living for two periods. Each period, a cohort of elderly dies, while a new young cohort enters the economy. In the first stage of their life, when young, this cohort works and receives labour income. In the second stage, when old, it retires. It can only consume what is saved from the first stage. We apply the maximum convenience principle in modelling. Without loss of generality we set the rates of population growth and technological progress equal to zero (hence: $g = 0$). The size of each cohort is normalized to unity. The young have two options for storing resources for consumption in the second stage of life. Either they can invest resources in vineyards. These investments are depleted in one period and yield a physical return. Since investments are fully depleted, investment is equal to the capital stock. Since $g = 0$, investing is dynamically efficient as long as one unit of investment yields a return of at least one unit of output. Or they can buy a bubbly asset, which will be referred to as gold. Gold can neither be (re)produced nor is it depleted. Its supply is normalized to unity. Holding gold does not enter the utility function. Money cannot be used as a store of value, e.g. because inflation is that high that holding gold will be more attractive than holding money in any state of nature. Say’s law holds in this economy, expectations are rational, and markets clear and are perfectly competitive.
2.2 Assumptions for the basic model

In the basic model discussed in this section, the young save one unit of their income for consumption in the second stage of their life. The rest is consumed during the first stage. Since the income and the share of saving are fixed by assumption, consumption of the young is also fixed. Hence, we focus entirely on saving and consumption of the old. All agents are risk neutral. Hence, expected consumption of the old is a sufficient statistic for the welfare of a generation.

Each member of cohort owns a vineyard. When young, he chooses how much to invest in his vineyard. The relation between the input of capital in period \( t \) and output in period \( t+1 \) is given by a production function \( f(\cdot) \):

\[
\begin{align*}
  f(k_t + u_t) &= \ln (k_t + u_t) + g(u_t), \\
  f'(k_t + u_t) &= \frac{1}{k_t + u_t} = f'_t,
\end{align*}
\]

where \( k_t \) is capital per worker, \( 0 \leq k_t \leq 1 \) (since the capital stock can never exceed the available saving); where \( g(\cdot) \) is a function, \( g(\cdot) > 0 \); and where \( u_t \) is an i.i.d. technology shock with support \( u_t \in [u^-, u^+] \), with expectation \( \mathbb{E}[u_t] = \mu \) and variance \( \text{Var}[u_t] = \sigma^2 \). Allowing for an autocorrelation in \( u_t \) would complicate the derivations without altering the conclusions of the analysis. Moreover, we have not been explicit about our unit of time. The context of an overlapping generation model where people live for just two periods means that the appropriate unit of time is several decades. Then, the assumption of serial independence of subsequent values of \( u_t \) does not seem to be much of a problem.

We assume that for any increasing function \( h(\cdot) \), \( \mathbb{E}[h(u_t)] = h(\mu) + \frac{1}{2} h''(\mu) \sigma^2 \) and \( \text{Var}[h(u_t)] = h'(\mu)^2 \sigma^2 \). We must assume that for any market equilibrium, \( k_t + u_t > 0 \), such that \( \ln (k_t + u_t) \) exists and the gross marginal return on capital \( f'_t \) is positive. Since \( k_t \leq 1 \), a necessary condition for this is \( u_t > -1 \). We shall show that for all equilibria considered the condition \( k_t > -u_t \) is satisfied. Similarly, since there is no growth in this economy, the possibility of dynamic inefficiency requires that the gross marginal return on investment when all savings are invested, \( k_t = 1 \), is less than unity \( f'(k_t + u_t) = f'(1 + u_t) < 1 \) for some states of nature. Hence: \( u_t > 0 \) for some states of nature. This motivates the following assumption on the upper and lower support of \( u_t \):

\[
\begin{align*}
  u^- &> -1, \\
  u^+ &> 0.
\end{align*}
\]

Since \( \mu \) must be an interior point on the support of \( u_t \), \( -1 < u^- < \mu < u^+ \).

Equation (1) implies that returns to capital are diminishing: \( f''(\cdot) < 0 \). The technology shock \( u_t \) is additively capital augmenting: \( u_t \) is a perfect substitute for capital. Due to diminishing returns, a higher \( u_t \) reduces the marginal return on capital for a given level of \( k_t \). The function \( g(\cdot) \) does not play a crucial role in the analysis.

\[\text{This assumption says that } \mathbb{E}[h(u_t)] \text{ and } \text{Var}[h(u_t)] \text{ are equal to the expressions derived from second order expansions.}\]
role in the analysis. We just add it here to show that the output of vineyards $f(k_t + u_t)$ can always be made positive for any admissible value of $k_t + u_t$ by an appropriate choice of the functional form of $g(\cdot)$. The function $g(\cdot)$ can also be used to make total output to depend either positively or negatively on the technology shock, so that $u_t$ can be interpreted at will as a negative or a positive shock to total output. This will be irrelevant to the subsequent analysis. What matters is the effect of $u_t$ on the marginal productivity of capital. Hence, we set $g(\cdot) = 0$ for the sake of convenience and without loss of generality, realizing that the fact that $f(k_t + u_t)$ is negative for some admissible values of $k_t + u_t$ can be fixed easily by an appropriate choice of $g(\cdot)$.

The crucial feature of this economy is that $u_t$ captures ex ante investment risk. Ex ante risk differs from the standard ex post risk in that its realization is known at the moment the young decide on how much to invest in their vineyard. Ex post risk is irrelevant in the current model with risk neutrality. It will be introduced in Section 4, where we allow for risk aversion.

Since Say’s law holds, the sum of the investment of the young in their vineyards, $k_t$, and the consumption of the old, denoted $c_t$, must be equal to the sum of saving and the return on last period’s investment minus current investment:

$$c_t = 1 + f(k_{t-1} + u_{t-1}) - k_t.$$  \(3\)

### 2.3 Characterization of the equilibrium

Let $p_t$ be the price of gold in period $t$. Each period, each member of the young generation has to decide how much of its saving to invest in its vineyard, $k_t$. What is left is spent on gold. The young take this decision as to maximize their expected return in the second period, which satisfies:

$$k_t = \arg \max_k f(k + u_t) + (1 - k) \frac{E_t [p_{t+1}]}{p_t},$$

where $E_t [x]$ denotes the expectation of $x$ conditional on the information available at time $t$. The first order condition for the optimal portfolio composition reads:

$$f'(k_t + u_t) = \frac{E_t [p_{t+1}]}{p_t}.$$  \(4\)

The expected return on gold should be equal to the marginal return on capital. Market clearing on the market for gold requires that the young buy the whole stock of gold from the old. Since the young spend $1 - k_t$ on gold and since the supply of gold is equal to unity, we have:

$$p_t = 1 - k_t.$$  \(5\)

An equilibrium is a solution for $p_t$ and $k_t$ that satisfies the first order condition (4) and the market clearing condition (5).

**Proposition 1** \textit{The existence of equilibria}
1. There exists an equilibrium where $k_t = 1$ and $p_t = 0$ for all realizations of $u_t$.

2. If:
   \[ 0 < \mu < (u^+)^{-1}, \]
   then there exists a second equilibrium where:
   \[
   k_t = 1 - \frac{\mu}{1 + \mu} (1 + u_t), \quad (7)
   \]
   \[
   p_t = \frac{\mu}{1 + \mu} (1 + u_t). \]

3. If $\mu > (u^+)^{-1}$, then a similar equilibrium exists, but where investment is constrained by the non-negativity constraint $k_t \geq 0$ in some states of nature.

**Proof:**
Substitution of condition (5) into equation (4) yields:
\[
(1 - E_t [k_{t+1}]) (k_t + u_t) = 1 - k_t. \quad (8)
\]
$k_t$ depends on $u_t$ and $E_t [k_{t+1}]$ only. Hence:
\[
E_t [k_{t+1}] = E [k_t]. \quad (9)
\]
Using this result and taking expectations in equation (8) yields an expression for $E[k_t]$:
\[
0 = (1 - E [k_t]) (E [k_t] + \mu - 1). \]
This equation has two solutions, $E[k_t] = 1$ and $E[k_t] = 1 - \mu$ The first equilibrium follows immediately from the first solution and equation (5). The second equilibrium follows from substitution of the solution for $E[k_t]$ into equation (9) and (8), yielding a solution for $k_t$. The solution for $p_t$ follows immediately from equation (5), proving equation (7).

Since agents cannot be forced to sell their gold, the price of gold has to be positive in any state of nature. Hence $\mu > 0$. Investment cannot be negative and the young cannot invest more than they save: $0 \leq k_t \leq 1$. The second constraint is satisfied for all states of nature. The first requires $\mu < (u^+)^{-1}$.

This proves condition (6).

When $0 < \mu < (u^+)^{-1}$, there are two equilibria. In the first equilibrium, the young invest all their savings in vineyards, even when this is dynamically inefficient. Hence, the price of gold is zero. Nobody finds it attractive to buy gold, since its expected price is zero. Therefore, its current price is zero. There is a second equilibrium where people find it attractive to buy gold, to which we shall refer as the bubbly equilibrium. In this equilibrium, the young do not want to invest all their savings in vineyards when the return on this investment is low.
Instead, they buy gold as an alternative store of value. They do so, because the expected price of gold is positive. In equilibrium, the return on investment and the expected return on buying gold must be equal. When the return on capital is temporarily low, the price of gold is above its long run equilibrium since everybody wants to buy gold instead of investing in vineyards. Hence, the expected return on buying gold is also low, because people expect the price of gold to return to its long run equilibrium, \( E[p_t] = 1 - E[k_t] \), satisfying the return equivalence condition (4). Note that \textit{ex ante} risk in investment leads to \textit{ex post} risk in the price of gold: though the current return on investment is known, the return on gold is uncertain due to uncertainty about the future return on investment.

The price of gold is an increasing function of the technology shock \( u_t \) and hence a decreasing function of the investment in vineyards \( k_t \). People are willing to buy gold, because there will be demand for gold in the next period. This demand will be low if next period’s technology shock induces high investment in vineyards, but will be high, when there is an investment slump. Since the expected price of gold is fixed, the variation in the return on gold is driven by variation in its price.

We refer to the first equilibrium as the naive equilibrium, because it is quite unlikely that agents coordinate on this equilibrium when the bubbly equilibrium exists. When gold carries a positive price, there is no reason why they should avoid buying it. It is rational decision to do so even when everybody is aware that gold is bubbly asset. As we shall see in Section 4, it is even suboptimal for youngsters not to diversify their portfolio by including bubbly assets in the presence of risk aversion. As we shall argue in Section 5, it is unlikely that no asset will emerge that can serve as alternative stores of value.

There are two conditions for the existence of the bubbly equilibrium. The first condition is that \( \mu > 0 \). This condition is stricter then condition (2), \( u^+ > 0 \). Just the existence of some states of nature where investment is dynamically inefficient is insufficient to sustain a bubbly equilibrium. Inefficiency should occur rather frequently. The second condition is that \( \mu < (u^+)^{-1} \). If this condition is violated, a bubbly equilibrium exists, but its description is more complicated, since investment will be bound by the non-negativity constraint in some states of nature. In these states, all savings are used to buy gold. The description of this equilibrium is straightforward in principle, but more messy than the unconstrained equilibrium. Hence, we leave it our here. Note that there is no constraint on \( u^- \). In any bubbly equilibrium, the young spend at least part of their saving on buying gold, even if the return on capital is far above unity, since the return on gold goes to infinity when the young buy no gold at all, because expected future price of gold is positive. Hence, while the lower bound on the investment in vineyards might very well be binding, the upper bound never is. In what follows we shall assume condition (6) to apply.

Tirole (1985) shows that in a non-stochastic context, bubbles can only be sustained when the economy is dynamically inefficient. Does an amended version of this condition carry over to the current model? Since there is no growth in this economy, \( g = 0 \), the condition for dynamic efficiency reads \( r < 0 \). Since
all capital is depleted in one period, \( r < 0 \) implies that the return on capital should be less than unity, \( f'(\cdot) < 1 \). Since the return on capital varies over time, we have to account for this variability. Hence, we investigate whether the expected return on capital satisfies this condition. Similarly, Abel et.al. (1987) have shown that permanent bubbles and permanent dynamic inefficiency would require the capital sector to be a net sink: investment should exceed capital outlays at any time. The subsequent proposition shows that the expected return on capital is positive, \( r > 0 \), and that the capital sector has on average net positive outlays.

**Proposition 2** The expected dynamic efficiency of investment

1. \( E \left[ \left( f_t' \right)^{-1} \right] = 1. \)

2. \( E \left[ f'_t \right] > 1. \)

3. \( E \left[ f'_t \cdot k_t \right] - E[k_t] > 0. \)

**Proof:**

The first statement follows directly from equation (4); the second from Jensen’s inequality: \( E_t \left[ \left( f_t' \right)^{-1} \right] E_t \left[ f'_t \right] > 1 \); the third from:

\[
E \left[ f'_t \cdot k_t \right] - E[k_t] = \left( E \left[ \frac{1 + \mu}{1 + u_t} \right] - 1 \right) E[k_t] + \mu Cov \left[ \frac{1}{1 + u_t}, -u_t \right].
\]

The first term is positive by Jensen’s inequality, the second term is positive since both stochastics depend negatively on \( u_t \).

Proposition 2 shows that the results of Tirole and Abel do not apply in expectation in this economy. Rational bubbles exist even though the expected return on capital is positive in expectation and even though the capital sector has positive net outlays on average. The intuition behind the third result is that a technology shock is only partially undone by lower investment. Hence, investment is high when the return on capital is high and the other way around. This positive correlation between investment and its return makes that the capital sector has a positive net outlay.

### 2.4 Welfare comparison

The first best policy is defined as a policy rule that maximizes expected utility before the veil of ignorance about the ex ante risk \( u_t \) is lifted. Since the consumption of the young is fixed, the first best policy maximize expected consumption of the elderly. Since we have no instruments to transfer wealth between periods other than investment in vineyards, the only decision we have to take is how much of the output generated in a particular period should be
invested in vineyards and how much should be consumed by the elderly. Since the former favours the young while the latter favours the elderly, any change in the allocation implies a transfer of wealth between generations.

Since the first best investment rule \( k(u_t) \) maximizes \( E[c_t] \), equation (3) implies:

\[
k(u_t) = \arg \max_k \left[ 1 + \ln \left( k + u_t \right) - k \right].
\]

Note that both \( k_t \) and \( k_{t-1} \) enter equation (3), which depend on different realizations of the technology shock, \( u_t \) and \( u_{t-1} \) respectively. However, the rule \( k(u) \) should apply likewise to both \( k_t \) and \( k_{t-1} \). Since both terms enter additively, we can take expectations for each term separately and add up the expectations. This makes this formulation of the first best policy applicable.² The first order condition implies:

\[
k(u) = 1 - u. \tag{10}
\]

This equation is the same as condition for dynamic efficiency. Since \( 0 \leq k(u) \leq 1 \), this condition applies unconstrained in all states of nature only if \( 0 \leq u^- < u^+ \leq 1 \). This is a more stringent constraint than equation (2) and (6). Again dealing with the truncations at \( k(u) = 0 \) and \( k(u) = 1 \) is straightforward in principle, but it messes up notation and does not provide new insights. Hence, we assume this more stringent condition to hold in this section.

The consumption \( c_t \) that goes with this investment rule, see equation (3), depends on the investment opportunities of the young. This outcome is similar to a form of intergenerational insurance of the return on capital. If a generation faces a low marginal return on capital due to a high realization of \( u_t \), then the current generation of elderly is "prepared" to absorb the excess saving by consuming it. Being "prepared" deserves inverted commas, because this additional consumption is equivalent to a windfall profit. The young benefit from this obligation to hand over part of their savings to the current elderly without proper compensation, because the same rule that forces them to do so will apply next period when they are the potential beneficiaries.

The bubbly equilibrium is a compromise between the naive market equilibrium and the first best allocation: the coefficient on \( u_t \) in the expression for \( k_t \) is equal to zero in the naive equilibrium and equal to unity in first best equilibrium, while it is in between zero and unity in the bubbly equilibrium, \( 0 < \frac{\delta}{1 + \rho} < 1 \), see Proposition 1. The bubbly equilibrium provides "partial insurance" for productivity shocks. If a generation faces a low marginal return on capital, it invests less in its vineyards and spends more on bubbly assets. This reduces the volatility in the return on capital, but does not eliminate it. Stabilization of the return on capital requires that \( k_t \) should vary. Then, the market clearing condition \( p_t = 1 - k_t \) implies that the price of gold should

²A more formal treatment would observe that the accumulated welfare of all future generations satisfies:

\[
\sum_{t=0}^{\infty} E[c_t] = \sum_{t=0}^{\infty} \left( E[\ln (k_{t-1} + u_{t-1})] + E[1 - k_t] \right).
\]

Taking the derivative of this expression with respect to \( k_t \) yields the same expression. The problem with this specification is that \( \sum_{t=0}^{\infty} E[c_t] \) does not converge. One could interpret this specification as the limiting case for the discount rate going to zero.
vary. Since the expected return on gold varies inversely to the price of gold and since the return on capital should equal to the expected return on gold, the variability in the price of gold implies that the return on capital should vary. Hence, it is less volatile than in the market equilibrium, but it is more volatile than in planner’s optimal allocation, where the return on capital is always equal to unity. A peak in gold prices leads to a boost in the consumption of the elderly. The reverse holds for a trough in gold prices. Fluctuations in gold prices are therefore a means for adjusting the consumption of the elderly to investment. When the marginal return on investment is high, consumption should be low, and the other way around. One would expect that this type of intergenerational transfers could not be implemented without enforcement by the government. However, bubbles are partial substitute for enforcement by the government. The desire of the young to avoid useless investment in their vineyards by buying gold as an alternative store of value provides an alternative mechanism for enforcement of at least a partial implicit insurance contract. In a decentralized bubbly economy, the market provides a second best substitute for efficient intergenerational transfers that were lacking in the naive market equilibrium.

**Proposition 3** The trade off between expected welfare and its variability.

1. First best yields the highest expected welfare and the naive equilibrium the lowest.

2. The ordering of the variability of welfare is the same.

3. The naive equilibrium yields the highest mean level of investment; mean investment is the same in the first best and the bubbly equilibrium.

**Proof:**

Remember that expected consumption of the old is a sufficient statistic for welfare. Hence, we can use the expressions for $E[c_t]$ and $\text{Var}[c_t]$ to prove the first two statements. Using equation (3), Proposition 1, and equation (10), one can derive the expressions presented in Table 1. Some simple calculation using these expressions proofs the proposition.

| Table 1 Expectation and variance of welfare and investment |
|---|---|---|---|
| equilibrium | naive | bubbly | first best |
| $c_t$ | $\ln(1+u_t)$ | $\ln\left(\frac{1+u_t}{1+\mu}\right) + \frac{\mu}{1+\mu}(u_{t+1}+1)$ | $u_{t-1}$ |
| $E[c_t]$ | $\ln(1+\mu) - \frac{1}{2} \left(\frac{\sigma}{1+\mu}\right)^2$ | $\mu - \frac{1}{2} \left(\frac{\sigma}{1+\mu}\right)^2$ | $\mu$ |
| $\text{Var}[c_t]$ | $\left(\frac{1}{1+\mu}\right)^2 \sigma^2$ | $\frac{1+\mu}{(1+\mu)^2} \sigma^2$ | $\sigma^2$ |
| $k_t$ | 1 | $\frac{1}{1+\mu} - \frac{\mu}{1+\mu} u_t$ | $1 - u_t$ |
| $E[k_t]$ | 1 | $1 - \mu$ | $1 - \mu$ |
| $\text{Var}[k_t]$ | 0 | $\left(\frac{\mu}{1+\mu}\right)^2 \sigma^2$ | $\sigma^2$ |
There is a trade off between expected welfare and its variability. Agents are risk neutral, so the variability does not come at a price in this economy. This will change when we allow for risk aversion in Section 4. The sources of variation differ between equilibria. In the naive market equilibrium, the variation comes from the return on capital. In the first best equilibrium, the variation comes from the investment in vineyards. The variation in the bubbly equilibrium is a mixture of both. Investment is the highest in the naive market equilibrium, since in that equilibrium agents have no alternative store of value. Remarkably, average investment is the same in the bubbly and the first best equilibrium, though investment is more volatile in the first best equilibrium. Hence, in the bubbly equilibrium, there is overinvestment when the return on capital is low and underinvestment when the return is high.

3 Intertemporal substitution in consumption

3.1 Assumptions

The basic model of Section 2 serves the purpose of showing how bubbles can fill a missing market for intergenerational transfers. It also shows why bubbles enhance expected welfare. However, by restricting the allocation problem to the use of current revenues for either investment in vineyards (to the benefit of the young) or consumption (to the benefit of the elderly), we constrain the choice set. In practice, consumption can be transferred between stages of life. In this section, we extend the model by introducing a trade off for the young between consumption now and saving for future consumption. We can only achieve an analytical solution by using a simplification compared to most of the literature. In an economy with constant returns to scale, investment of the previous generation yields a positive externality to the wage rate faced by the next generation. This externality is the equal to the difference between the marginal and the intra-marginal return on capital, or equivalently, to the rent on the vineyard. This positive externality is a source of persistence: higher investment today yields higher wages and hence higher savings next period. In order to avoid this complication we consider an economy where this externality is absent. For this purpose, we introduce a separate class of rentiers, who own the vineyards and who consume all the rents derived from their property, but who play no further role in the economy. We assume that there is no market for vineyards.

To fix ideas, suppose that agents have a Cobb Douglas utility function with expected consumption in both periods as its arguments, like in Olivier Blanchard and Philippe Weil (1992):

$$U = (1 - \eta) \ln E[c_t^y] + \eta \ln E[c_{t+1}].$$

where $c_t^y$ is the consumption of the young of the cohort entering the economy at $t$ and $c_{t+1}$ is their consumption when they have grown old at $t+1$. The parameter $\eta$ is the budget share of consumption in the second period, $0 < \eta < 1$. Equation
decouples the rate of intertemporal substitution from the degree of risk aversion, leaving the expected utility framework. For the moment, we maintain the assumption of risk neutrality. In the next section, we will allow for constant relative risk aversion within each stage of life. For the special of the degree of relative risk aversion being equal to unity, this brings us back in the expected utility framework. In the current framework with within period risk-neutrality, the Cobb Douglas function for the degree of intertemporal substitution implies that income and substitution effects cancel, so that variations in the (expected) return on capital do not affect the budget share that the young set apart for future consumption. Total labour income earned in the first period of life is assumed to be equal to $\eta^{-1}$, so that savings are again equal to unity in a market equilibrium, both in the naive and the bubbly equilibrium.

Capital productivity is modelled exactly the same as in the previous section, except that vineyards are owned not by the population at large, but by a separate class of rentiers. The return on capital is therefore not equal to $f(k_t + u_t)$ as in the previous section, but to $f'(k_t + u_t) k_t$. The difference between the marginal and the intra-marginal return, $f(k_t + u_t) - f'(k_t + u_t) k_t$ is the income of the rentier class.

3.2 Fiscal policy

Since income and substitution effects cancel in the intertemporal trade off of youngsters, the naive and the bubbly equilibrium are exactly the same as in the previous section, apart from the difference due to the introduction of a separate rentier class. However, the extension of the model with intertemporal substitution in consumption allows us to study the effect of fiscal policy. We consider a simple policy rule, where the government issues bonds at time $t$ that pay back one unit of consumption per bond at time $t + 1$. The government commits to issue $b$ bonds of this type each period. It sells them at a price $r_t$, which differs between periods. Hence, each period, the government has to repay its debt $b$, but it receives $br_t$ from new bond issuance. The difference between two is covered by a tax $z_t$ on labour income (or: subsidy, if $r_t > 1$), which satisfies:

$$z_t = b (1 - r_t).$$

Lifetime wealth is equal to gross labour income $\eta^{-1}$ minus the tax on labour income. Hence, agents consume an amount $\frac{1}{1 - \eta} (1 - \eta z_t)$ when young and save an amount $1 - \eta z_t$ for future consumption. Hence, expected consumption of the elderly at time $t + 1$ evaluated at time $t$ satisfies:

$$E_t [c_{t+1}] = (1 - \eta z_t) E_t [R_t + 1],$$

$$R_t = 1 - \frac{r_t}{r_t} + s \left( f_t - \frac{1}{r_t} \right) + g \left( \frac{p_{t+1}}{p_t} - \frac{1}{r_t} \right),$$

where $s$ and $g$ are the shares of saving held in investment and gold respectively. Note that $R_t \neq E_t [R_t]$, since $R_t$ depends on $u_{t+1}$. The term $1 - \eta z_t$ is the saving set apart for consumption in the second period, $R_t + 1$ measures the return on
that savings. Since saving $1 - \eta z_t$ is fixed, the young choose the composition of their portfolio as to maximize $R_t$:

$$g_t, s_t = \arg \max_{g, s} E_t [R_t + 1].$$

The first order conditions of this problem require the expected return on the three available assets to be equal:

$$f'(k_t + u_t) = \frac{E_t [p_{t+1}]}{p_t} = \frac{1}{r_t},$$

(13)

Market clearing requires investment $k_t$ to be equal to its share $s_t$ in total savings, and likewise for bonds and gold, implying:

$$k_t = (1 - \eta z_t) s_t,$$

$$b r_t = (1 - \eta z_t)(1 - g_t - s_t),$$

$$p_t = (1 - \eta z_t) g_t = 1 - \beta (\eta_0 + r_t) - k_t,$$

(14)

where $\beta = (1 - \eta) b$ and $\eta_0 \equiv \eta / (1 - \eta)$. An equilibrium is a quintuple $g_t, s_t, p_t, k_t,$ and $r_t$ that solves equation (13) and the market clearing conditions (14).

**Proposition 4** The effect of fiscal policy in the bubbly equilibrium

1. A bubbly equilibrium exists when $b < \min \left[ \mu, (1 + u^-) \eta^{-1} \right]$.
2. The expected level of investment does not depend on $b$.
3. The expected price of gold is lower for higher $b$.
4. Investment and the price of gold are less volatile for a higher $b$, but the price of bonds is more volatile.
5. Expected utility reaches a maximum for $b = \min \left[ \frac{1 - \mu}{\eta^-}, \mu, (1 + u^-) \eta^{-1} \right]$.

**Proof:**

Substitution of equation (13) into the market clearing condition (14) for $r_t$ yields:

$$p_t = 1 - \beta (\eta_0 + u_t) - (1 + \beta) k_t,$$

$$E_t [p_{t+1}] = 1 - \beta (\eta_0 + \mu) - (1 + \beta) E_t [k_{t+1}],$$

which can then by used to eliminate $p_t$ from condition (14):

$$(k_t + u_t) [1 - \beta (\eta_0 + \mu) - (1 + \beta) E_t [k_{t+1}]] = 1 - \beta (\eta_0 + u_t) - (1 + \beta) k_t,$$

This equation shows that $k_t$ depends on $u_t$ and $E_t [k_{t+1}]$ only. Hence, equation (9) applies. Taking expectation in the final equation yields:

$$0 = (1 - \beta (\eta_0 + \mu) - (1 + \beta) E [k_t]) (E [k_t] + \mu - 1).$$
This equation has two solutions. The solution setting the first factor equal to zero corresponds to naive equilibrium. We focus on the second solution corresponding to the bubbly equilibrium:

\[ E[k_t] = 1 - \mu. \]

Some calculation yields expressions for investment and the prices of bonds and gold:

\[ k_t = \frac{1 - \eta b - (\mu - \eta b) u_t}{1 - \eta b + \mu}, \quad (15) \]

\[ r_t = \frac{1 - \eta b + u_t}{1 - \eta b + \mu}, \]

\[ p_t = (\mu - b) \frac{1 - \eta b + u_t}{1 - \eta b + \mu}. \]

\( p_t \) should be positive for any state of nature for an equilibrium to exist, proving statement 1. The expected prices of bonds and gold follow immediately:

\[ E[r_t] = 1, E[p_t] = \mu - b, \]

proving statement 2 and 3. Statement 4 follows from equation (15).

Consumption for the young and the elderly satisfies:

\[ E[c_{yt}] = E\left[ \frac{1 - \eta}{\eta} (1 - \eta z_t) \right] = \frac{1 - \eta}{\eta}, \]

\[ E_t[c_{t+1}] = \frac{1 - \eta z_t}{r_t} = (1 - \eta b) r_t^{-1} + \eta b, \]

\[ E[c_{t+1}] = 1 + (1 - \eta b) \left( \frac{d r_t}{d u_t} \right)^2 \sigma^2 = 1 + \frac{1 - \eta b}{(1 - \eta b + \mu)^2} \sigma^2. \]

Substitution in expected welfare yields:

\[ U = (1 - \eta) \ln \frac{1 - \eta}{\eta} + \eta \ln \left( 1 + \frac{1 - \eta b}{(1 - \eta b + \mu)^2} \sigma^2 \right). \quad (16) \]

The first order condition for the optimal value of \( b \) implies:

\[ \frac{d}{db} \left( \frac{1 - \eta b}{(1 - \eta b + \mu)^2} \right) = \eta \frac{1 - \mu - \eta b}{(1 - \eta b + \mu)^3} = 0. \]

\( b = \frac{1 - \mu}{\eta} \) solves this equation. However, \( b < \min \left[ \mu, (1 + u^-) \eta^{-1} \right] \) for a bubbly equilibrium to exist, see statement 1. Hence, the optimal value is \( b = \min \left[ \frac{1 - \mu}{\eta}, \mu, (1 + u^-) \eta^{-1} \right] \), proving statement 5. ■

The first four statements of Proposition 4 are similar to Proposition 3: a bubbly equilibrium yields the same average level of investment as first best, but
investment is less sensitive to shocks to the return on capital. Similarly, fiscal policy does not affect the average level of investment, but makes investment less sensitive to the return on capital. Hence, it moves the investment rule further away from first best. Nevertheless it improves welfare compared to the equilibrium without fiscal policy, since fiscal policy allows transferring consumption between the two stages of life. This cannot be achieved in a bubbly equilibrium without fiscal policy, since the young always consume a share $1 - \eta$ of their lifetime income. Hence, if lifetime income is constant, so will be consumption in the first stage of life. The only way to change this is to introduce a policy that changes life time income and hence current consumption. When investment is low due to a low return on capital, demand for government bonds is high, leading to a high price and hence low interest rates. Hence, $z_t$ will be negative, which raises life time income and hence consumption of the young. While without fiscal policy, only the elderly increase their consumption when during an investment slump, part of the extra consumption is shifted to the young with fiscal policy. The young pay for this by a low interest rate on their holding of bonds, which will reduce their consumption when old. Stated differently, fiscal policy uses the income effect of negative taxes to boost current consumption and the substitution effect of a low interest rate to reduce future consumption, thereby accommodating the temporary excess demand storing resources for future consumption.

Statement 3 of Proposition 4 says that fiscal policy stabilizes financial markets in the sense that it reduces both the average price of gold and its volatility. There is less demand for gold as a store of value since sovereign debt serves as a substitute. The only difference between holding gold and holding government bonds is that holding gold is risky since the future price of gold depends on the future return on capital, while holding government bonds is not. However, since agents are risk neutral in this economy, this difference is irrelevant here. Hence, the expected return on gold is exactly equal to the interest rate on government bonds. When fiscal policy satisfies $b = \mu$, bubbles cease to exist. This result can be understood easily. Since the expected level of investment is $1 - \mu$, the expected demand for stores of value is equal to $\mu$. Hence, the demand for gold vanishes when the government supplies that amount of sovereign debt on average.

As long as there is a positive demand for bubbly assets, $b < \mu$, fiscal policy is most effective when investment in vineyards is rather efficient on average, that is, when $\mu$ is low. The intuition is that for high levels of average efficiency, the burden of variations in current consumption due to shocks in the return on capital can be absorbed completely by shifting consumption between the two stages of life of the current generation of youngsters. This is what fiscal policy does. For lower levels of efficiency, it becomes attractive to shift part of that burden to the previous generation, the current elderly. They will consume more when the return on capital is low, but in exchange, they will have to consume less when the return is high. This is what bubbly assets achieve. The smaller $\eta$, the less attractive it becomes to let all the variations in consumption be absorbed by the elderly. Hence, there is a larger role for fiscal policy to shift
consumption between the stages of life. Hence, contrary to standard view of fiscal policy being about intergenerational transfers, fiscal policy implements intragenerational transfers of consumption between the stages of life of a generation, while trade in bubbly assets implements the intergenerational transfers. When $\frac{1-\mu}{\mu} < \eta$, welfare is maximized by some combination of both institutions; if not, fiscal policy alone can do the job.

The proposed fiscal policy applies an income tax to cover the deficits or surpluses from the government’s public debt operations. Hence, only the young pay. One could generalize this policy by allowing for a combination of income and consumption taxes, thereby spreading the burden between young and the old. A proper combination of income and consumption taxes implements a constrained efficient allocation and would therefore eliminate completely the demand for bubbly assets.

Even when allowing for both an income and a consumption tax, the type of fiscal policy considered here is simple. Would more complicated policy rules allow for a further improvement of welfare? The answer is definitely yes. One can show that a first best allocation would require more complicated investment and consumption rules, which depend not only on $u_t$, but also on $k_{t-1}$ and $u_{t-1}$. A more complicated and activist fiscal policy would therefore improve welfare beyond the constrained optimum considered in Proposition 4. We do not present this first best allocation here, since it has no analytical solution and is therefore hard to characterize, while it contributes little to understanding the relevant mechanisms. Moreover, more complicated rules are more difficult to commit to and hence less credible. From a policy perspective, the simple rule might therefore be more relevant.

4 Risk aversion and the risk free rate

4.1 Assumptions

Though the model in the previous section allows for intertemporal substitution, it assumes risk neutrality. In this section, we relax this assumption, while maintaining the Cobb Douglas structure for intertemporal substitution:

$$U = \frac{1-\eta}{1-\gamma} \ln E \left( (c^\eta_y)^{1-\gamma} \right) + \frac{\eta}{1-\gamma} \ln E \left( c^\gamma_{t+1} \right).$$

The parameter $\gamma$ is the degree of relative risk aversion. For $\gamma = 1$, the utility function simplifies:

$$U = E \left[ \ln ((1-\eta) c^\eta_t + \eta c_{t+1}) \right].$$

In that case, we are back in the standard expected utility framework.

Till sofar, the riskiness of the investment in vineyards was irrelevant. Under risk neutrality, the only thing that mattered was ex ante investment risk. Since the realization of this factor is known at the moment of investment, the investment itself is risk free in an economy with only this type risk. However, investment is risky in reality. In an economy with risk aversion, this uncertainty
should be taken into account. We therefore extend the production function with an additional random variable accounting for *ex post* investment risk:

\[
f(k_t + u_t, v_{t+1}) = (1 - v_{t+1}) \ln (k_t + u_t),
\]

where \(v_t\) is an i.i.d. technology shock with \(E[v_t] = 0, \text{Var}[v_t] = \chi^2\) and \(\text{Cov}[u_t, v_{t+1}] = \text{Cov}[u_t, v_t] = 0\). \(v_{t+1}\) and \(u_t\) are independent by construction. \(v_{t+1}\) captures the new information that is coming in at \(t + 1\). Would that information be correlated to \(u_t\), \(u_t\) would contain information about the future value of \(v_{t+1}\), and hence \(v_{t+1}\) would not be news.\(^3\) The more substantive assumption is that \(u_t\) and \(v_t\) are uncorrelated. One would expect the expected return on future investment to be correlated to the realized return on current investment. Allowing for this correlation is straightforward in principle, but would mess up subsequent derivations. Hence, it is ruled out by assumption. The production function implies:

\[
f'(k_t + u_t, v_{t+1}) = \frac{1 - v_{t+1}}{k_t + u_t} \equiv f'_t,
\]

\[
E_t[f'(k_t + u_t, v_{t+1})] = \frac{1}{k_t + u_t},
\]

\[
\text{Var}_t[f'(k_t + u_t, v_{t+1})] = \frac{\chi^2}{(k_t + u_t)^2}.
\]

Note that \(f'_t \neq E_t[f'_t]\) due to *ex post* investment risk \(v_{t+1}\).

Fiscal policy is the same as in the previous section. Note that \(E_t[r_t^{-1}] = r_t^{-1}\) and hence \(\text{Var}_t[r_t^{-1}] = 0\): governments bonds are risk free.

### 4.2 Characterization of the equilibrium

Like in the model with risk neutrality, youngsters save \(1 - \eta z_t\) for consumption in the second stage of life. Hence, equation (12) for \(c_{t+1}\) applies. Agents choose \(g_t\) and \(s_t\) as to maximize \(E_t \left[ c_{t+1}^{1/\gamma} \right]^{1/(1-\gamma)}\). Since we can factor out \(E_t[\gamma (1 - \eta z_t)^\gamma] = (1 - \eta z_t)^\gamma\) and since only \(R_t\) depends on \(g_t\) and \(s_t\), the problem can be written as:

\[
g_t, s_t = \arg \max_{g,s} E_t \left[ (R_t + 1)^{1/\gamma} \right]^{1/(1-\gamma)}. \tag{17}
\]

An equilibrium is a quintuple \(g_t, s_t, p_t, k_t, r_t\) that solves equation (17) and the market clearing conditions (14).

A full characterization of this equilibrium is too difficult a task. Hence, we retreat one step. We approximate the optimal portfolio for small deviations of \(u_t\) and \(v_t\) from their expected value. In particular, we specify:

\[
\sigma = h \cdot \sigma_0, \quad \chi = h \cdot \chi_0, \tag{18}
\]

\(^3\)Strictly, this argument would apply only when \(u_t\) and \(v_t\) would enter additively: \(f(\cdot) = \ln (k_t + u_t + v_{t+1})\). Up to a second order term, this specification is identical to the specification in the text. The latter specification is somewhat more convenient in the subsequent analysis.
where we consider the equilibrium for the limiting case \( \lim h \to 0 \). Hence: 
\[ u_t - \mu = O(h) \] and 
\[ v_t = O(h) \]. The following Proposition allows a Taylor approximation of the market returns.

**Proposition 5**  
*Posit equation (18). There is an equilibrium with following properties:*  

1. \( r_t - 1 = O(h) \).  
2. \( f_t' - 1 = O(h) \).  
3. \( p_t/E_t[p_{t+1}] = 1 + \pi (u_t - \mu) + O(h^2) \),  
   where \( \pi \equiv E_t[p_{t+1}]^{-1} dp_t/du_t + O(h^2) \).

**Proof:** see Appendix. \( \blacksquare \)

The proposition states that the rates of return on the three assets do not deviate from unity (their value in the first best equilibrium) by more than a term of order \( h \). This is what one would expect since the stochastic terms \( u_t - \mu \) and \( v_t \) are of order \( h \). The expression for \( p_t/E_t[p_{t+1}] \) requires some further explanation. It states that \( p_t \) is a linear function of \( u_t \) up to a term of \( O(h^2) \). Hence, equation (9) applies up to a term of \( O(h^2) \), implying \( E[k_t/E_t[k_{t+1}] = 1 + O(h^2) \). The proof of the proposition shows that the effect of shifts in the variation in savings, \( 1 - \eta z_t \), and deviations of \( k_t \) and \( p_t \) from their expectation on the composition of the portfolio can be ignored in a second order approximation. Hence, we can replace \((1 - \eta z_t) s_t \) by \( E[k_t] \) and \((1 - \eta z_t) g_t \) by \( E[p_t] \). These approximation allow the following proposition.

**Proposition 6**  
*In the bubbly equilibrium the following relations hold:*  

\[ r_t - k_t - u_t \cong \psi_k, \]  
\[ r_t - 1 - \pi (u_t - \mu) \cong \psi_p, \]

where the symbol \( \cong \) implies that terms of \( O(h^3) \) are ignored and where \( \psi_k = \gamma \chi^2 E[k_t] \) and \( \psi_p = \gamma \sigma^2 \pi^2 E[p_t] \).

**Proof:** see Appendix. \( \blacksquare \)

The inverse risk free rate minus the inverse rate of return on risky assets (capital investments and gold) are equal to the risk premium on holding these assets (\( \psi_k \) and \( \psi_p \) respectively). An equilibrium is the solution for \( k_t, p_t, \) and \( r_t \) to the asset return equations (19) and the market clearing condition (14). This solution is characterized in the next proposition.

**Proposition 7**  
*Risk aversion and fiscal policy*

1. A bubbly equilibrium exist when \( b < \min \left[ \mu, \left( 1 + \psi_k - (1 + \beta) \psi_p + u^- \right) \eta^{-1} \right] \).
2. The expected level of investment is lower for higher \( b \).

3. The expected price of gold and of bonds is lower for higher \( b \).

4. Investment and the price of gold are less volatile for a higher \( b \), but the price of bonds is more volatile.

5. Fiscal policy increases welfare.

**Proof:** See Appendix.

As can be checked easily, for \( \gamma = 0 \) the solution to this system is equivalent to equation (15), see equation (25) in the Appendix. The expected value of investment in vineyards deviates from that in the risk neutral economy depending on which is the higher risk premium: that on gold or that on investment in vineyards. The former depends on \( \text{ex ante} \) risk, the latter on \( \text{ex post} \) risk. When the former dominates, risk aversion leads to higher investment, since alternative of holding gold is more risky. The higher the risk premium on gold, the higher the price of risk free bonds. Bonds and gold are almost perfect substitutes, apart from the fact that gold is a risky asset due to next period’s \( \text{ex ante} \) risk. In this zero growth economy, the expected return on safe assets is negative. The term \( \psi_r \) is the insurance premium for holding safe assets. Fiscal policy has a negative effect on the price of gold: by providing sovereign debt as an alternative store of value the average demand for gold goes down. However, this initial effect is partly offset by a reduction in the risk premium on gold, since fiscal policy stabilizes the price of gold.

The volatility of investment and the price of gold goes down by the introduction of fiscal policy. Fiscal policy is a less effective means of shifting resources between investment and consumption. Hence, investment is less responsive to variations in the return on capital. The price of gold is less volatile because government bonds are available as an alternative store of value. The higher \( b \), the more volatile the price of bonds, because the riskiness of gold declines and hence the more the return on government bonds will be equal to the return on gold.

Risk aversion makes fiscal policy even more effective. However, it is hard to tell what is the net effect of risk aversion pushes on the optimal level of \( b \), though it is more easy to find combinations of \( \gamma, \mu \) and \( \eta \) for which risk aversion pushes the optimal value of \( b \) up rather than down.

5 How can bubbles arise?

In the three models considered in the previous sections, there were usually at least two equilibria: with and without bubbly assets. The bubbly equilibrium was shown to be Pareto superior to the so-called naive equilibrium, since bubbles allow intergenerational transfers to accommodate variations in the market return on capital. Is there a natural reason for agents to coordinate on the
bubbly instead of the naive equilibrium? We argue there is. An asset can serve its role as a bubbly store of value best if it is irreproducible and hence in fixed supply. Suppose to the contrary that its supply grows at a rate \( y \). Since savings grow at a rate \( g \), the price of the bubbly asset increases at a rate \( g - y \). Hence, the lower \( y \), the higher the return on the bubbly asset and the more attractive it is as a store of value. Reproducible stores of value are less efficient because part of their potential return is spoiled on compensating the producers of additional bubbly assets.

Suppose that the supply of gold is indeed fixed, so that it is a perfect bubbly store of value. How would people come to coordinate on gold as the bubbly asset? Till so far, we have assumed that holding gold does not yield a utility stream. Let us refer this class of assets as pure bubbly assets. However, historically gold has always been a status item. Wearing golden jewelry is a way for the upper class to distinguish itself from the ordinary people. Hence, people derive some utility from holding gold as a status symbol, either directly by showing off, or indirectly, since status yields power. We refer to this class as almost-pure bubbly assets, see also Rhee (1991). How would this almost-pure bubbly asset be priced? The standard formulas for asset pricing would apply. The price of gold would be equal to the net present value of utility stream derived from wearing the jewelry. When \( r \) falls below \( g \), the price of gold would go up due to the lower discount rate. All of the previous analysis applies almost unamended. As soon as people have coordinated on gold being the store of value, then Fort Knox becomes a viable institution, where a large supply of gold is stored for decades for no other purpose than to serve as a store of value.

Gold is not the only asset that satisfies closely the criteria for almost-pure bubbly asset. Residential real estate in inner cities is an alternative. It is in fixed supply due to land scarcity in city centers and it yields regular flow utility, for which users pay rents. The price of central city real estate has increased steeply over the past four decades, see the analysis of superstar cities in the United States by Joseph Gyourko, Christopher Mayer, and Todd Sinai (2013), the high prices in Amsterdam’s canal zone in the Netherlands by Coen Teulings, Ioulia Ossokina and Henri de Groot (2014). Katharina Knoll, Moritz Schularick, and Thomas Steger (2014) have documented house prices having been largely flat from 1870 till 1950, but having increased afterwards. Thomas Piketty (2014) documented the rise in the value-share of real estate in the total capital stock in France (http://piketty.pse.ens.fr/files/capital21c/pdf/G3.2.pdf) and the United Kingdom (.../G3.1.pdf). The steep price increases in central city real estate coincided with the decline in the real interest rate.

The fact that almost-pure bubbly assets yield a positive utility stream makes them even more suitable to serve as temporary store of value than pure bubbly assets. Pure bubbly assets command a positive price only when the economy is sufficiently dynamic ineﬃcient, see equation (6) in Proposition 1. This constraint does not apply to almost-pure bubbly assets, since these assets will always be positively priced due to the utility stream that is derived from them. The separation between pure and almost-pure bubbly assets is likely to be gradual in practice, since it is hard to conceive a bubbly asset of which the ownership...
does not generate any pleasure at all. When almost-pure bubbly assets are available, the naive equilibrium becomes unstable, see also Tirole (1985). For this reason, the equilibrium without bubbly stores of value might be qualified as naive. Bubbles are the rule, not the exception.

6 Conclusion

We analyzed a world where bubbles are a means for implementing intergenerational transfers to accommodate temporary fluctuations in the return on capital. In this situation, bubbly assets serve as an alternative store of value in presence of dynamic inefficiency. Despite these temporary episodes of dynamic inefficiency, the capital will be productive on average in the sense that average outlays of the capital sector exceed inflow, a criterion very similar to that derived by Abel et.al. (1987), and which is equivalent to Tirole’s (1985) condition for the feasibility of bubbles, to Diamond’s (1965) condition for dynamic inefficiency, and Aaron’s condition for PAYG being more efficient than funded pension systems.

Fiscal policy has a profound effect on financial stability, as measured by the mean and the variance of the price of bubbly assets. A simple fiscal policy that issues a fixed quantity bonds with a fixed future pay out cures this problem. These bonds serve as an alternative store of value when investment demand is low. Hence, they reduce the price of bubbly assets. Variation in the price of these bonds provide a means for adjusting consumption to investment demand. Remarkably, where trade in bubbly assets shifts resources between investment and consumption by intergenerational transfers, this fiscal policy does this by intragenerational transfers, shifting resources between current and future consumption. This runs against the common wisdom that only the government can enforce intergenerational transfers.

Our analysis applies a Cobb Douglas function for the degree of intertemporal substitution. When the degree of intertemporal substitution is lower than implied by the Cobb Douglas function, the effect of investment slumps becomes even stronger. A fall in the return on capital will then lead to an increase in the budget share that the young set apart for future consumption, putting greater strains on the ability of the capital market to absorb these savings.

Our analysis puts the full burden of variation in $r - g$ on investment demand. This might be counterfactual. The analysis in IMF (2014, Chapter 3) and Teulings and Baldwin (2014, Table 1) suggests that a substantial share of the variation might be due to demographic shocks, in particular the increase in life expectancy over the past three decades while the retirement age has not kept pace with this trend, leading to an increase in the supply of savings. However, the impact of this phenomenon is equivalent to a downward shock in the return on capital. A higher life-expectancy under a constant retirement age is equivalent to an increase of the share $\eta$ of old age consumption in the life time income.\footnote{Note that a fall in $g$ per se, as hypothesized by Robert Gordon (2014),}

\footnote{In the notation in the paper, it is convenient to normalize lifetime income equal to $\eta^{-1}$.}
does not pose much of a problem from the point of view of the analysis in this paper. It would push the economy towards rather than away from a dynamically efficient equilibrium. Problems arise only when \( r \) falls relative to \( g \), not the other way around.

In this type of economies, there are always two equilibria, one with and another without bubbles. When there exist an almost-pure bubbly asset, which like a pure bubbly asset, is irreproducible, but unlike a pure bubbly asset, generates some utility, then the equilibrium without bubbles becomes unstable. Bubbles are the rule, a world without bubbles is the exception. For this reason, we refer to the equilibrium without bubbles as naive.

Our analysis treats \( u_t \) as if it reflects exogenous shocks that are beyond the control of policy makers. One might be uncomfortable with this assumption. Low investment demand might be due -if only partially- to some kind of coordination failure rather than to a bad state of nature. If this is the case, policy makers can play a larger role in handling the investment slump than envisaged in the current analysis. They can try to solve the coordination problem, by shifting beliefs from a low to a high investment equilibrium. In a neoclassical world without this type of coordination failures, policy makers face a trade off during an episode of a low return on capital. Either they allow the real interest rate to fall temporarily into the negative territory, assuming that the zero lower bound does not impose a binding constraint on monetary policy. This allows bubbles to inflate. Or they revert to fiscal policy by letting the government issue more sovereign debt as an alternative store of value. These policies have profoundly different implications for the intergenerational distribution of wealth. Relying on monetary policy favours the elderly as they gain windfall profits on their holding of (almost) bubbly assets. Relying on fiscal policy favours the young, as they are enabled to increase their current consumption. No sensible neoclassical economist would recommend policy makers to revert to a third alternative, setting a minimum for the real interest rate as to avoid bubbles to inflate. Such an artificial floor for the real interest rate would prohibit the capital market from clearing and would thereby force the economy into a Keynesian recession.

7 References

http://www.nber.org/papers/w3992


- IMF (2014), *World Economic Outlook*.


Appendix Proofs

Proposition 5 and 6

The strategy of the proof is to conjecture that the approximations posited in Proposition 5 apply and then to characterize the equilibrium. We then show that this characterization satisfies Proposition 5. The symbol $\approx$ means that terms of $O(h^3)$ are ignored.

Lemma 8 Let $x$ be a random variable with $E[x] = 1 + \mu$ and $\text{Var}[x] = \sigma^2$ with $\mu = O(h)$ and $\sigma = O(h)$. Then:

$$E \left[ x^{1-\gamma} \right]^{1/(1-\gamma)} \approx 1 + \mu - \frac{1}{2} \gamma \left( \sigma^2 + \mu^2 \right).$$

Proof of the lemma:
Define $m \equiv \ln (1 + \mu)$ and $\bar{\gamma} = 1 - \gamma$. We have:

$$m \approx \mu - \frac{1}{2} \mu^2 + O \left( h^3 \right) = O \left( h \right);$$
$$\mu \approx m + \frac{1}{2} m^2;$$
$$E[\ln x] \approx m - \frac{1}{2} \sigma^2;$$
$$\text{Var} [\ln x] \approx \sigma^2.$$
Hence:

\[
E \left[ e^{\gamma \ln x} \right]^{1/\gamma} = \left[ e^{\gamma E[\ln x]} + \frac{1}{2} \gamma^2 e^{2\gamma E[\ln x]} \text{Var}[\ln x] + O(h^3) \right]^{1/\gamma} \\
= \left[ 1 + \gamma E[\ln x] + \frac{1}{2} \gamma^2 E[\ln x]^2 + \frac{1}{2} \gamma^2 \text{Var}[\ln x] + O(h^3) \right]^{1/\gamma} \\
= \left[ 1 + \gamma m + \frac{1}{2} \gamma^2 m^2 - \frac{1}{2} \gamma^2 \sigma^2 + O(h^3) \right]^{1/\gamma} \\
\cong 1 + m + \frac{1}{2} \gamma^2 m^2 - \frac{1}{2} \gamma^2 \sigma^2 \cong 1 + \mu - \frac{1}{2} \gamma (\mu^2 + \sigma^2).
\]

This Lemma and the assumption in equation (18) allow a Taylor expansion of equation (17):

\[
g_t, s_t \cong \arg \max_{g,s} \left( E_t[R_t] - \frac{1}{2} \gamma E_t[R_t]^2 - \frac{1}{2} \gamma \text{Var}[R_t] \right).
\]

Using \( p_t^{-1}dp_{t+1}/du_{t+1} = \pi + O(h) \) and \( df_i'/du_{t+1} = -1 \), the variance term can be written as:

\[
\text{Var}_t[R_t] \cong \chi^2 s_t^2 + \sigma^2 \pi^2 g_t^2
\]

Hence, the first order conditions read:

\[
(1 - \gamma E_t[R_t]) \left( f_i' - \frac{1}{r_t} \right) \cong \gamma \chi^2 s_t, \tag{21}
\]

\[
(1 - \gamma E_t[R_t]) \left( p_t^{-1}E_t[p_{t+1}] - \frac{1}{r_t} \right) \cong \gamma \sigma^2 \pi^2 g_t.
\]

Since \( f_i'^{-1} - 1 = k_t + u_t - 1 = O(h) \) and \( u_t - \mu = O(h) \), \( k_t - E[k_t] = O(h) \).
Since \( z_t = 1 - r_t = O(h) \) and \( p_t - E[p_t] = O(h) \), equation (14) can be written as:

\[
s_t = (1 - \eta z_t)^{-1} k_t = E[k_t] + O(h), \tag{22}
\]

\[
g_t = (1 - \eta z_t)^{-1} p_t = E[p_t] + O(h).
\]

Consider equation (21): i) multiply by \((1 - \gamma E_t[R_t]) r_t\); ii) multiply by \((f_i')^{-1} = k_t + u_t\) and \( p_t E_t[p_{t+1}]^{-1} = 1 + \pi (u_t - \mu) + O(h^2) \) respectively; iii) substitute equation (22) for \( s_t \) and \( g_t \); and iv) drop all terms of \( O(h^3) \). This yields equation (19) in Proposition 6. Hence, Proposition 6 holds if Proposition 5 holds.

Equation (14) for \( p_t \) and equation (19) is a system of three linear equations with three unknowns: \( k_t, p_t, \) and \( r_t \). The solution to this system reads:

\[
k_t \cong 1 - \mu + (\pi - 1) (u_t - \mu) + \psi, \tag{23}
\]

\[
p_t \cong \mu - \beta (\eta_0 + 1) + [1 - \pi (1 + \beta)] (u_t - \mu) + \psi_b,
\]

\[
r_t \cong 1 + \pi (u_t - \mu) + \psi_p.
\]
where \( \psi \equiv \psi_p - \psi_k = O(h^2) \) and \( \psi_b \equiv \psi_k - (1 + \beta) \psi_p = O(h^2) \) since \( \psi_k = O(h^2) \) and \( \psi_p = O(h^2) \), with \( d\psi/db < 0 \) and \( d\psi_b/db > 0 \). This proves Proposition 5 and hence Proposition 6.\( \blacksquare \)

**Proposition 7**

Equation (23) and using the definitions of \( \beta \) and \( \eta_b \) implies:

\[
\begin{align*}
E[p_t] &\cong \mu - b + \psi_b, \\
\pi &\equiv \frac{dp_t}{E[p_t] du_t} \cong \frac{1 - \pi [1 + (1 - \eta) b]}{\mu - b + \psi_b} \cong \frac{1}{1 + \mu - \eta b + \psi_b}.
\end{align*}
\] (24)

Substitution of the value for \( \pi \) in equation (23) yields:

\[
\begin{align*}
k_t &\cong 1 - \mu - \frac{\mu - \eta b + \psi_b}{1 + \mu - \eta b + \psi_b} (u_t - \mu) + \psi, \\
p_t &\cong \mu - b + \frac{\mu - b + \psi_b}{1 + \mu - \eta b + \psi_b} (u_t - \mu) + \psi_b, \\
r_t &\cong 1 + \frac{1}{1 + \mu - \eta b + \psi_b} (u_t - \mu) + \psi_p.
\end{align*}
\] (25)

Statement 1-4 follow immediately.

By equation (20) and (22) and using \( E[k_t] = 1 - \mu + O(h) \) and \( E[p_t] = \mu - b + O(h) \) yields:

\[
\begin{align*}
\gamma \text{Var}_t [R_t] &\cong E[\psi_k s_t + \psi_p g_t] \cong \Psi, \\
\Psi &\equiv \psi_k (1 - \mu) + \psi_p (\mu - b).
\end{align*}
\]

Equation (25) implies \( E[r_t] \cong 1 + \psi_p \). Using this and equation (21), the definition of \( R_t \) in equation (12) implies:

\[
\begin{align*}
E_t [R_t] &\cong \frac{1 - r_t}{r_t} + \Psi, \\
E [R_t] &\cong -\psi_p + \text{Var} [r_t] + \Psi, \\
\text{Var} [r_t] &\cong \left( \frac{\sigma}{1 + \mu - \eta b} \right)^2.
\end{align*}
\] (26)

where we use equation (24) in the final step.

By Lemma 1 and equation (12), the certainty equivalents of consumption of the young and the elderly satisfy:

27
\[
E_t \left[ (c_t^y)^{1-\gamma} \right]^{1/(1-\gamma)} = \frac{1-\eta}{\eta} (1-\eta z_t),
\]
\[
E \left[ (c_t^y)^{1-\gamma} \right]^{1/(1-\gamma)} \approx \frac{1-\eta}{\eta} (1+\eta b \psi_t),
\]
\[
E_t \left[ (c_{t+1}^y)^{1-\gamma} \right]^{1/(1-\gamma)} \approx (1-\eta z_t) \left( E_t [R_t + 1] - \frac{1}{2} \gamma \text{Var}_t [R_t] - \frac{1}{2} \gamma E_t [R_t]^2 \right)
\]
\[
= (1-\eta b + \eta b r_t) \left( \frac{1}{r_t} + \frac{1}{2} \psi - \frac{1}{2} \gamma \left( \frac{1-r_t}{r_t} \right)^2 \right),
\]
\[
E \left[ (c_{t+1}^y)^{1-\gamma} \right]^{1/(1-\gamma)} \approx 1 - (1-\eta b) \psi_p + \left( 1 - \eta b - \frac{1}{2} \gamma \right) \text{Var} [r_t] + \frac{1}{2} \psi.
\]
In the final step we use \( \Psi = O \left( h^2 \right) \), \( E[r_t] = 1 + O \left( h^2 \right) \), and:
\[
(1+\eta b x) \left\{ \frac{1}{1+x} - \frac{1}{2} \gamma \left( \frac{-x}{1+x} \right)^2 \right\} = 1 - (1-\eta b) x + \left( 1 - \eta b - \frac{1}{2} \gamma \right) x^2 + O \left( x^3 \right),
\]
where \( x \equiv r_t - 1 \). Hence:
\[
U \approx \frac{1}{\eta} \left[ (1-\eta) \ln \frac{1-\eta}{\eta} + \Delta \sigma^2 + \frac{1}{2} \psi_k (1-\mu) \right],
\]
\[
\Delta \equiv \frac{1-\eta b}{(1-\eta b + \mu)^2} + \frac{1}{2} \frac{(\mu-b)^2 - (1-2\eta b) (\mu-b) - \frac{1}{2} \gamma}{(1-\eta b + \mu)^2}.
\]
We have:
\[
\frac{d}{db} \left( \frac{1}{2} \frac{(\mu-b)^2 - (1-2\eta b) (\mu-b) - \frac{1}{2}}{(1-\eta b + \mu)^2} \right) =_{b=0} \frac{1-\eta}{(1-\eta)^3} + \frac{2\eta \mu^2}{(1+\mu)^3} > 0.
\]
This proves statement 7. ■