Subjective expected utility representations for Savage preferences on topological spaces*

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Abstract

In many decisions under uncertainty, technological constraints restrict both the acts which an agent can perform and the events which she can observe. To address such situations, we assume that the set of possible states of the world and the set of possible outcomes each have a topological structure. The only feasible acts are continuous functions from states to outcomes, and the only observable events are regular open subsets. We axiomatically characterize a Subjective Expected Utility (SEU) representation of ex ante preferences in this environment. Our SEU representation involves a continuous utility function on the outcome space (unique up to positive affine transformations), and a unique probability measure on the Boolean algebra of regular open subsets of the state space. With additional topological hypotheses, we obtain a unique Borel or Radon probability measure, along with an auxiliary apparatus called a *liminal structure*, which describes the agent's informational constraints, and generalizes the classical SEU representation.

Keywords: Subjective expected utility; topological spaces; technological feasibility constraints; continuous acts; regular open sets; Borel probability measures.
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Natura non facit saltum. —Linnaeus

1 Introduction

Economic decisions under uncertainty often face technological constraints. For instance, consider a farmer who must plant crops in the early spring, without knowing the meteorological conditions for the rest of the year. The crop yields of his various planting strategies are thus uncertain at the moment of choice. But slight variations of the meteorological conditions will only result in slight variations in yields. The constraints of the agricultural technology imply that the only strategies available to the farmer are those where crop yields depend *continuously* on the unpredictable meteorological conditions.

Continuity constraints manifest in many other decision problems under uncertainty; in particular, they arise in most economic production activities which depend upon natural resource extraction, weather conditions, or any other interaction with unpredictable features of the natural environment. They also arise in medical decisions, where the uncertainty concerns the patient's medical condition, and the outcome is her prognosis. Anthropogenic climate change generates a plethora of such decision problems; there is uncertainty about the values of many parameters in climate models, which leads to uncertainty about the response of weather patterns (e.g. temperature, rainfall, floods, droughts) to rising CO_2 concentrations. There is also uncertainty about the social and economic impact of these weather patterns, as well as the proposed policies to reduce CO_2 emissions. Finally, there is hour-to-hour uncertainty about the electricity output of solar and wind-power facilities. But in all these cases, the outcome varies continuously with the unknown variables.

Continuity constraints also arise in many financial and economic decision problems. For example, the income stream arising from an individual's investment in education or a firm's investment in physical capital is a continuous function of future economic conditions. The value of a portfolio is a linear combination of the values of its constituent assets. In most financial derivatives (e.g. futures, options), the payoff for both buyer and seller is a continuous function of the price of the underlying assets. In most insurance contracts, the indemnity is a continuous function of the loss. Finally, the future real value of a savings instrument is a continuous function of future real interest rates. In these examples, continuity restrictions can be interpreted as a kind of market incompleteness.

Furthermore, feasibility considerations constrain not only the possible actions, but also the information available to an agent. The limitations of her measurement technology may restrict the events that an agent can observe, form beliefs about, or employ for Bayesian updating. In particular, measurement devices are often *stable*, meaning that they are relatively insensitive to small perturbations. This is especially common in devices which convert "anolog" to "digital" signals. For example, a digital thermometer may lack the precision that is required to detect a small change of temperature. Measurement devices are also *approximate*: an observation cannot always be taken at face value, and sometimes only provides some approximate value. For instance, the statistical estimation of unknown parameters in climate or economic models often only yields an interval of possible values.

The Subjective Expected Utility (SEU) model is the standard paradigm to describe

decision-making under uncertainty. A key feature of the classic axiomatic foundations of Savage (1954) is that the agent has preferences over *all* possible functions from states into outcomes, and can condition on *all* subsets of the state space. This makes sense in decision problems where the natural topology on the state space is *discrete* (e.g. bets on coin flips, urn experiments, sports games, or Arrow-Debreu economies). One *could* apply the Savage approach to technologically constrained decision problems, but this would require the agent to rank infeasible acts and condition on unobservable events, which would undermine both the normative and the descriptive content of the preference relation, the axioms, and the resulting SEU representation. At a normative level, an agent might feel uncomfortable about formulating preferences over infeasible acts, or conditioning on unobservable events; hence she might be reluctant to apply the Savage axioms to such preferences. These axioms are supposed to impose some "internal consistency" on preferences. But why should preferences over *feasible* acts be consistent with, and sometimes even determined by, preferences over *infeasible* acts? At a descriptive level, it is impossible by definition to observe an agent's preference over infeasible acts, or her preferences conditional on unobservable events. Such acts and events can still play a role in thought experiments. But since they are impossible to properly incentivize, one can seriously question the meaning of such preferences and their relevance to the elicitation of utility and beliefs. For these reasons, technological constraints make it desirable to depart from the Savage framework and restrict preferences to feasible acts and observable events.

This paper studies decision-making under uncertainty with technological constraints, and axiomatically characterizes SEU representations of *ex ante* preferences in such an environment. The consequences of the decision range over a topological space of *outcomes*; these may be crop yields, health status, production levels, income streams, or consumption bundles. The underlying uncertainty is represented by a topological space of *states of the world*; this encodes all the meteorological, physiological, geophysical, or financial variables on which the outcome (continuously) depends. The feasible alternatives are given by a set of continuous functions, or *acts*, from the state space onto the outcome space; these could be production plans, medical interventions, climate policies, financial portfolios, or insurance contracts. The observable events are given by the Boolean algebra of *regular subsets* –a specific subdomain of the state space topology. These represent the information that can be obtained by measuring meteorological conditions, estimating unknown climate parameters, or performing medical tests. Furthermore, the domain of feasible acts need not contain *all* continuous acts, and the observable events need not consist of *all* regular subsets; our framework can incorporate further technological restrictions on acts and information. Finally, the agent's preferences only rank feasible acts conditional upon observable events.

Despite these restrictions on feasible acts and observable events, we obtain a *unique* SEU representation. Thus, uniqueness does not require the universal domains of acts and events. Moreover, our SEU representation is compatible with the topological structure of the states and the outcomes. In our representation, utility is a continuous function of the outcome. Thus, "similar outcomes" yield "similar utility levels"; this makes our SEU representation particularly relevant to applications in economics and finance, which usually take continuous utility for granted. Finally, under economically plausible assumptions on

the state space topology, the *ex ante* beliefs of the agent involve a Baire charge or a Borel probability measure. But these beliefs do not consist *only* of such a measure; they are complemented with what we call a *liminal structure*, with which the agent compensates for her informational constraints. Thanks to the restriction of preference to feasible acts, it provides a "dynamically consistent" and "consequentialist" updating rule for *ex ante* beliefs that generalizes the classical Bayes rule.

For a general topological state space, or when the agent faces further informational constraints, beliefs are described by what we call a *credence* – a structure like a finitely-additive probability measure on the algebra of regular sets of the state space topology. As we later explain in greater detail, a credence implicitly combines both the Borel probability measure and the liminal structure.

The remainder of this paper is organized as follows: Section 2 introduces the notation and terminology used in the paper. Section 3 introduces the axioms we use to characterize the SEU representations. Section 4 presents our first SEU representation result, based on a special kind of probability measure called a *Baire charge*. Next, Sections 5 and 6 present *liminal* SEU representations, which are somewhat more informative. All of these representation theorems require mild assumptions on the state space topology, and assume that the agent's "observation technology" is maximally sophisticated. Section 7 removes these assumptions, and presents our most general SEU representation (Theorem 11), where the agent's beliefs are represented by a credence. Section 8 reviews prior literature, and discusses some other applications and interpretations of our results.

All the proofs are in the Appendices. Appendix A provides the proof of Theorem 11, which is the lynchpin result of the paper. Appendix B then derives the SEU representations of Sections 4, 5 and 6, using Theorem 11. Finally Appendix C discusses the topological implications of some of our assumptions.

2 Basic framework

Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces. Elements of $\mathcal{S}$ are called *states of the world* and describe the various possible resolutions of uncertainty. Elements of $\mathcal{X}$ are called *outcomes* and represent the various possible consequences of decisions. We will assume $\mathcal{X}$ is connected.

Acts. Like Savage, we will suppose that the agent can choose from a menu of *acts*, where each act is a function from the state space onto the outcome space. This function describes the outcome that would result from the choice of this act at each possible state of the world. Unlike Savage, we will focus on situations where feasibility imposes *continuous* relationships between states and outcomes. We will also suppose that the only feasible acts are *bounded* functions. This captures the intuition that no feasible act results in arbitrarily large "gains" or "losses".

Recall that a subset $\mathcal{Y} \subseteq \mathcal{X}$ is *relatively compact* if its closure $\text{clos}(\mathcal{Y})$ is compact. (It follows that any continuous, real-valued function on $\mathcal{X}$ is bounded when restricted to $\mathcal{Y}$.) For example, if $\mathcal{X}$ is a metric space, then $\mathcal{Y}$ is relatively compact if and only if $\mathcal{Y}$ is a bounded subset of $\mathcal{X}$. A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *bounded* if its image $\alpha(\mathcal{S})$ is

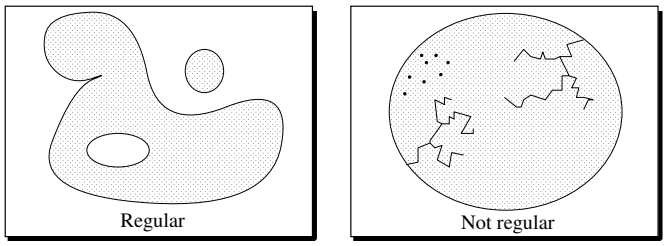

Figure 1: Regular and non-regular open subsets.

relatively compact in $\mathcal{X}$. If $\mathcal{X}$ is a metric space, then this agrees with the usual definition of "bounded". But this definition makes sense even if $\mathcal{X}$ is nonmetrizable. Let $\mathcal{C}(\mathcal{S}, \mathcal{X})$ be the set of all continuous functions from $\mathcal{S}$ into $\mathcal{X}$, and let $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$ be the set of all *bounded* continuous functions from $\mathcal{S}$ into $\mathcal{X}$.

There may be additional feasibility restrictions on acts, beyond continuity and boundedness. Thus, we introduce an exogenously given subset $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$; this is the set of *feasible acts*. If technological constraints only entail continuity and boundedness, then $\mathcal{A} = \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. But in general, $\mathcal{A}$ could be much smaller. For instance, if feasible production plans must be infinitely differentiable, then we could take and define $\mathcal{A}$ to be the set of all infinitely differentiable functions from $\mathcal{S}$ to $\mathcal{X}$. However, the collection $\mathcal{A}$ cannot be *too* small; it must be large enough to satisfy structural condition (Rch) below, and must contain all constant acts; these represents *riskless* alternatives. The inclusion of such acts in $\mathcal{A}$ means that we can risklessly obtain any outcome by a feasible act.

Information. We assume that the only events which the agent can observe are regular subsets. A *regular subset* of a topological space $\mathcal{Y}$ is an open subset $\mathcal{E} \subseteq \mathcal{Y}$ which is the interior of its own closure.[1] For example, let $\mathcal{Y} = \mathbb{R}$; then $(0, 1)$ is a regular subset, but $(0, 1) \cup (1, 2)$ is not. Heuristically, an open set is regular if it does not have any "pinholes" or "cracks" (Figure 1).

Clearly, the intersection of two regular subsets is another regular subset. Given any two regular subsets $\mathcal{D}, \mathcal{E} \subseteq \mathcal{Y}$, we define $\mathcal{D} \vee \mathcal{E}$ to be the smallest regular subset of $\mathcal{Y}$ containing both $\mathcal{D}$ and $\mathcal{E}$ —that is, the interior of the closure of $\mathcal{D} \cup \mathcal{E}$. For example, if $\mathcal{Y} = \mathbb{R}$, then $(0, 1) \vee (1, 2) = (0, 2)$. Meanwhile, we define $\mathcal{D} \wedge \mathcal{E} := \mathcal{D} \cap \mathcal{E}$. Finally, given a regular subset $\mathcal{D}$, we define its *complement* $\neg\mathcal{D}$ to be the interior of $\mathcal{Y} \setminus \mathcal{D}$. This is another regular subset. The set $\mathfrak{R}(\mathcal{Y})$ of all regular subsets of $\mathcal{Y}$ forms a Boolean algebra under these three operations (Fremlin, 2004, Theorem 314P).

Now let $\mathcal{R} \in \mathfrak{R}(\mathcal{Y})$. A *finite regular partition* of $\mathcal{R}$ is a collection $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ (for some $N \in \mathbb{N}$) of disjoint regular subsets of $\mathcal{Y}$ such that $\mathcal{R} = \mathcal{R}_1 \vee \cdots \vee \mathcal{R}_N$. Equivalently: for all $m \in [1 \ldots N]$, the set $\mathcal{R}_m$ is the interior of $\mathcal{R} \setminus \bigcup_{\substack{n=1 \\ n \neq m}}^{N} \mathcal{R}_n$. This implies that $\bigcup_{n=1}^{N} \mathcal{R}_n$ is dense in $\mathcal{R}$. For instance, if $\mathcal{Y} = \mathbb{R}$, then $\{(0, 1), (1, 2)\}$ is a regular partition of $(0, 2)$. Regular

[1] These are sometimes called "regular open" subsets.

partitions share some properties with the partitions used in standard Bayesian decision theory: cells must be nonempty and represent mutually exclusive events. But they depart from the standard model in that cells are not fully exhaustive: they only cover "almost all" of the space. A regular partition is nonetheless *observationally exhaustive*: the agent always *thinks* she observes some cell of the regular partition, even if the true state lies in the space between the cells. Suppose the agent makes an observation corresponding to the regular partition $\mathfrak{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, and let $s \in \mathcal{S}$ be the true state of the world. If $s \in \mathcal{R}_n$, then the agent will observe $\mathcal{R}_n$. But if $s \in \partial\mathcal{R}_n \cap \partial\mathcal{R}_m$, then the agent might "observe" $\mathcal{R}_n$ *or* $\mathcal{R}_m$ (or indeed, any other cell of $\mathfrak{R}$ whose closure contains s).

For example, suppose $\mathcal{S} = [-1, 1]$, let $\mathcal{L} := [-1, 0)$, and let $\mathcal{R} := (0, 1]$. Then, $\{\mathcal{L}, \mathcal{R}\}$ is a regular partition of $\mathcal{S}$. If the true state is negative, then the agent will observe $\mathcal{L}$, while if it is positive, she will observe $\mathcal{R}$. If the true state is zero, then she might "observe" either $\mathcal{L}$ or $\mathcal{R}$, even though neither event truly holds. Therefore, observing $\mathcal{L}$ (resp. $\mathcal{R}$) only provides the knowledge that the state is nonpositive (resp. nonnegative).

Regular partitions of the state space satisfy the aforementioned "stability" property, because their cells are open sets: if $s \in \mathcal{R}_n$, then the agent will observe $\mathcal{R}_n$, and a sufficiently small perturbation of s will not modify this observation. Regular partitions can also be interpreted as "approximate" measurements, which include the possibility of *measurement errors*: the observation $\mathcal{R}_n$ does not mean that the true state lies in $\mathcal{R}_n$, but only that it lies inside the *closure* of $\mathcal{R}_n$. A measurement error occurs precisely when the true state lies in $\partial\mathcal{R}_n$ (and thus, outside of $\mathcal{R}_n$ itself), yet the agent nonetheless "observes" $\mathcal{R}_n$.

Technological constraints might impose additional informational restrictions beyond mere regularity. Let $\mathfrak{B}$ be some Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$; we interpret $\mathfrak{B}$ as the algebra of "observable events". We will say that a regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of an event $\mathcal{R} \in \mathfrak{B}$ is a $\mathfrak{B}$-*partition* if $\mathcal{R}_n \in \mathfrak{B}$ for all $n \in [1 \ldots N]$. In Sections 4, 5 and 6, we will assume that *all* regular open subsets are observable (i.e. $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$). But in the most general version of our model (Section 7), we will suppose that the only observable measurements are those arising some Boolean subalgebra $\mathfrak{B}$ of $\mathfrak{R}(\mathcal{S})$. Because of this, we will formulate all the axioms of Section 3 in terms of such a Boolean subalgebra. However, for the purposes of Sections 4, 5 and 6, the reader can just assume that $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ while reading Section 3.

Preferences. We assume that the agent has a system of *conditional preference orders*. However, these preferences can rank only *feasible acts*, and can only be conditioned on *observable* events. Thus, in terms of its primitive behavioral data, our model is not directly comparable to the Savage (1954) theory: while Savage assumes a *single* preference order on the universal domain of acts, our approach relies on a *collection* of preference orders on a more restrictive domain. However, compared to other conditional versions of SEU (for instance, see Ghirardato, 2002), our approach typically requires less behavioral data, both in terms of the number of preference orders and their domain.

For any regular subset $\mathcal{B} \in \mathfrak{B}$, and any act $\alpha \in \mathcal{A}$, let $\alpha_{\restriction\mathcal{B}}$ denote the restriction of α to a function on $\mathcal{B}$. Let $\mathcal{A}(\mathcal{B}) := \{\alpha_{\restriction\mathcal{B}};\ \alpha \in \mathcal{A}\}$ be the set of acts conditional upon $\mathcal{B}$. For

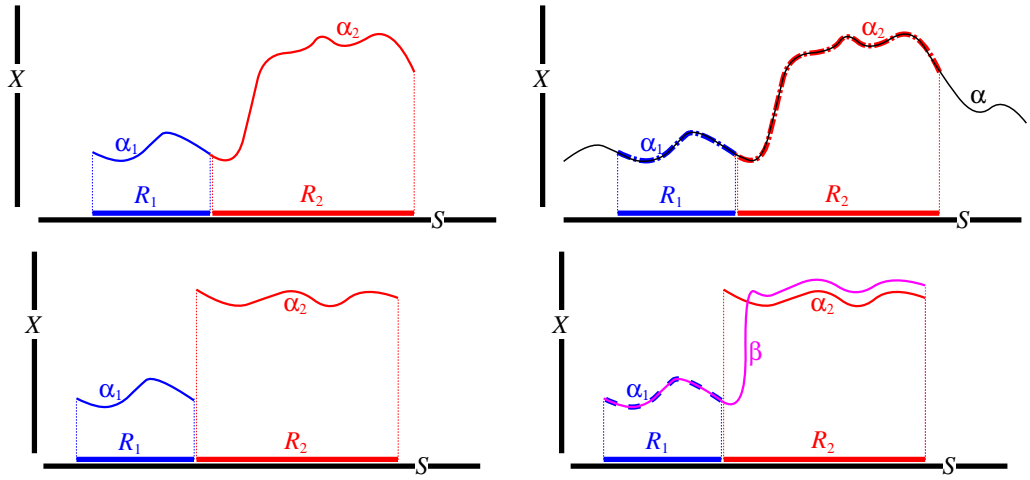

Figure 2: *Top row.* α_1 is compatible with α_2. *Bottom row.* The richness condition.

example, if $\mathcal{X}$ is compact and $\mathcal{A} = \mathcal{C}(\mathcal{S}, \mathcal{X})$, and $\mathcal{B}$ is a *retract* of $\mathcal{S}$, then $\mathcal{A}(\mathcal{B}) = \mathcal{C}(\mathcal{B}, \mathcal{X})$.[2] For a concrete example, suppose $\mathcal{S} = \mathbb{R}^N$ and $\mathcal{X} = [0,1]^M$ (for some $N, M \geq 1$), and let $\mathcal{B}$ be the open unit ball in $\mathbb{R}^N$. If $\mathcal{A} = \mathcal{C}(\mathbb{R}^N, [0,1]^M)$, then $\mathcal{A}(\mathcal{B}) := \mathcal{C}(\mathcal{B}, [0,1]^M)$. Likewise, if $\mathcal{A}$ is the set of all differentiable (resp. Lipschitz, Hölder, or bounded variation) functions from $\mathbb{R}^N$ to $[0,1]^M$, then $\mathcal{A}(\mathcal{B})$ is the set of all differentiable (resp. Lipschitz, Hölder, or bounded variation) functions from $\mathcal{B}$ to $[0,1]^M$.

For all $\mathcal{B} \in \mathfrak{B}$, let $\succeq_{\mathcal{B}}$ be a preference order on $\mathcal{A}(\mathcal{B})$. We interpret $\succeq_{\mathcal{B}}$ as the *conditional preferences* over $\mathcal{A}(\mathcal{B})$ of an agent who has observed the event $\mathcal{B}$. We will therefore refer to the system $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ as a *conditional preference structure*; this will be the primitive data of the model. Our goal is to axiomatically characterize an SEU representation for the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$.

3 Axioms

Savage's (1954) theory of SEU maximization depends on the fact that, for any two feasible acts α and β, and any admissible event $\mathcal{B}$, the "spliced" act $\alpha_{\mathcal{B}}\beta$ (which is equal to α on $\mathcal{B}$ and to β on the complementary event $\mathcal{B}^{\complement}$) is also feasible. However, such "spliced" acts are often discontinuous, hence, inadmissible in our framework. Therefore, we cannot apply most of the Savage axioms in their classic form. To compensate for this problem, we will require the set $\mathcal{A}$ of feasible acts to satisfy a "Richness" condition with respect to the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$.

[2]However, if $\mathcal{B}$ is *not* a retract of $\mathcal{S}$, then not every continuous function on $\mathcal{B}$ can be extended to a continuous function on $\mathcal{S}$. Thus, $\mathcal{A}(\mathcal{B})$ is usually a proper subset of $\mathcal{C}(\mathcal{B}, \mathcal{X})$. The same holds for differentiable functions, Lipschitz functions, etc.

The richness condition. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ be disjoint regular subsets of $\mathcal{S}$. For any $\alpha_1 \in \mathcal{A}(\mathcal{B}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{B}_2)$, say that α_1 and α_2 are *compatible* if there is some $\alpha \in \mathcal{A}$ with $\alpha_{\restriction \mathcal{B}_1} = \alpha_1$ and $\alpha_{\restriction \mathcal{B}_2} = \alpha_2$. We need $\mathcal{A}$ to satisfy the following condition:

(Rch) For any disjoint regular subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, and any $\alpha_1 \in \mathcal{A}(\mathcal{B}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{B}_2)$, there is an act $\beta_2 \in \mathcal{A}(\mathcal{B}_2)$ which is compatible with α_1, such that $\alpha_2 \approx_{\mathcal{B}_2} \beta_2$.

In other words, the values of an act on a regular subset $\mathcal{B}_1$ do not restrict the indifference class of that act conditional upon the disjoint regular subset $\mathcal{B}_2$, in spite of the continuity requirement on feasible acts. If there is a "gap" between $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathcal{S}$, then (Rch) is generally trivial, since then *every* element of $\mathcal{A}(\mathcal{B}_2)$ is compatible with α_1. The nontrivial case of (Rch) is when $\mathcal{B}_1$ and $\mathcal{B}_2$ are "touching" –e.g. when $\mathcal{B}_1 = \neg\mathcal{B}_2$. In particular, (Rch) provides the following weak version of Savage's act splicing: For any regular event $\mathcal{B} \in \mathfrak{B}$, and any feasible acts $\alpha, \beta \in \mathcal{A}$, there exists a feasible act $\gamma \in \mathcal{A}$ that is equal to α on $\mathcal{B}$ and *indifferent* to $\beta_{\restriction \neg\mathcal{B}}$ conditional on $\neg\mathcal{B}$. (Rch) is also is somewhat similar to the *solvability* condition often employed in axiomatizations of additive utility representations.

$\mathcal{A}$ need not actually contain *all* bounded and continuous functions from $\mathcal{S}$ to $\mathcal{X}$, as long as it satisfies (Rch) (and contains all constant acts). For example, suppose $\mathcal{S}$ and $\mathcal{X}$ are differentiable manifolds (e.g. open subsets of Euclidean spaces $\mathbb{R}^N$ and $\mathbb{R}^M$, for some $N, M \geq 1$), and let $\mathcal{A}$ be the set of all *differentiable* functions from $\mathcal{S}$ to $\mathcal{C}$; then a conditional preference structure on $\mathcal{A}$ can easily satisfy (Rch) along with our other axioms.[3] Alternatively, let $\mathcal{S}$ and $\mathcal{X}$ be metric spaces, let $c \in (0, 1]$, and let $\mathcal{A}$ be the set of all *c-Hölder-continuous* functions from $\mathcal{S}$ to $\mathcal{X}$; then (Rch) is easily satisfied.[4] Or, let $\mathcal{S}$ be a bounded interval in $\mathbb{R}$, let $\mathcal{X}$ be a path-connected metric space, and let $\mathcal{A}$ be the set of all continuous functions from $\mathcal{S}$ into $\mathcal{X}$ having *bounded variation*; then again (Rch) is easily satisfied.[5] However, suppose $\mathcal{S}$ and $\mathcal{X}$ are open subsets of Euclidean spaces, and let $\mathcal{A}$ be a collection of *analytic* functions from $\mathcal{S}$ to $\mathcal{X}$; then a conditional preference structure on $\mathcal{A}$ *cannot* satisfy condition (Rch).[6] In particular, if $\mathcal{A}$ is a collection of *polynomial* functions from $\mathcal{S}$ to $\mathcal{X}$, then it cannot satisfy (Rch).

The ordering axiom. For the rest of this paper, we will assume that each order $\succeq_{\mathcal{B}}$ in the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ is *complete* (for any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, at least one of $\alpha \succeq_{\mathcal{B}} \beta$ or $\beta \succeq_{\mathcal{B}} \alpha$ holds), *transitive* (for any $\alpha, \beta, \gamma \in \mathcal{A}(\mathcal{B})$, if $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$, then $\alpha \succeq_{\mathcal{B}} \gamma$), and *nontrivial* (there exist $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ such that $\alpha \succ_{\mathcal{B}} \beta$).

[3] The same is true if $\mathcal{A}$ is the set of N-times differentiable functions, for any $N \in [2 \ldots \infty]$.

[4] A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *c-Hölder-continuous* if there is some constant $K > 0$ such that $d[f(s_1), f(s_2)] \leq K \cdot d(s_1, s_2)^c$ for all $s_1, s_2 \in \mathcal{S}$. In the special case when $c = 1$, these are called *Lipschitz-continuous* functions. Any continuously differentiable function is Lipschitz.

[5] A function $\alpha : [0, S] \longrightarrow \mathcal{X}$ has *bounded variation* if its "total variation" $\sup\{\sum_{n=1}^{N} d[\alpha(s_n), \alpha(s_{n-1})]$; $N \in \mathbb{N}$ and $0 \leq s_0 < s_1 < \cdots < s_N \leq S\}$ is finite. Heuristically, this means that α does not oscillate too violently; it describes a path through $\mathcal{X}$ of finite total length.

[6] An infinitely differentiable function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *analytic* if it is the limit of its own Taylor series in a neighbourhood around each point in $\mathcal{S}$. For example, any polynomial function is analytic. In effect, an analytic function can be completely reconstructed from its behaviour in an "infinitesimal" neighbourhood around any point in its domain. Thus, analytic functions are extremely "rigid": an analytic function defined on some open subset $\mathcal{R} \subseteq \mathcal{S}$ has at most *one* extension to an analytic function on all of $\mathcal{S}$.

These assumptions are more natural in our framework than in Savage's: they only require a transitive ordering on *feasible* acts, and not on all the theoretically conceivable acts. To understand the interplay between feasibility and transitivity, consider a case where an agent observes event $\mathcal{B} \in \mathfrak{B}$, and must make a choice between two feasible acts α and γ in $\mathcal{A}(\mathcal{B})$. Say momentarily that she has preferences over unfeasible acts, and that she realizes that there is an unfeasible act β such that $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$. A blind application of the transitivity rule would lead to the conclusion that α is preferred to γ conditional on $\mathcal{B}$. But the unfeasibility of β undermines the meaningfulness of both rankings $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$. Why should these two rankings be relevant for the choice between α and γ? By restricting preferences to feasible acts, we eliminates these problematic cases.

The separability axioms. Additive separability over disjoint events is a characteristic feature of SEU theories. In a Savage framework, it is captured by the Sure Thing Principle. In Ghirardato's (2002) conditional version of the Savage theory, where an agent is endowed with a collection of conditional preferences, separability is instead captured by the axiom of Dynamic Consistency. But it is not straightforward to adapt either of these axioms to our topological setting: the Sure Thing Principle requires spliced acts, while Dynamic Consistency relies on partitions. The next axiom captures separability through a version of Dynamic Consistency that only applies to regular partitions of a regular event.

(Sep) For any event $\mathcal{B} \in \mathfrak{B}$, any disjoint events $\mathcal{D}, \mathcal{E} \in \mathfrak{B}$ such that $\mathcal{D} \vee \mathcal{E} = \mathcal{B}$, and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ with $\alpha_{|\mathcal{D}} \approx_{\mathcal{D}} \beta_{|\mathcal{D}}$, we have $\alpha \succeq_{\mathcal{B}} \beta$ if and only if $\alpha_{|\mathcal{E}} \succeq_{\mathcal{E}} \beta_{|\mathcal{E}}$.

In (Sep), the "forward implication" (from $\alpha \succeq_{\mathcal{B}} \beta$ to $\alpha \succeq_{\mathcal{E}} \beta$) captures the idea that a feasible act that was deemed optimal conditional on $\mathcal{B}$ will be still be optimal conditional on $\mathcal{E}$. The "backward implication" captures the idea that a more-informed decision is more reliable than a less-informed decision; thus, whenever possible, decisions based on inferior information should be guided by the hypothetical decisions that *would* have been made with *superior* information. In this case, the agent is willing to choose α over β given inferior information ($\mathcal{B}$), because she recognizes that she *would* be willing to choose α given superior information (either $\mathcal{D}$ or $\mathcal{E}$).

Just as the restriction to feasible acts strengthens the appeal of the ordering axiom, the restriction to regular events strengthens the appeal of (Sep) —more specifically, its "backward implication". To see this, suppose the agent must choose between two feasible acts α and β, conditional on some regular event $\mathcal{B}$. Say that she has preferences conditional upon nonregular (hence, unobservable) events $\mathcal{D}$ and $\mathcal{E}$, with $\mathcal{D} \vee \mathcal{E} = \mathcal{B}$, such that $\alpha_{|\mathcal{D}} \approx_{\mathcal{D}} \beta_{|\mathcal{D}}$ and $\alpha_{|\mathcal{E}} \succeq_{\mathcal{E}} \beta_{|\mathcal{E}}$. A naïve application of separability would then yield $\alpha \succeq_{\mathcal{B}} \beta$. But $\mathcal{D}$ and $\mathcal{E}$ are *unobservable* events, so it is not clear that the preferences conditional on $\mathcal{D}$ and $\mathcal{E}$ are even meaningful, much less that they should determine the choice between α and β. The restriction to regular events eliminates this problem.

It is easy to see that the logical equivalence in Axiom (Sep) also holds for indifference and for strict preference: for any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ with $\alpha_{|\mathcal{D}} \approx_{\mathcal{D}} \beta_{|\mathcal{D}}$, we have:

(i) $\alpha \succ_{\mathcal{B}} \beta$ if and only if $\alpha_{|\mathcal{E}} \succ_{\mathcal{E}} \beta_{|\mathcal{E}}$; and

(ii) $\alpha \approx_{\mathcal{B}} \beta$ if and only if $\alpha_{\restriction \mathcal{E}} \approx_{\mathcal{E}} \beta_{\restriction \mathcal{E}}$.

Statement (i) means that *no regular event is null*. Thus, any SEU representation must give nonzero probability to all regular events in $\mathfrak{B}$. Conversely, statement (ii) says that *the boundary of any regular event is null*: the behaviour of α and β on that small part of $\mathcal{B}$ that is not covered by $\mathcal{D} \cup \mathcal{E}$ is irrelevant for decisions conditional on $\mathcal{B}$. This seems to suggest that the SEU representation must give *zero probability* to the boundary of any regular set. But $\mathcal{A}$ is a set of *continuous* functions; thus, the behaviour of α and β on the open sets $\mathcal{D}$ and $\mathcal{E}$ entirely determines their behaviour on the common boundary $\partial\mathcal{D} \cap \partial\mathcal{E}$. Thus, statement (ii) does not mean that we *ignore* the behaviour of α and β on $\partial\mathcal{D} \cap \partial\mathcal{E}$, as if $\partial\mathcal{D} \cap \partial\mathcal{E}$ had zero probability; it just means that we have *already* implicitly accounted for this behaviour in our rankings of $\alpha_{\restriction \mathcal{D}}$ versus $\beta_{\restriction \mathcal{D}}$ and $\alpha_{\restriction \mathcal{E}}$ versus $\beta_{\restriction \mathcal{E}}$. This implicit account will become explicit with the "liminal structures" of Sections 5 and 6.

If $\mathcal{D}, \mathcal{E} \in \mathfrak{B}$ are disjoint and $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$, then Axiom (Sep) says that the $\succeq_{\mathcal{B}}$-ranking of two acts $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ is partly determined by the $\succeq_{\mathcal{D}}$-ranking of $\alpha_{\restriction \mathcal{D}}$ versus $\beta_{\restriction \mathcal{D}}$ and the $\succeq_{\mathcal{E}}$-ranking of $\alpha_{\restriction \mathcal{E}}$ versus $\beta_{\restriction \mathcal{E}}$. The next axiom says that this dependency is continuous.

(PC) Let $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$ as in axiom (Sep). Let $\underline{\beta}, \alpha, \overline{\beta} \in \mathcal{A}(\mathcal{B})$ be three acts with $\underline{\beta} \prec_{\mathcal{B}} \alpha \prec_{\mathcal{B}} \overline{\beta}$. Then there exist $\underline{\delta}, \overline{\delta} \in \mathcal{A}(\mathcal{D})$ and $\underline{\epsilon}, \overline{\epsilon} \in \mathcal{A}(\mathcal{E})$, with $\underline{\delta} \prec_{\mathcal{D}} \alpha_{\restriction \mathcal{D}} \prec_{\mathcal{D}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{E}} \alpha_{\restriction \mathcal{E}} \prec_{\mathcal{E}} \overline{\epsilon}$ such that, for any $\alpha' \in \mathcal{A}(\mathcal{B})$, if $\underline{\delta} \prec_{\mathcal{D}} \alpha'_{\restriction \mathcal{D}} \prec_{\mathcal{D}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{E}} \alpha'_{\restriction \mathcal{E}} \prec_{\mathcal{E}} \overline{\epsilon}$ then $\underline{\beta} \prec_{\mathcal{B}} \alpha' \prec_{\mathcal{B}} \overline{\beta}$.

The intuition here is that a small variation in $\alpha_{\restriction \mathcal{D}}$ and $\alpha_{\restriction \mathcal{E}}$ (relative to the order topologies on $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\mathcal{E})$) should not affect the $\succeq_{\mathcal{B}}$- ranking of α versus $\underline{\beta}$ and $\overline{\beta}$.

Continuity of *ex post* preferences. For any $x \in \mathcal{X}$, let κ^x be the constant x-valued act on $\mathcal{S}$. Let $\mathcal{K} := \{\kappa^x;\ x \in \mathcal{X}\}$. We have assumed $\mathcal{K} \subseteq \mathcal{A}$, so the preference order $\succeq_{\mathcal{S}}$, restricted to $\mathcal{K}$, induces a preference order $\succeq_{\text{xp}}$ on $\mathcal{X}$ as follows: for any $x, y \in \mathcal{X}$,

$$\Big(x \succeq_{\text{xp}} y\Big) \quad \Longleftrightarrow \quad \Big(\kappa^x \succeq_{\mathcal{S}} \kappa^y\Big). \tag{1}$$

In effect, $\succeq_{\text{xp}}$ describes the *ex post* preferences of the agent on outcomes in $\mathcal{X}$, in a situation with no uncertainty. The next axiom says that these preferences are compatible with the underlying topology on the space of outcomes.

(C) The *ex post* order $\succeq_{\text{xp}}$ is continuous in the topology on $\mathcal{X}$. That is: for all $x \in \mathcal{X}$, the contour sets $\{y \in \mathcal{X};\ y \succeq_{\text{xp}} x\}$ and $\{y \in \mathcal{X};\ y \preceq_{\text{xp}} x\}$ are closed subsets of $\mathcal{X}$.

Certainty equivalents. For any $\mathcal{B} \in \mathfrak{B}$ and $x \in \mathcal{X}$, let $\kappa^x_{\mathcal{B}} := (\kappa^x)_{\restriction \mathcal{B}}$; this is the constant x-valued act, conditional on $\mathcal{B}$. Given an act $\alpha \in \mathcal{A}(\mathcal{B})$, we say x is a *certainty equivalent* for α on $\mathcal{B}$ if $\kappa^x_{\mathcal{B}} \approx_{\mathcal{B}} \alpha$. The next axiom is a mild richness condition on $\mathcal{X}$.

(CEq) For any event $\mathcal{B} \in \mathfrak{B}$, any act $\alpha \in \mathcal{A}(\mathcal{B})$ has a certainty equivalent on $\mathcal{B}$.

Axiom (CEq) may appear somewhat implausible. But it is a logical consequence of the following axiom of "constant continuity" which may seem more natural.

(CC) For any event $\mathcal{B} \in \mathfrak{B}$ and any act $\alpha \in \mathcal{A}(\mathcal{B})$, the sets $\{x \in \mathcal{X},\ \kappa^x_{\mathcal{B}} \succeq_{\mathcal{B}} \alpha\}$ and $\{x \in \mathcal{X},\ \alpha \succeq_{\mathcal{B}} \kappa^x_{\mathcal{B}}\}$ are closed in $\mathcal{X}$.

If $\mathcal{X}$ is connected and $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, then (CC) is equivalent to the conjunction of (C) and (CEq). So we could state our results with a single axiom (CC) in place of (CEq) and (C).

The statewise dominance axiom. Our next axiom imposes some consistency between the agent's conditional preference structure and her *ex post* preferences. It says that the agent always prefers a statewise dominating act.

(Dom) For any $\mathcal{B} \in \mathfrak{B}$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, if $\alpha(b) \succeq_{\mathrm{xp}} \beta(b)$ for all $b \in \mathcal{B}$, then $\alpha \succeq_{\mathcal{B}} \beta$. Furthermore, if $\alpha(b) \succ_{\mathrm{xp}} \beta(b)$ for all $b \in \mathcal{B}$, then $\alpha \succ_{\mathcal{B}} \beta$.

Recall that the fact that the agent "observes" the event $\mathcal{B} \in \mathfrak{B}$ does not necessarily mean that the true state belongs to $\mathcal{B}$ —it only means that it lies within the *closure* of $\mathcal{B}$. But since α and β are continuous functions, they have unique extensions to the closure of $\mathcal{B}$, and these extensions preserve the statewise dominance relation. For this reason, it is sufficient to check for statewise dominance over $\mathcal{B}$ to obtain statewise dominance over the set of all states that remain possible; that is, the closure of $\mathcal{B}$.

(Dom) appears similar to (Sep), and thus to Savage's axiom P2. The difference is that (Sep) applies to regular partitions, while (Dom) applies to partitions into singleton sets, which, in general, are *not* regular. Thus, (Dom) cannot be obtained as a special case of (Sep). Axiom (Dom) is also related to Savage's axioms P3 and P7. Axiom P3 requires the ranking of outcomes to be independent of the events that yield the outcomes. (Dom) entails a similar form of state independence: it implies that $\succeq_{\mathcal{S}}$ can be replaced by $\succeq_{\mathcal{B}}$ for any $\mathcal{B} \in \mathfrak{B}$, in formula (1). Thus, the *ex post* preference orders obtained from different conditional preference orders must agree with one other. Finally, to see how (Dom) and P7 overlap, consider the special case of (Dom) where one of α or β is a constant act.

Tradeoff consistency. Our last axiom is a version of the *Cardinal Coordinate Independence* axiom used in Wakker's (1988) axiomatization of SEU. We need some preliminary definitions. Let $\mathcal{B} \in \mathfrak{B}$, and let $\mathcal{Q} := \neg\mathcal{B}$. Consider an outcome $x \in \mathcal{X}$ and an act $\alpha \in \mathcal{A}(\mathcal{Q})$. Structural condition (Rch) yields an act $(x_{\mathcal{B}}\alpha) \in \mathcal{A}$ with two properties:

(B1) $(x_{\mathcal{B}}\alpha)_{\restriction \mathcal{B}} \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$,

(B2) $(x_{\mathcal{B}}\alpha)_{\restriction \mathcal{Q}} \approx_{\mathcal{Q}} \alpha$.

We will call $(x_{\mathcal{B}}\alpha)$ an (x, α)-*bet* for $\mathcal{B}$; if $\mathcal{B}$ obtains, this bet is indifferent to the outcome x, while it is indifferent to α conditional on the complement of $\mathcal{B}$.[7] Note that $(x_{\mathcal{B}}\alpha)$ is not uniquely defined by (B1) and (B2). But if $(x_{\mathcal{B}}\alpha)$ and $(x_{\mathcal{B}}\alpha)'$ are two acts satisfying (B1) and (B2), then axiom (Sep) implies that $(x_{\mathcal{B}}\alpha) \approx_{\mathcal{S}} (x_{\mathcal{B}}\alpha)'$.

Fix now four outcomes $x, y, v, w \in \mathcal{X}$, and a regular subset $\mathcal{B} \in \mathfrak{B}$. Let $\mathcal{Q} := \neg\mathcal{B}$. We write $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ if there exist $\alpha, \beta \in \mathcal{A}(\mathcal{Q})$, an (x, α)-bet $(x_{\mathcal{B}}\alpha) \in \mathcal{A}$, a (y, β)-bet

[7]

$(y_{\mathcal{B}}\beta) \in \mathcal{A}$, a (v,α)-bet $(v_{\mathcal{B}}\alpha) \in \mathcal{A}$ and a (w,β)-bet $(w_{\mathcal{B}}\beta) \in \mathcal{A}$ such that $(x_{\mathcal{B}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}}\beta)$ while $(v_{\mathcal{B}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}}\beta)$. By the remark in the previous paragraph, this implies that for *any* such bets $(x_{\mathcal{B}}\alpha), (y_{\mathcal{B}}\beta), (v_{\mathcal{B}}\alpha), (w_{\mathcal{B}}\beta) \in \mathcal{A}$, we have $(x_{\mathcal{B}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}}\beta)$ and $(v_{\mathcal{B}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}}\beta)$. To understand this, note that $(x_{\mathcal{B}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}}\beta)$ means that the "gain" obtained by changing x to y on $\mathcal{B}$ is at least enough to compensate for the "loss" incurred by changing α to β on $\mathcal{Q}$. In contrast, $(v_{\mathcal{B}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}}\beta)$ means that the gain obtained by changing v to w on $\mathcal{B}$ is at *most* enough to compensate for the loss incurred by changing α to β on $\mathcal{Q}$. Together, these two observations imply that the gain obtained from changing x to y on $\mathcal{B}$ is at least as large as the gain obtained from changing v to w on $\mathcal{B}$; hence the notation $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$. If $\succeq_{\mathcal{S}}$ has an SEU representation with utility function u, then $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ means that $u(y) - u(x) \geq u(w) - u(v)$.

Conversely, we write $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ if there exist $\gamma, \delta \in \mathcal{A}(\mathcal{Q})$, an (x,γ)-bet $(x_{\mathcal{B}}\gamma) \in \mathcal{A}$, a (y,δ)-bet $(y_{\mathcal{B}}\delta) \in \mathcal{A}$, a (v,γ)-bet $(v_{\mathcal{B}}\gamma) \in \mathcal{A}$ and a (w,δ)-bet $(w_{\mathcal{B}}\delta) \in \mathcal{A}$ such that $(x_{\mathcal{B}}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{B}}\delta)$ while $(v_{\mathcal{B}}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{B}}\delta)$. Again, this implies that for any such bets $(x_{\mathcal{B}}\gamma), (y_{\mathcal{B}}\delta), (v_{\mathcal{B}}\gamma), (w_{\mathcal{B}}\delta) \in \mathcal{A}$, we have $(x_{\mathcal{B}}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{B}}\delta)$ and $(v_{\mathcal{B}}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{B}}\delta)$. If $\succeq_{\mathcal{S}}$ had an SEU representation, then this would imply that the utility difference between x and y is *less than* the utility difference between v and w. We now come to our final axiom:

(TC) For any two regular subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, there are no $x, y, v, w \in \mathcal{X}$ such that $(x \overset{\mathcal{B}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}_1}{\rightsquigarrow} w)$ while $(x \overset{\mathcal{B}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}_2}{\rightsquigarrow} w)$.

In the case where $\mathcal{B}_1 = \mathcal{B}_2$, (TC) requires outcome trading attitudes to be well-defined, independently of the acts that are used to reveal them. In the case where $\mathcal{B}_1 \neq \mathcal{B}_2$, this axiom requires outcome trading attitudes at different regular subsets to be consistent with each other: they must be independent of the regular subset over which outcomes are traded. In other words, tradeoffs between outcomes can be evaluated independently from the choice situation from which they have been revealed.

4 SEU representations on Baire state spaces

In this section and the next two sections, we will suppose that *all* regular subsets of $\mathcal{S}$ are observable. Thus, the agent's conditional preference structure takes the form $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$. We will see that, with mild assumptions on the topology of the state space, the axioms described in Section 3 yield a unique subjective expected utility representation for this conditional preference structure. But first we need some topological preliminaries.[8]

A subset $\mathcal{N} \subseteq \mathcal{S}$ is *nowhere dense* if $\text{int}[\text{clos}(\mathcal{N})] = \emptyset$. For example, for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, the boundary $\partial\mathcal{R}$ is nowhere dense in $\mathcal{S}$. A subset $\mathcal{M} \subseteq \mathcal{S}$ is *meager* if it is a countable union of nowhere dense sets. Heuristically, meager sets are "small". For example, the set $\mathbb{Q}$ of rational numbers is meager in the space $\mathbb{R}$.

A subset $\mathcal{B} \subseteq \mathcal{S}$ has the *Baire property* if $\mathcal{B} = \mathcal{O} \triangle \mathcal{M}$ for some open $\mathcal{O} \subseteq \mathcal{S}$ and some meager $\mathcal{M} \subset \mathcal{S}$. Heuristically, this means that $\mathcal{B}$ is "almost open" in $\mathcal{S}$. Let $\mathfrak{Bai}(\mathcal{S})$ be

[8]For further explanation of these and all other topological concepts in this paper, see Willard (2004). Other good sources are Chapter 2 of Aliprantis and Border (2006) and Chapter 8 of Royden (1988).

the collection of all subsets of $\mathcal{S}$ with the Baire property; then $\mathfrak{Bai}(\mathcal{S})$ is a Boolean algebra under the standard set-theoretic operations. Observe that $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{Bai}(\mathcal{S})$ as sets, but the Boolean algebra operations are different.

A *probability charge* on $\mathfrak{Bai}(\mathcal{S})$ is a function $\nu : \mathfrak{Bai}(\mathcal{S}) \longrightarrow [0,1]$ such that (1) $\nu[\mathcal{S}] = 1$ and (2) $\nu[\mathcal{A} \sqcup \mathcal{B}] = \nu[\mathcal{A}] + \nu[\mathcal{B}]$ for any disjoint $\mathcal{A}, \mathcal{B} \in \mathfrak{Bai}(\mathcal{S})$. We say that ν is a *Baire charge* if, furthermore, $\nu[\mathcal{M}] = 0$ for any meager $\mathcal{M} \subset \mathcal{S}$. We say that ν has *full support* if $\nu(\mathcal{R}) > 0$ for any nonempty regular subset $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. In particular, if every nonempty open subset of $\mathcal{S}$ contains a nonempty regular subset (i.e. $\mathcal{S}$ is *quasiregular*), then ν has full support if and only if $\nu(\mathcal{O}) > 0$ for any nonempty open subset $\mathcal{O}$.

The topological space $\mathcal{S}$ is a *Baire space* if the intersection of any countable family of open dense sets is dense. For example, any open subset of the Euclidean space $\mathbb{R}^N$ is a Baire space. More generally, any completely metrizable space is Baire (Willard, 2004, Corollary 25.4).[9] In particular, every topological manifold is Baire.[10] Finally, any locally compact Hausdorff space is Baire.[11] Intuitively, *non*-Baire spaces are extremely "sparse" or "porous"; they are unlikely to arise naturally in economic models. (For example, a countable Hausdorff space is not Baire. Also, the product topology on $\mathbb{Q} \times \mathbb{R}$ is not Baire.) Finally, $\mathcal{S}$ is *nondegenerate* if it contains a nonempty open subset which is *not* dense —or equivalently, a proper closed subset with nonempty interior. This means that $\mathfrak{R}(\mathcal{S})$ is not trivial. Nondegeneracy is a very mild condition; for example, any nonsingleton Hausdorff space is nondegenerate (Lemma B2(a)). Our first result gives an SEU representation for conditional preference structures on any nondegenerate Baire state space.

Theorem 1 *Let $\mathcal{S}$ be a nondegenerate Baire space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC) *and* (TC) *if and only if there exists a Baire charge ν on $\mathcal{S}$ with full support, and a continuous function $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$, we have*

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \left(\int_{\mathcal{R}} u \circ \alpha \, \mathrm{d}\nu \geq \int_{\mathcal{R}} u \circ \beta \, \mathrm{d}\nu\right). \tag{2}$$

Furthermore, ν is unique, and u is unique up to positive affine transformation.

One possible objection to Theorem 1 is that Baire charges are generally not *normal*. Normality is a "continuity" condition for measures; it requires the measure of any set to be well-approximated from above by open sets and well-approximated from below by closed sets. It would be better to obtain an SEU representation in terms of a *normal* charge. The next section provides such a representation. This representation will also give an insight into how the agent copes with her informational constraints —an insight which is lacking

[9] A topological space is *metrizable* if its topology can be generated by a metric. It is *completely metrizable* if this metric is *complete*, meaning that every Cauchy sequence converges.

[10] A Hausdorff space is an N-dimensional *topological manifold* if every point has an open neighbourhood which is homeomorphic to $\mathbb{R}^N$. Thus, it is "locally Euclidean" near every point. A sphere is an example.

[11] See Section 6 for the definition of "locally compact".

in Theorem 1. Heuristically, this is because a Baire charge assigns probability zero to the boundary of any regular set, as if the agent thinks that such boundary events "never" occur. But a more sophisticated representation of the agent's beliefs shows that she recognizes that such boundary events *can* occur; this is encoded in what we call a *liminal* structure.

5 SEU representations on normal state spaces

A topological space $\mathcal{S}$ is ***Hausdorff*** if any pair of points in $\mathcal{S}$ can be placed in two disjoint open neighbourhoods. A Hausdorff space $\mathcal{S}$ is ***normal*** (or "T_4") if, for any disjoint closed subsets $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{S}$, there exist disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{S}$ with $\mathcal{C}_1 \subseteq \mathcal{O}_1$ and $\mathcal{C}_2 \subseteq \mathcal{O}_2$. For example, any subset of $\mathbb{R}^N$ is normal. More generally, every metrizable space is normal. In particular, every topological manifold is normal. Also, every compact Hausdorff space is normal. Finally, the order topology on any strictly ordered set is both Hausdorff and normal. (However, if $\mathcal{I}$ is an uncountably infinite set, then the topology of pointwise convergence on $\mathbb{R}^{\mathcal{I}}$ is *not* normal.) Thus, almost all topological spaces which would arise naturally in economic applications are normal.

Charges and liminal charge structures. Let $\mathfrak{A}(\mathcal{S})$ be the Boolean algebra generated by the open subsets of $\mathcal{S}$. Thus, $\mathfrak{A}(\mathcal{S})$ contains all open subsets, all closed subsets, and all finite unions and intersections of such sets. A function $\nu : \mathfrak{A}(\mathcal{S}) \longrightarrow [0,1]$ is a ***charge*** if it is finitely additive —i.e. $\nu[\mathcal{A} \sqcup \mathcal{B}] = \nu[\mathcal{A}] + \nu[\mathcal{B}]$ for any disjoint $\mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{S})$. We say ν is a ***probability*** charge if, furthermore, $\nu(\mathcal{S}) = 1$. Another charge ρ is ***absolutely continuous*** relative to ν if $\rho[\mathcal{B}] = 0$ whenever $\nu[\mathcal{B}] = 0$.

Let ν be a probability charge on $\mathfrak{A}(\mathcal{S})$. For any $\mathcal{B} \in \mathfrak{A}(\mathcal{S})$, let $\nu_{\mathcal{B}}$ be the restriction of ν to a charge on the elements of $\mathfrak{A}(\mathcal{B})$. A ***liminal charge structure subordinate to*** ν is a collection $\{\rho_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$, where, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{R})$, $\rho_{\mathcal{R}}$ is a charge on $\mathfrak{A}(\partial\mathcal{R})$ which is absolutely continuous with respect to ν, such that, for any regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, we have

$$\rho_{\mathcal{R}_1} + \cdots + \rho_{\mathcal{R}_N} \quad = \quad \nu_{\partial\mathcal{R}_1 \cup \cdots \cup \partial\mathcal{R}_N}. \tag{3}$$

In the model we develop in this section, the agent's "beliefs" are represented by a charge ν, along with a liminal charge structure $\{\rho_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ subordinate to ν. Heuristically, ν describes the agent's *ex ante* beliefs, while $\{\rho_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ describes how she copes with her informational constraints. Recall that, when the agent "observes" a regular event $\mathcal{R}$, there is a chance that the state of the world is not actually in $\mathcal{R}$, but instead on its boundary $\partial\mathcal{R}$. The charge $\rho_{\mathcal{R}}$ describes the agent's beliefs about the latter possibility. To be precise, for any $\mathcal{U} \in \mathfrak{A}(\mathcal{S})$, the conditional probability she assigns to $\mathcal{U}$, given that she has "observed" $\mathcal{R}$, is the following ratio:

$$\frac{\nu(\mathcal{U} \cap \mathcal{R}) + \rho_{\mathcal{R}}(\mathcal{U} \cap \partial\mathcal{R})}{\nu(\mathcal{R}) + \rho_{\mathcal{R}}(\partial\mathcal{R})}. \tag{4}$$

Note that this expression may leave some conditional probability *outside* $\mathcal{R}$. Thus, "observing" $\mathcal{R}$ increases the probability that $\mathcal{R}$ actually holds to at least $\nu(\mathcal{R})/\nu[\text{clos}(\mathcal{R})]$, but not necessarily to certainty. On the other hand, this "spillover" probability is confined

to the *closure* of $\mathcal{R}$; formula (4) implies that the probability of $\mathrm{clos}(\mathcal{R})$ given $\mathcal{R}$ always equals one. The intuition is straightforward: because of the technological constraints on her information, an agent observing that $\mathcal{R}$ holds only knows for sure that the true state lies in the closure of $\mathcal{R}$.

For example, suppose the agent is about to make an observation corresponding to a regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$. For all $n \in [1 \ldots N]$, her *ex ante* probability of "observing" $\mathcal{R}_n$ is $\nu[\mathcal{R}_n] + \rho_{\mathcal{R}_n}[\partial\mathcal{R}_n]$. Now suppose that she *has* observed $\mathcal{R}_n$; then she believes that the true state is *really* in $\mathcal{R}_n$ with probability $\nu[\mathcal{R}_n]/(\nu[\mathcal{R}_n] + \rho_{\mathcal{R}_n}[\partial\mathcal{R}_n])$, whereas it is on the *boundary* of $\mathcal{R}_n$ with probability $\rho_{\mathcal{R}_n}[\partial_{\mathcal{R}_n}]/(\nu[\mathcal{R}_n] + \rho_{\mathcal{R}_n}[\partial\mathcal{R}_n])$. Furthermore, $\rho_{\mathcal{R}_n}$ tells her *where* the state is likely to be on $\partial\mathcal{R}_n$, given that this latter case occurs.

In particular, suppose $\mathcal{S} = [-1, 1]$, and consider the partition $\{\mathcal{L}, \mathcal{R}\}$, where $\mathcal{L} := [-1, 0)$ and $\mathcal{R} := (0, 1]$. Thus, $\partial\mathcal{L} = \partial\mathcal{R} = \{0\}$, and formula (3) says that $\rho_{\mathcal{L}}\{0\} + \rho_{\mathcal{R}}\{0\} = \nu\{0\}$. Let $\mu[\mathcal{L}] := \nu(\mathcal{L}) + \rho_{\mathcal{L}}\{0\}$, while $\mu[\mathcal{R}] := \nu(\mathcal{R}) + \rho_{\mathcal{R}}\{0\}$; these are the agent's subjective probabilities of observing $\mathcal{L}$ and $\mathcal{R}$. Suppose that $\nu\{0\} > 0$. If the true state is $s = 0$, then agent believes that she will observe the event $\mathcal{L}$ with probability $\rho_{\mathcal{L}}\{0\}/\nu\{0\}$, whereas she will observe the event $\mathcal{R}$ with probability $\rho_{\mathcal{R}}\{0\}/\nu\{0\}$. Once she has observed $\mathcal{L}$, she thinks that actually $s = 0$ with probability $\rho_{\mathcal{L}}\{0\}/\mu(\mathcal{L})$, whereas $s < 0$ with probability $1 - \rho_{\mathcal{L}}\{0\}/\mu(\mathcal{L})$. On the other hand, if she observed $\mathcal{R}$, she thinks that actually $s = 0$ with probability $\rho_{\mathcal{R}}\{0\}/\mu(\mathcal{R})$, whereas $s > 0$ with probability $1 - \rho_{\mathcal{R}}\{0\}/\mu(\mathcal{R})$.

A charge ν is *normal* if, for every $\mathcal{B} \in \mathfrak{A}(\mathcal{S})$, we have $\nu[\mathcal{B}] = \sup\{\nu[\mathcal{C}];\ \ \mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{C}$ closed in $\mathcal{S}\}$ and $\nu[\mathcal{B}] = \inf\{\nu[\mathcal{O}];\ \ \mathcal{B} \subseteq \mathcal{O} \subseteq \mathcal{S}$ and $\mathcal{O}$ open in $\mathcal{S}\}$. A liminal charge structure $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is *normal* if $\rho_{\mathcal{R}}$ is a normal charge on $\partial\mathcal{R}$ for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. Finally, a charge ν is said to have *full support* if $\nu(\mathcal{O}) > 0$ for any open set $\mathcal{O}$ in $\mathcal{S}$.

Liminal charge SEU representation. Let $\mathcal{X}$ be another topological space, let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A *liminal charge SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is given by a charge ν on $\mathfrak{A}(\mathcal{S})$, a liminal charge structure $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ subordinate to ν, and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \left(\int_{\mathcal{R}} u \circ \alpha \ \mathrm{d}\nu + \int_{\partial\mathcal{R}} u \circ \alpha \ \mathrm{d}\rho_{\mathcal{R}} \geq \int_{\mathcal{R}} u \circ \beta \ \mathrm{d}\nu + \int_{\partial\mathcal{R}} u \circ \beta \ \mathrm{d}\rho_{\mathcal{R}}\right). \quad (5)$$

(Here we implicitly use the fact $u \circ \alpha$ and $u \circ \beta$ have unique extensions to $\partial\mathcal{R}$, by continuity.) In this representation, the value of an act conditional on a regular event $\mathcal{R}$ has two components, which correspond to the two ways in which $\mathcal{R}$ could be the outcome of an observation, as explained above.

Note that the "boundary" terms in the SEU representation (5) do *not* violate consequentialism or dynamic consistency. Let $\alpha, \beta \in \mathcal{A}$. Because α, β and u are all continuous, the values of $u \circ \alpha$ and $u \circ \beta$ on $\partial\mathcal{R}$ are completely determined by their values on $\mathcal{R}$. Thus, (5) satisfies *consequentialism*: if $\alpha_{\restriction\mathcal{R}} = \beta_{\restriction\mathcal{R}}$, then α and β must have the same expected utility, conditional on $\mathcal{R}$. Meanwhile, equation (3) ensures that the expected utility of α on $\mathcal{S}$ is a weighted average of the conditional-expected utilities of α on $\mathcal{R}$ and on $\neg\mathcal{R}$ (as expressed on the right side of (5)), and likewise for β. Thus, if the conditional-expected

utility of α is greater than that of β on both $\mathcal{R}$ and also on $\neg\mathcal{R}$, then it must be greater on all of $\mathcal{S}$, in accord with the principle of dynamical consistency.

If $\mathcal{S} = \mathcal{R}$, then (5) simplifies to

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \left(\int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\nu \geq \int_{\mathcal{S}} u \circ \beta \, \mathrm{d}\nu\right). \tag{6}$$

In other words, the unconditional preference order $\succeq_{\mathcal{S}}$ has always a "classical" SEU representation. The liminal charge structure only emerges in the representation of the *conditional* preference orders, and essentially serves in the updating of preferences and beliefs as new information is acquired. Our second result provides a characterization of liminal charge SEU representations in terms of the axioms of Section 3.

Theorem 2 *Let $\mathcal{S}$ be a (nonsingleton) normal Hausdorff space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC) *and* (TC) *if and only if it admits a liminal charge SEU representation* (5)*, where ν is a normal probability charge on $\mathfrak{A}(\mathcal{S})$ with full support, and $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is normal liminal charge structure. Furthermore, ν and $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ are unique, and u is unique up to positive affine transformation.*

The representation in Theorem 2 has two advantages over Theorem 1: the charge ν is now *normal*, and the liminal structure explicitly encodes how the agent deals with the limits of her observational technology. But both representations have a shortcoming: the agent's beliefs are represented by *finitely* additive charges, rather than *countably* additive measures. This stymies the approximation arguments which play a central role in much of real analysis. It would be better to obtain a representation with a countably additive measure. In fact, the next section delivers a representation in terms of a *Borel* measure.

6 SEU representations on (locally) compact state spaces

Borel measures and liminal density structures. Let $\mathfrak{Bor}(\mathcal{S})$ be the Borel sigma-algebra of $\mathcal{S}$ —that is, the smallest sigma-algebra containing all open and closed sets. Let ν be a *Borel probability measure* on $\mathcal{S}$ — that is, is a (countably additive) probability measure on $\mathfrak{Bor}(\mathcal{S})$. For any $\mathcal{B} \in \mathfrak{Bor}(\mathcal{S})$, let $\nu_{\mathcal{B}}$ denote the restriction of ν to a Borel measure on the subsets of $\mathcal{B}$, and let $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}})$ denote the Banach space of real-valued, $\nu_{\mathcal{B}}$-integrable functions on $\mathcal{B}$, modulo equality $\nu_{\mathcal{B}}$-almost everywhere. Finally, let $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}}; [0,1])$ denote the set of $[0,1]$-valued functions in $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}})$.

A *liminal density structure subordinate to* ν is a collection $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$, where, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{R})$, $\phi_{\mathcal{R}} \in \mathbb{L}^1(\partial\mathcal{R}, \nu_{\partial\mathcal{R}}; [0,1])$ is a function such that, for any regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, we have

$$\phi_{\mathcal{R}_1} + \cdots + \phi_{\mathcal{R}_N} \quad = \quad 1, \quad \nu\text{-almost everywhere on } \partial\mathcal{R}_1 \cup \cdots \cup \partial\mathcal{R}_N. \tag{7}$$

As with the liminal structures introduced in Section 5, the liminal density structure describes how the agent copes with her informational limitations; once she has "observed" the event $\mathcal{R}$, the density $\phi_{\mathcal{R}}$ describes how much probability she conditionally assigns to $\partial\mathcal{R}$, and how this probability is distributed. To be precise, she assigns the following conditional probability to any subset $\mathcal{B} \in \mathfrak{Bor}(\mathcal{S})$:

$$\frac{\nu(\mathcal{B} \cap \mathcal{R}) + \displaystyle\int_{\mathcal{B}\cap\partial\mathcal{R}} \phi_{\mathcal{R}} \, \mathrm{d}\nu}{\nu(\mathcal{R}) + \displaystyle\int_{\partial\mathcal{R}} \phi_{\mathcal{R}} \, \mathrm{d}\nu}. \tag{8}$$

Liminal density SEU representation. Let $\mathcal{X}$ be another topological space, let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A *liminal density SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is given by a Borel probability measure ν on $\mathfrak{Bor}(\mathcal{S})$ and a liminal density structure $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ subordinate to ν, along with a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\begin{aligned} &\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \quad \Longleftrightarrow \\ &\quad \left(\int_{\mathcal{R}} u\circ\alpha \, \mathrm{d}\nu + \int_{\partial\mathcal{R}} (u\circ\alpha)\, \phi_{\mathcal{R}} \, \mathrm{d}\nu \;\geq\; \int_{\mathcal{R}} u\circ\beta \, \mathrm{d}\nu + \int_{\partial\mathcal{R}} (u\circ\beta)\, \phi_{\mathcal{R}} \, \mathrm{d}\nu\right). \end{aligned} \tag{9}$$

If $\mathcal{S}$ is a compact Hausdorff space, then Theorem 2 can be strengthened to the following.

Theorem 3 *Let $\mathcal{S}$ be a (nonsingleton) compact Hausdorff space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC) *and* (TC) *if and only if it admits a liminal density SEU representation* (9)*, where ν is a normal Borel probability measure with full support. Furthermore, ν is unique, the elements of $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ are unique* (ν-almost everywhere)*, and u is unique up to positive affine transformation.*

Theorem 3 provides an SEU representation where the agent's beliefs are represented by a (countably additive) Borel measure, in the case when the state space $\mathcal{S}$ is *compact*. Unfortunately, many state spaces are not compact. For example, let d be a metric which induces the topology on $\mathcal{S}$. Then $\mathcal{S}$ is compact if and only if it is both *complete* and *totally bounded*. A metric space $(\mathcal{S}, d)$ is *complete* if every Cauchy sequence in $\mathcal{S}$ converges to some limit in $\mathcal{S}$. Meanwhile, $(\mathcal{S}, d)$ is *totally bounded* if, for any $\epsilon > 0$, $\mathcal{S}$ can be covered by a finite collection of ϵ-balls. For example, the open interval $(0, 1)$ (with the standard metric) is totally bounded, but not complete. On the other hand, the real line $\mathbb{R}$ is complete, but not totally bounded. The closed interval $[0, 1]$ is both complete and totally bounded, and hence, compact.

Total boundedness is plausible in many decision problems; for example, it is plausible when $\mathcal{S}$ represents uncertainty about natural resources whose total supply is bounded. Meanwhile, incompleteness is easily remedied: if $(\mathcal{S}, d)$ is an incomplete metric space, then

there is a (unique) complete metric space $(\overline{\mathcal{S}}, \overline{d})$ (called the *completion* of $(\mathcal{S}, d)$) such that $\mathcal{S}$ is embedded in $\overline{\mathcal{S}}$ as a dense subset (with the same metric). (For example, the completion of $(0,1)$ is $[0,1]$, with the standard metric.) For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a unique $\overline{\mathcal{R}} \in \mathfrak{R}(\overline{\mathcal{S}})$ such that $\overline{\mathcal{R}} \cap \mathcal{S} = \mathcal{R}$ (Pivato and Vergopoulos, 2016, Lemma 5.12(a)). If $(\mathcal{X}, d')$ is another complete metric space, then any *uniformly continuous* function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ has a unique extension to a continuous function $\overline{\alpha} : \overline{\mathcal{S}} \longrightarrow \mathcal{X}$ (Willard, 2004, Theorem 39.10). Thus, if *every* feasible act is uniformly continuous, then any incompleteness of the state space is merely an artifact of poor modeling choices, which can be easily remedied by adding extra states. This allows us to obtain an SEU representation, as we now explain.

A metric space $(\mathcal{S}, d)$ is *locally complete* if every point in $\mathcal{S}$ has a neighbourhood which is complete. (For example, $(0,1)$ is locally complete. But $\mathbb{Q}$ is not locally complete.) Let $\mathcal{C}_{\mathrm{U}}(\mathcal{S}, d; \mathcal{X}, d')$ be the set of uniformly continuous functions from $(\mathcal{S}, d)$ to $(\mathcal{X}, d')$. Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{U}}(\mathcal{S}, d; \mathcal{X}, d')$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. Let $(\overline{\mathcal{S}}, \overline{d})$ be the completion of $\mathcal{S}$. A *completion SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ is given by a normal Borel probability measure $\overline{\nu}$ on $\mathfrak{Bor}(\overline{\mathcal{S}})$ and a liminal density structure $\{\overline{\phi}_{\overline{\mathcal{R}}}\}_{\overline{\mathcal{R}} \in \mathfrak{R}(\overline{\mathcal{S}})}$ subordinate to $\overline{\nu}$, along with a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \quad \Longleftrightarrow \tag{10}$$
$$\left(\int_{\overline{\mathcal{R}}} u \circ \overline{\alpha} \; \mathrm{d}\overline{\nu} + \int_{\partial\overline{\mathcal{R}}} (u \circ \overline{\alpha}) \cdot \overline{\phi}_{\overline{\mathcal{R}}} \; \mathrm{d}\overline{\nu} \;\geq\; \int_{\overline{\mathcal{R}}} u \circ \overline{\beta} \; \mathrm{d}\overline{\nu} + \int_{\partial\overline{\mathcal{R}}} (u \circ \overline{\beta}) \cdot \overline{\phi}_{\overline{\mathcal{R}}} \; \mathrm{d}\overline{\nu}\right).$$

Theorem 4 *Let $(\mathcal{S}, d)$ be a totally bounded, locally complete metric space, let $(\mathcal{X}, d')$ be a connected, complete metric space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{U}}(\mathcal{S}, d; \mathcal{X}, d')$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC) *and* (TC) *if and only if it admits a completion SEU representation* (10)*, where $\overline{\nu}$ is a Borel probability measure with full support on $\overline{\mathcal{S}}$. Furthermore, $\overline{\nu}$ is unique, the elements of $\{\overline{\phi}_{\overline{\mathcal{R}}}\}_{\overline{\mathcal{R}} \in \mathfrak{R}(\overline{\mathcal{S}})}$ are unique* ($\overline{\nu}$-almost everywhere)*, and u is unique up to positive affine transformation.*

Theorem 4 obtains an SEU representation by "augmenting" the state space with extra states. This strategy works whenever $\mathcal{S}$ is a totally bounded metric space.[12] But some applications require an *unbounded* state space. In other applications, the state space has no metric structure at all, so Theorem 4 is unapplicable. But we shall now see that this strategy of "augmenting" the state space works for much larger class of topologies.

A Hausdorff space $\mathcal{S}$ is *locally compact* if every point in $\mathcal{S}$ has a compact neighbourhood. For example, every compact Hausdorff space is locally compact. Every topological manifold is locally compact. In particular, any open subset of $\mathbb{R}^N$ is locally compact. (However, the set of rational numbers is *not* locally compact.) Every totally bounded, locally complete metric space is locally compact. In short: most topological spaces which would arise naturally in economic applications are locally compact.

[12] Actually, Theorem 4 can be extended to any totally bounded *uniform* space. Uniform spaces are a generalization of metric spaces (Willard, 2004, Ch.9).

For any Hausdorff space $\mathcal{X}$, let $\mathcal{C}_{\mathrm{L}}(\mathcal{S},\mathcal{X})$ be the set of all continuous functions $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ which converge to some limit "at infinity" in the following sense: there exists $x \in \mathcal{X}$ such that, for any open neighbourhood $\mathcal{O} \subseteq \mathcal{X}$ around x, there is a compact subset $\mathcal{K} \subseteq \mathcal{S}$ such that $\alpha(\mathcal{S} \setminus \mathcal{K}) \subseteq \mathcal{O}$. When it exists, this limit x is unique and denoted $\lim_{\infty} f$.

Let $\mathcal{S}$ be a locally compact Hausdorff space. The *Alexandroff* (or *one-point*) *compactification* $\mathcal{S}^*$ is the set $\mathcal{S} \sqcup \{\infty\}$ (where ∞ represents a "point at infinity") equipped with the smallest topology such that every open subset of $\mathcal{S}$ remains open in $\mathcal{S}^*$, while the open neighbourhoods of ∞ are the sets $\mathcal{S}^* \setminus \mathcal{K}$, where $\mathcal{K}$ is any compact subset of $\mathcal{S}$. For example, the Alexandroff compactification of $\mathbb{R}$ is homeomorphic to a circle. The Alexandroff compactification of $\mathbb{R}^2$ is homeomorphic to a sphere. In general, Alexandroff compactifications have the following properties:

(i) $\mathcal{S}^*$ is compact and Hausdorff. If $\mathcal{S}$ is noncompact, then $\mathcal{S}$ is a dense open subset of $\mathcal{S}^*$. (Otherwise, if $\mathcal{S}$ is already compact, then ∞ is an isolated point of $\mathcal{S}^*$.)

(ii) For any Hausdorff space $\mathcal{X}$, any function $\alpha \in \mathcal{C}_{\mathrm{L}}(\mathcal{S},\mathcal{X})$ has a unique extension $\alpha^* \in \mathcal{C}(\mathcal{S}^*,\mathcal{X})$ defined by $\alpha^*_{|\mathcal{S}} = \alpha$ and $\alpha^*(\infty) = \lim_{\infty} \alpha$.

(iii) For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a unique $\mathcal{R}^* \in \mathfrak{R}(\mathcal{S}^*)$ such that $\mathcal{R}^* \cap \mathcal{S} = \mathcal{R}$.[13]

Let $\mathcal{X}$ be a Hausdorff space. Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{L}}(\mathcal{S},\mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. An *Alexandroff SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is given by a normal Borel probability measure ν^* on $\mathfrak{Bor}(\mathcal{S}^*)$ and a liminal density structure $\{\phi^*_{\mathcal{R}^*}\}_{\mathcal{R}^*\in\mathfrak{R}(\mathcal{S}^*)}$ subordinate to ν^*, along with a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \Big(\int_{\mathcal{R}^*} u\circ\alpha^* \,\mathrm{d}\nu^* + \int_{\partial\mathcal{R}^*} (u\circ\alpha^*)\cdot\phi^*_{\mathcal{R}^*} \,\mathrm{d}\nu^* \;\geq\; \int_{\mathcal{R}^*} u\circ\beta^* \,\mathrm{d}\nu^* + \int_{\partial\mathcal{R}^*} (u\circ\beta^*)\cdot\phi^*_{\mathcal{R}^*} \,\mathrm{d}\nu^*\Big). \tag{11}$$

A key difference between the SEU representations obtained so far and the Alexandroff SEU representation lies in the unconditional preference order $\succeq_{\mathcal{S}}$. While both liminal charge and liminal density SEU representations for $\succeq_{\mathcal{S}}$ simplify into a "classical" representation of type (6), formula (11) rather simplifies into the following: for any $\alpha, \beta \in \mathcal{A}$,

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \Big(\int_{\mathcal{S}} u\circ\alpha \,\mathrm{d}\nu^* + \nu^*\{\infty\}\cdot\lim_{\infty}(u\circ\alpha) \;\geq\; \int_{\mathcal{S}} u\circ\beta \,\mathrm{d}\nu^* + \nu^*\{\infty\}\cdot\lim_{\infty}(u\circ\beta)\Big). \tag{12}$$

Thus, *ex ante* beliefs consist of two elements, a Borel probability measure on $\mathcal{S}$ and an additional coefficient weighting the asymptotic behavior of the feasible utility profiles.

[13] See Lemma 5.12(a) in Pivato and Vergopoulos (2016).

Theorem 5 *Let $\mathcal{S}$ be a noncompact, locally compact Hausdorff space, let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{L}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC) *and* (TC) *if and only if it admits an Alexandroff SEU representation* (11)*, where ν^* is a Borel probability measure with full support on $\mathcal{S}^*$. Furthermore, ν^* is unique, the elements of $\{\phi^*_{\mathcal{R}^*}\}_{\mathcal{R}^*\in\mathfrak{R}(\mathcal{S}^*)}$ are unique* (ν^*-almost everywhere)*, and u is unique up to positive affine transformation.*

The main advantage of Theorem 5 is that it applies when $\mathcal{S}$ is a *locally* compact space like $\mathbb{R}^N$, whereas Theorems 3 and 4 do not. The main *disadvantage* of Theorem 5 is that the conditional preference structure can only compare acts which converge "at infinity". The problem is that a conditional preference structure defined over a larger domain of acts may be sensitive to the asymptotic behaviour of these acts in a way which eludes an Alexandroff SEU representation. For example, if $\mathcal{S} = \mathbb{R}^2$, then the conditional preference structure could be sensitive in different ways to the asymptotic behaviour of acts along different curves, like $y = x^2$ and $y = x^3$. Intuitively, to capture such sensitivity with an SEU representation, we would need to introduce distinct "endpoints" for these two curves, and then assign different probabilities to these endpoints. But *no such distinct endpoints exist* in $\mathbb{R}^2$, or in its (spherical) Alexandroff compactification. To solve this problem, we must add a plethora of new states to $\mathcal{S}$, each acting like a distinct "point at infinity". To be precise, we must extend $\mathcal{S}$ to its *Stone-Čech compactification*.

Let $\mathcal{S}$ is a locally compact Hausdorff space. There is a unique compact Hausdorff space $\check{\mathcal{S}}$, called the *Stone-Čech compactification* of $\mathcal{S}$, with the following properties.

(i) $\mathcal{S}$ is an open, dense subset of $\check{\mathcal{S}}$, and the native topology of $\mathcal{S}$ is the same as the subspace topology it inherits from $\check{\mathcal{S}}$.

(ii) For any compact Hausdorff space $\mathcal{K}$, and any continuous function $f : \mathcal{S} \longrightarrow \mathcal{K}$, there is a unique continuous function $\check{f} : \check{\mathcal{S}} \longrightarrow \mathcal{K}$ such that $\check{f}_{\restriction\mathcal{S}} = f$.

(iii) For any regular set $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a unique regular set $\check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})$ such that $\check{\mathcal{R}} \cap \mathcal{S} = \mathcal{R}$. Furthermore, the mapping $\mathfrak{R}(\mathcal{S}) \ni \mathcal{R} \mapsto \check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})$ is a Boolean algebra isomorphism.[14]

Let $\mathcal{X}$ be another Hausdorff space. For any $\alpha \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, assertion (ii) says there is a unique function $\check{\alpha} \in \mathcal{C}(\check{\mathcal{S}}, \mathcal{X})$ such that $\check{\alpha}_{\restriction\mathcal{S}} = \alpha$. Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A *Stone-Čech SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is given by a normal Borel probability measure $\check{\nu}$ on $\mathfrak{Bor}(\check{\mathcal{S}})$ and a liminal density structure $\{\check{\phi}_{\check{\mathcal{R}}}\}_{\check{\mathcal{R}}\in\mathfrak{R}(\check{\mathcal{S}})}$ subordinate to $\check{\nu}$, along with a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such

[14] Property (ii) holds if $\mathcal{S}$ is any *Tychonoff* space; see e.g. Theorem 19.5 of Willard (2004) or Theorem 2.79 of Aliprantis and Border (2006). But property (iii) only holds for the somewhat smaller class of locally compact Hausdorff spaces (Pivato and Vergopoulos, 2016, Lemma 5.12(a)).

that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \Big(\int_{\check{\mathcal{R}}} u \circ \check{\alpha} \, \mathrm{d}\check{\nu} + \int_{\partial\check{\mathcal{R}}} (u \circ \check{\alpha}) \, \check{\phi}_{\check{\mathcal{R}}} \, \mathrm{d}\check{\nu} \;\geq\; \int_{\check{\mathcal{R}}} u \circ \check{\beta} \, \mathrm{d}\check{\nu} + \int_{\partial\check{\mathcal{R}}} (u \circ \check{\beta}) \, \check{\phi}_{\check{\mathcal{R}}} \, \mathrm{d}\check{\nu}\Big). \tag{13}$$

Similarly to formula (12), the Stone-Čech representation of $\succeq_{\mathcal{S}}$ reveals *ex ante* beliefs that may assign some weight outside of $\mathcal{S}$. More precisely, for any $\alpha, \beta \in \mathcal{A}$, we have

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \Big(\int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\check{\nu} + \int_{\check{\mathcal{S}}\setminus\mathcal{S}} u \circ \check{\alpha} \, \mathrm{d}\check{\nu} \;\geq\; \int_{\mathcal{S}} u \circ \beta \, \mathrm{d}\check{\nu} + \int_{\check{\mathcal{S}}\setminus\mathcal{S}} u \circ \check{\beta} \, \mathrm{d}\check{\nu}\Big). \tag{14}$$

The set $\check{\mathcal{S}} \setminus \mathcal{S}$ is called the *corona* —intuitively, this is the set of "points at infinity". These points play an essential role in the Stone-Čech representation. It is straightforward to construct examples of SEU representations where much of the probability weight lies in the corona. But to save space we will not present these examples here. We refer the interested reader to Examples 3.13 and 5.6 of Pivato and Vergopoulos (2016). The next theorem generalizes Theorem 5 by providing an SEU representation on locally compact state spaces without requiring feasible acts to have a limit at infinity.

Theorem 6 *Let $\mathcal{S}$ be a (nonsingleton) locally compact Hausdorff space, let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC) *and* (TC) *if and only if it admits a Stone-Čech SEU representation* (13)*, where $\check{\nu}$ is a Borel probability measure with full support on $\check{\mathcal{S}}$. Furthermore, $\check{\nu}$ is unique, the elements of $\{\check{\phi}_{\check{\mathcal{R}}}\}_{\check{\mathcal{R}}\in\mathfrak{R}(\check{\mathcal{S}})}$ are unique* ($\check{\nu}$-almost everywhere)*, and u is unique up to positive affine transformation.*

The results in this section can be summarized as follows. When $\mathcal{S}$ is compact, it is sufficient to formulate beliefs on $\mathcal{S}$ itself to represent preferences over continuous acts. When $\mathcal{S}$ is a totally bounded metric space and all acts are uniformly continuous, we must instead formulate these beliefs on the metric completion of $\mathcal{S}$. When $\mathcal{S}$ is only *locally* compact, we can formulate beliefs on the Alexandroff compactification of $\mathcal{S}$ to represent preferences over *convergent* continuous acts. But we must formulate beliefs on the (much larger) Stone-Čech compactification of $\mathcal{S}$ to represent preferences over all bounded continuous acts.

The importance of boundaries. It might seem that the "boundary" integrals in the SEU representations (5), (9), (11), and (13) are likely to be zero in most economic applications, where probability is typically distributed "smoothly" over the state space. But this is not the case, for two reasons. First, the probability distributions describing many

economic or strategic interactions naturally concentrate around certain values. For example, consider a principal-agent interaction, where the principal has imperfect monitoring technology, and the agent wants to extract as much surplus as possible without crossing a threshold which would trigger a reprisal from the principal. Likewise, consider a regulated market, in which firms must meet or exceed threshold values of certain metrics to keep their license or to receive some desired certification. For example, to comply with the Basel III requirements, a bank must maintain a "Common Equity Tier 1 Ratio" of at least 4.5%, and a "Leverage Ratio" of at least 3%. It is likely that many banks will seek to exceed these minimum thresholds by *as small a margin as possible*, thereby maximizing expected profits while technically complying with the rules.

Similar phenomena occur in games where a player's strategy ranges continuously over a topological space, but other players can only observe this strategy through a regular partition. For example, professional boxers must compete with other boxers in the same "weight class"; in particular, the weight range is 63.5 kg - 66.7 kg for "Welterweight" class, and 66.7 kg - 69.9 kg for "Super Welterweight". During a fight, more weight is better. So each boxer wants to be as close as possible to the upper weight limit for his class. Thus, if you learn that a boxer is in the "Welterweight" class, then your conditional probability should not be smoothly distributed over the interval $[63.5, 66.7]$; instead, most of the probability should be concentrated at 66.7. On the other hand, upon learning that a boxer is "Super Welterweight", most of your conditional probability should be concentrated at 69.9. A liminal structure can naturally accommodate this kind of reasoning.[15]

Similar examples arise in any economic interaction where an actor's true "quality" is an element of a continuum, but it is summarized publicly by some discrete "grade". For example, a complete description of the credit-worthiness of a debtor might require a huge array of real numbers, but it is summarized by Fitch, Standard & Poor and other rating agencies with a code like AAA, AA-, or BB+. Likewise, the academic quality of students, the gastronomic quality of food products, and the energy efficiency of home appliances are summarized by letter grades like A, B+, or C-. It is likely that many debtors, food producers, appliance manufacturers and university students will find ways to just barely exceed the quality threshold necessary to achieve a certain grade.

Concentration of probability also occurs in natural systems *without* such strategic considerations. For example, the theory of *self-organized criticality* suggests that many systems evolve gradually but inexorably toward the boundary of a "critical" region, where a small perturbation can trigger an arbitrarily large "crisis" (Jensen, 1998). After the crisis has run its course, the system will find itself far from the critical boundary, and the process will begin anew. Standard examples include mountain snow packs, tectonic fault lines, forests, and financial markets; the resulting "crises" are, respectively, avalanches, earthquakes, forest fires, and financial crashes. If such a system is viewed as a stochastic process, then it will not *always* be on the critical boundary (for example, shortly after a crisis), but it will be very close to it, with high probability.

Aside from these concrete examples, there is a second, purely mathematical reason why the "boundary" integrals in the representations (9), (11) and (13) are unavoidable: if the

[15]Formally, in this example, we would have $\mathcal{S} = \mathbb{R}_+$, $\mathcal{R}_{\text{welt}} = (63.5, 66.7)$, $\mathcal{R}_{\text{superwelt}} = (66.7, 69.9)$, etc.

space $\mathcal{S}$ satisfies mild topological conditions, then the liminal density structure *cannot* be zero for *all* regular sets (Pivato and Vergopoulos, 2016, Proposition 5.11). Thus, these boundary integrals should not be waved away as a mere technicality or annoyance; they are an essential part of the representation, and are both economically meaningful and mathematically inevitable.

Compactification, proximity, and fallibility. Theorems 4, 5 and 6 are in fact three special cases of a large family of results. Let $\overline{\mathcal{S}}$ be any *compactification* of $\mathcal{S}$ —that is, a compact Hausdorff space in which $\mathcal{S}$ is embedded as an dense subset. Let $\mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S}, \mathcal{X})$ be the set of continuous functions in $\mathcal{C}(\mathcal{S}, \mathcal{X})$ which can be extended to a continuous function in $\mathcal{C}(\overline{\mathcal{S}}, \mathcal{X})$. (For example, let $\mathcal{S} := \mathbb{R}$. Then $\overline{\mathbb{R}} := [-\infty, \infty]$ is a compactification, and $\mathcal{C}_{\overline{\mathbb{R}}}(\mathbb{R}, \mathcal{X})$ is the set of continuous functions from $\mathbb{R}$ to $\mathcal{X}$ which converge to (possibly different) limits at $\pm\infty$.) If $\mathcal{A} \subseteq \mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S}, \mathcal{X})$, and $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is a conditional preference structure on $\mathcal{A}$ which satisfies (Rch), (CEq), (C), (Dom), (Sep), (PC) and (TC), then we can obtain a normal Borel probability measure $\overline{\nu}$ on $\mathfrak{Bor}(\overline{\mathcal{S}})$, a liminal density structure $\{\overline{\phi}_{\overline{\mathcal{R}}}\}_{\overline{\mathcal{R}}\in\mathfrak{R}(\overline{\mathcal{S}})}$ subordinate to $\overline{\nu}$, and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, which yield an SEU representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ analogous to the representations (11) and (13).

One way to interpret such a "compactified" SEU representation is in terms of a *proximity* on $\mathcal{S}$: a symmetric, reflexive binary relation $\sim$ on the power set of $\mathcal{S}$ which satisfies certain axioms.[16] For any $\mathcal{Q}, \mathcal{R} \subseteq \mathcal{S}$, heuristically, $\mathcal{Q} \sim \mathcal{R}$ means that $\mathcal{Q}$ and $\mathcal{R}$ are "touching". A proximity $\sim$ is *compatible* with the topology of $\mathcal{S}$ if $\mathrm{clos}[\mathcal{R}] = \{s \in \mathcal{S};\ \{s\} \sim \mathcal{R}\}$, for any $\mathcal{R} \subseteq \mathcal{S}$. There is a natural bijective correspondence between the compatible proximities on $\mathcal{S}$ and the compactifications of $\mathcal{S}$. Indeed, for any compactification $\overline{\mathcal{S}}$, the set $\mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S}, \mathcal{X})$ is precisely the set of functions which are "continuous" with respect to the proximity induced by $\overline{\mathcal{S}}$.

Thus, by using an SEU representation with respect to a particular compactification $\overline{\mathcal{S}}$, we are in effect augmenting the topology of $\mathcal{S}$ with a particular proximity relation $\sim$. Given two disjoint regular sets $\mathcal{Q}$ and $\mathcal{R}$, we have $\mathcal{Q} \sim \mathcal{R}$ if and only if the boundaries of $\mathcal{Q}$ and $\mathcal{R}$ intersect in $\overline{\mathcal{S}}$. This means that it is at least theoretically possible that the agent will encounter a situation where the true state of the world is in neither $\mathcal{Q}$ nor $\mathcal{R}$, but her observation technology will erroneously *report* the state as being in either $\mathcal{Q}$ or $\mathcal{R}$ with some nonzero probability (determined by the liminal density structure on $(\partial\mathcal{Q}) \cap (\partial\mathcal{R})$.) In other words, the use of a particular compactification —hence a particular proximity —encodes a particular model of the *fallibility* of the agent's observation technology.

7 SEU representations via credences

This section contains our most general SEU representation result. First, we relax the assumptions on the state space topology made in Sections 4, 5 and 6. Second, we now allow the possibility that the agent is subject to informational constraints, represented by two Boolean subalgebras $\mathfrak{B} \subseteq \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} \subseteq \mathfrak{R}(\mathcal{X})$. We suppose that the agent can

[16] See Example 5.9(f) of Pivato and Vergopoulos (2016) or §40-§41 of Willard (2004) for details.

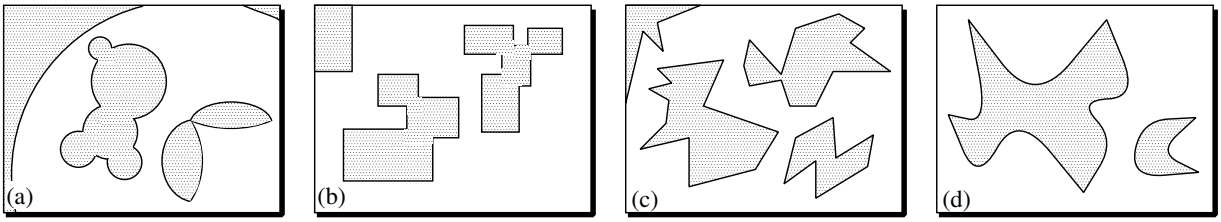

Figure 3: The Boolean subalgebras from Example 7. (a) A typical element of $\mathfrak{B}_{\text{ball}}(\mathbb{R}^2)$. (b) A typical element of $\mathfrak{B}_{\text{box}}(\mathbb{R}^2)$. (c) A typical element of $\mathfrak{B}_{\text{poly}}(\mathbb{R}^2)$. (d) A typical element of $\mathfrak{B}_{\text{smth}}(\mathbb{R}^2)$. (Note that in each case, the *negation* of the shaded set is also an element of the algebra in question.)

only observe events in the state space which correspond to elements of $\mathfrak{B}$, and she can only observe properties of outcomes which correspond to elements of $\mathfrak{D}$. Such Boolean subalgebras arise from natural constraints on the agent's observational technology, as the next example shows.

Example 7. In the following examples, we assume $\mathcal{S} = \mathbb{R}^N$ for some $N \in \mathbb{N}$.

(a) Any open ball in $\mathbb{R}^N$ is regular. So is the complement of any closed ball. Let $\mathfrak{B}_{\text{ball}}(\mathbb{R}^N)$ be the collection of all regular subsets of $\mathbb{R}^N$ constructed by taking joins and/or intersections of finite collections of open balls and/or the complements of their closures. Then $\mathfrak{B}_{\text{ball}}(\mathbb{R}^N)$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$. A typical element is shown in Figure 3(a). This Boolean algebra describes the information available to an agent whose observational technology allows her to check whether the true state of the world is within some specified distance of some target state. (For example, she can check the statement, "The true state is within distance 1.6 of the point $(0, 0, 0)$".)[17]

(b) A subset of $\mathbb{R}^N$ is an *open box* if it is a Cartesian product of open intervals. Any open box is regular. The intersection of two open boxes is also an open box (if it is nonempty). Let $\mathfrak{B}_{\text{box}}(\mathbb{R}^N)$ be the Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$ generated by open boxes. A typical element is shown in Figure 3(b). This Boolean algebra describes the information available to an agent whose observational technology allows her to check whether any particular *coordinate* of the true state satisfies some strict inequality. (For example, she can check the statement, "The horizontal coordinate of the state is strictly between 1.16 and 3.24.").[18]

(c) A subset $\mathcal{H} \subseteq \mathbb{R}^N$ is a *hyperplane* if there is a linear function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that $\mathcal{H} := \phi^{-1}\{r\}$ for some $r \in \mathbb{R}$. A regular subset $\mathcal{R} \subseteq \mathbb{R}^N$ is a *polyhedron* if there is a finite collection $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ of hyperplanes such that $\partial\mathcal{R} = (\mathcal{H}_1 \cap \partial\mathcal{R}) \cup \cdots \cup (\mathcal{H}_N \cap \partial\mathcal{R})$. (Heuristically, each of the sets $\mathcal{H}_n \cap \partial\mathcal{R}$ is one of the "faces" of the polyhedron. Note that we do not require these polyhedra to be convex, or even connected.) Let $\mathfrak{B}_{\text{poly}}(\mathbb{R}^N)$ be the set of regular polyhedra; then $\mathfrak{B}_{\text{poly}}(\mathbb{R}^N)$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$. A typical element is shown in Figure 3(c). This Boolean algebra describes the information available

[17]This construction also works if we replace $\mathbb{R}^N$ with any metric space. But not all open balls in a general metric space are regular, so we must restrict our attention to the collection of *regular* open balls.

[18]This construction also works if we replace $\mathbb{R}^N$ with any Cartesian product of linearly ordered sets, each endowed with its order topology.

to an agent whose observational technology allows her to check whether the state satisfies any finite collection of strict linear inequalities.[19]

(d) A subset $\mathcal{H} \subseteq \mathbb{R}^N$ is a *smooth hypersurface* if there is a differentiable function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that $\mathcal{H} := \phi^{-1}\{r\}$ for some $r \in \mathbb{R}$, and such that $\mathrm{d}\phi(h) \neq 0$ for all $h \in \mathcal{H}$. We will say that a regular subset $\mathcal{R} \subseteq \mathbb{R}^N$ has a *piecewise smooth boundary* if there is a finite collection $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ of smooth hypersurfaces such that $\partial\mathcal{R} = (\mathcal{H}_1 \cap \partial\mathcal{R}) \cup \cdots \cup (\mathcal{H}_N \cap \partial\mathcal{R})$. Let $\mathfrak{B}_{\mathrm{smth}}(\mathbb{R}^N)$ be the set of regular subsets of $\mathbb{R}^N$ with piecewise smooth boundaries; then $\mathfrak{B}_{\mathrm{smth}}(\mathbb{R}^N)$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$. A typical element is shown in Figure 3(d).[20]

This Boolean algebra describes the information available to an agent whose observational technology allows her to check whether the state satisfies any finite collection of strict inequalities based on differentiable functons. We normally assume that the output of any scientific instrument is a differentiable function of the true state of the world; thus, this Boolean algebra describes the sort of information available through such scientific instruments. We can also construct Boolean subalgebras of $\mathfrak{B}_{\mathrm{smth}}(\mathbb{R}^N)$ based on more restrictive classes of functions (e.g. polynomials).

(e) In the case $N = 1$, the Boolean algebras in parts (a)-(d) take a particularly simple form. Let us say that a subset $\mathcal{B} \subseteq \mathbb{R}$ is *basic* if $\mathcal{B} = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \cdots \sqcup (a_N, b_N)$ for some $-\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N \leq \infty$. Let $\mathfrak{Bas}(\mathbb{R})$ be the set of all basic open subsets of $\mathbb{R}$. Then $\mathfrak{Bas}(\mathbb{R})$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R})$, and it is easy to verify that $\mathfrak{B}_{\mathrm{ball}}(\mathbb{R}) = \mathfrak{B}_{\mathrm{box}}(\mathbb{R}) = \mathfrak{B}_{\mathrm{poly}}(\mathbb{R}) = \mathfrak{B}_{\mathrm{smth}}(\mathbb{R}) = \mathfrak{Bas}(\mathbb{R})$. ♢

All the statements in this section hold when $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$. So on a first reading, the reader may assume $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$ for simplicity.

Comeasurability. The Boolean subalgebras $\mathfrak{B}$ and $\mathfrak{D}$ and the set of feasible acts $\mathcal{A}$ must satisfy a natural consistency requirement. Consider a feasible act $\alpha \in \mathcal{A}$ and a regular subset $\mathcal{D} \in \mathfrak{D}$. Together, they define an observation of the state, obtained by first performing the act α, and then making the observation of the outcome corresponding to the partition $\{\mathcal{D}, \neg\mathcal{D}\}$ to see whether the true state s has the property that $\alpha(s)$ lies in $\mathcal{D}$. But if such an observation of s is feasible, then there must be a $\mathfrak{B}$-partition representing it. This motivates the following requirement. A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *comeasurable* with respect to $\mathfrak{B}$ and $\mathfrak{D}$ if $\mathrm{int}\,(\alpha^{-1}[\mathrm{clos}(\mathcal{D})]) \in \mathfrak{B}$ for all $\mathcal{D} \in \mathfrak{D}$. For example, if α is any continuous function, then α is comeasurable with respect to $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{R}(\mathcal{X})$ (because the interior of any closed set is regular). Thus, all feasible acts in Sections 5 and 6 implicitly satisfied the comeasurability requirement. Let $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$ denote the set of bounded, $(\mathfrak{B}, \mathfrak{D})$-comeasurable and continuous functions from $\mathcal{S}$ to $\mathcal{X}$. From now on, we assume that $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$ and that $\mathcal{A}$ contains all constant functions.

[19] This construction also works if we replace $\mathbb{R}^N$ with any topological vector space. But we must then stipulate that ϕ is *continuous* as well as linear.

[20] This construction also works if we replace $\mathbb{R}^N$ with any differentiable manifold.

The measurability axiom. We will need one more axiom in addition to those which appeared in Section 3. For any $x \in \mathcal{X}$, the *open upper contour set* of x is $\{y \in \mathcal{X};\, y \succ_{\text{xp}} x\}$; the *open lower contour set* of x is $\{y \in \mathcal{X};\, y \prec_{\text{xp}} x\}$. Axiom (C) already ensures that these are open subsets of $\mathcal{X}$. The next axiom goes further.

(M) All open contour sets of $\succeq_{\text{xp}}$ are elements of $\mathfrak{D}$.

The intuition here is straightforward. The agent is presumably aware of her own *ex post* preferences. Thus, for any $y \in \mathcal{X}$, she is able to discern whether y satisfies the properties "$y \succ_{\text{xp}} x$" or "$y \prec_{\text{xp}} x$" —i.e. whether y belongs to the open upper or lower contour set of x. In other words, these open contour sets must be "observable" subsets of the outcome space. But then they must belong to $\mathfrak{D}$.

Beliefs. We will represent the probabilistic "beliefs" of the agent by a *credence* —a structure like a finitely additive probability measure on the Boolean subalgebra $\mathfrak{B}$ of $\mathfrak{R}(\mathcal{S})$. To be precise, a *credence* on $\mathfrak{B}$ is a function $\mu : \mathfrak{B} \longrightarrow [0,1]$ such that $\mu[\mathcal{S}] = 1$ and such that, for any finite collection $\{\mathcal{B}_n\}_{n=1}^N$ of disjoint elements of $\mathfrak{B}$, we have

$$\mu\left[\bigvee_{n=1}^{N} \mathcal{B}_n\right] \quad = \quad \sum_{n=1}^{N} \mu[\mathcal{B}_n]. \tag{15}$$

A credence μ behaves like an ordinary probability measure. For example, for any $\mathcal{B} \in \mathfrak{B}$ with $\mu[\mathcal{B}] > 0$, we can define the *conditional credence* $\mu_{\mathcal{B}}$ by setting $\mu_{\mathcal{B}}[\mathcal{R}] := \mu[\mathcal{B} \cap \mathcal{R}]/\mu[\mathcal{B}]$ for all $\mathcal{R} \in \mathfrak{B}$. Then $\mu_{\mathcal{B}}$ also satisfies equation (15). But note an important difference from the usual definition of a measure: additivity is defined with respect to the operation $\vee$, rather than ordinary union. For any regular subset $\mathcal{B} \in \mathfrak{B}$, the number $\mu(\mathcal{B})$ should be understood as the probability of *observing* $\mathcal{B}$, rather than the probability that $\mathcal{B}$ *actually obtains*. Since $\mathfrak{B}$-partitions are observationally exhaustive, additivity with respect to $\vee$ as captured by equation (15) becomes a natural requirement. We say that a credence μ has *full support* if $\mu[\mathcal{B}] > 0$ for all nonempty $\mathcal{B} \in \mathfrak{B}$.

Example 8. (a) Let $\mathcal{S} := (0,1)$, and let $\mathfrak{Bas}(0,1)$ be the Boolean algebra of basic open subsets of $(0,1)$, as defined in Example 7(e). For any $\mathcal{B} \in \mathfrak{Bas}(0,1)$, if $\mathcal{B} = (a,b)$ for some $a < b$, then let $\mu[\mathcal{B}] := b - a$. Next, if $\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_N$ for some disjoint open intervals $\mathcal{B}_1, \ldots, \mathcal{B}_N$, then define $\mu[\mathcal{B}] := \mu[\mathcal{B}_1] + \cdots + \mu[\mathcal{B}_N]$. Then μ is a credence on $\mathfrak{Bas}(0,1)$.

(b) Let $\mathfrak{Bai}(\mathcal{S})$ be the Boolean algebra of sets with the Baire property, and let $\nu : \mathfrak{Bai}(\mathcal{S}) \longrightarrow [0,1]$ be a Baire charge, as defined in Section 4. Recall that $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{Bai}(\mathcal{S})$, but $\mathfrak{R}(\mathcal{S})$ is not a *subalgebra* of $\mathfrak{Bai}(\mathcal{S})$, because the Boolean algebra operations are different (i.e. $\vee$ vs. $\cup$). But if we restrict ν to $\mathfrak{R}(\mathcal{S})$, then we get a credence on $\mathfrak{R}(\mathcal{S})$.[21] It follows that ν also defines a credence when restricted to any Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$. $\diamondsuit$

We can also construct credences on $\mathfrak{R}(\mathcal{S})$ from the liminal charge structures of Section 5 or the liminal density structures of Section 6; see Pivato and Vergopoulos (2016).

[21] *Proof.* For any disjoint $\mathcal{R}, \mathcal{Q} \in \mathfrak{R}(\mathcal{S})$, we have $\mathcal{R} \vee \mathcal{Q} = \mathcal{R} \sqcup \mathcal{Q} \sqcup \mathcal{M}$, where $\mathcal{M}$ is meager. Thus, $\nu[\mathcal{R} \vee \mathcal{Q}] = \nu[\mathcal{R}] + \nu[\mathcal{Q}] + \nu[\mathcal{M}] = \nu[\mathcal{R}] + \nu[\mathcal{Q}]$, because $\nu[\mathcal{M}] = 0$ because ν is a Baire charge.

Conditional expectation structures. From any credence, we can construct a system of *expectation functionals*, which assign "expected values" to the utility profiles of feasible acts. Recall that $\mathfrak{Bas}(\mathbb{R})$ is the Boolean algebra of all basic subsets of $\mathbb{R}$. A function $f : \mathcal{S} \to \mathbb{R}$ is $\mathfrak{B}$-*comeasurable* if it is comeasurable with respect to $\mathfrak{B}$ and $\mathfrak{Bas}(\mathbb{R})$. Equivalently, f is $\mathfrak{B}$-*comeasurable* if $\operatorname{int}(f^{-1}(-\infty, r]) \in \mathfrak{B}$ and $\operatorname{int}(f^{-1}[r, \infty)) \in \mathfrak{B}$ for all $r \in \mathbb{R}$.

Example 9. (a) If $f : \mathcal{S} \longrightarrow \mathbb{R}$ is continuous, then f is $\mathfrak{R}(\mathcal{S})$-comeasurable.

(b) Let $\mathcal{S} = \mathbb{R}$. A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is $\mathfrak{Bas}(\mathbb{R})$-comeasurable if there is a finite sequence of points $-\infty = r_0 < r_1 < r_2 < r_3 < \cdots < r_N = \infty$ such that for each $n \in [1 \dots N]$, either f is non-increasing on (r_{n-1}, r_n), or f is non-decreasing on (r_{n-1}, r_n). In particular, any polynomial is $\mathfrak{Bas}(\mathbb{R})$-comeasurable, as is any non-decreasing or non-increasing continuous function. But $f(x) = \sin(x)$ is not $\mathfrak{Bas}(\mathbb{R})$-comeasurable.

(c) Let $\mathcal{S} = \mathbb{R}^N$, and let $\mathfrak{B}_{\text{poly}}(\mathbb{R}^N)$ be the Boolean algebra of regular polyhedra, from Example 7(c). A function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is *affine* if $f = f_0 + r$ for some linear function $f_0 : \mathbb{R}^N \longrightarrow \mathbb{R}$ and some constant $r \in \mathbb{R}$. We say f is *piecewise affine* if there is a partition $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_N\}$ of $\mathbb{R}^N$ into regular polyhedra, and a collection $f^1, \dots, f^N : \mathbb{R}^N \longrightarrow \mathbb{R}$ of affine functions, such that $f_{\restriction \mathcal{R}_n} = f^n_{\restriction \mathcal{R}_n}$ for all $n \in [1 \dots N]$. Any piecewise affine function is $\mathfrak{B}_{\text{poly}}(\mathbb{R}^N)$-comeasurable.

(d) Let $\mathcal{S} = \mathbb{R}^N$, and let $\mathfrak{B}_{\text{smth}}(\mathbb{R}^N)$ be the Boolean algebra of regular sets with piecewise smooth boundaries, from Example 7(d). If $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is any differentiable function such that $\mathrm{d}f(s) \neq 0$ for all $s \in \mathbb{R}^N$, then f is $\mathfrak{B}_{\text{smth}}(\mathbb{R}^N)$-comeasurable. ♢

Let $\mathcal{C}(\mathcal{S}, \mathbb{R})$ denote the vector space of all continuous, real-valued functions on $\mathcal{S}$. Let $\mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$ be the Banach space of *bounded*, continuous, real-valued functions, with the uniform norm $\|\cdot\|_\infty$. Let $\mathcal{C}_{\mathfrak{B}}(\mathcal{S})$ be the set of all $\mathfrak{B}$-comeasurable functions in $\mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$. This set is not necessarily closed under addition. So, let $\mathcal{G}_{\mathfrak{B}}(\mathcal{S})$ be the linear subspace of $\mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$ spanned by $\mathcal{C}_{\mathfrak{B}}(\mathcal{S})$. (If $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$, then $\mathcal{G}_{\mathfrak{B}}(\mathcal{S}) = \mathcal{C}_{\mathfrak{B}}(\mathcal{S}) = \mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$.) For any subset $\mathcal{B} \subseteq \mathcal{S}$, let $\mathcal{G}_{\mathfrak{B}}(\mathcal{B}) := \{g_{\restriction \mathcal{B}};\ \ g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{S})\}$. This is a linear subspace of $\mathcal{C}_{\text{b}}(\mathcal{B}, \mathbb{R})$. An *expectation functional* on $\mathcal{B}$ is a linear functional $\mathbb{E} : \mathcal{G}_{\mathfrak{B}}(\mathcal{B}) \longrightarrow \mathbb{R}$ which satisfies two properties:

- *Unimodularity.* $\|\mathbb{E}\|_\infty = 1$.
- *Weak monotonicity.* For any $f, g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, if $f(b) \leq g(b)$ for all $b \in \mathcal{B}$, then $\mathbb{E}[f] \leq \mathbb{E}[g]$.

Let $\mathbf{1}$ be the constant function with value 1. Note that *Unimodularity* not only implies that $\mathbb{E}$ is continuous; together with *Weak Monotonicity*, it also implies that $\mathbb{E}[\mathbf{1}] = 1$.

Now let μ be a credence on $\mathfrak{B}$. A *conditional expectation structure* for $\mathcal{S}$ that is *compatible with* μ is a collection $\mathbf{E} := \{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$, where, for all $\mathcal{B} \in \mathfrak{B}$,

- $\mathbb{E}_{\mathcal{B}}$ is an expectation functional on $\mathcal{G}_{\mathfrak{B}}(\mathcal{B})$; and

- If $\mu[\mathcal{B}] > 0$, then for any $\mathfrak{B}$-partition $\{\mathcal{B}_n\}_{n=1}^N$ of $\mathcal{B}$, and any $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, we have

$$\mathbb{E}_{\mathcal{B}}[g] \quad = \quad \frac{1}{\mu[\mathcal{B}]} \sum_{n=1}^{N} \mu[\mathcal{B}_n]\, \mathbb{E}_{\mathcal{B}_n}\left[g_{\restriction \mathcal{B}_n}\right]. \tag{16}$$

In particular, $\mathbb{E} = \mathbb{E}_{\mathcal{S}}$ is an expectation functional on $\mathcal{G}_{\mathfrak{B}}(\mathcal{S})$, and a version of equation (16) holds for every $\mathfrak{B}$-partition of $\mathcal{S}$. Equation (16) captures a key feature of Bayesianism: conditional expectations are additively separable over the disjoint events of a $\mathfrak{B}$-partition. Indeed, for any regular event $\mathcal{B} \in \mathfrak{B}$ with $\mu[\mathcal{B}] > 0$, the subcollection $\{\mathbb{E}_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{B}'}$, where $\mathfrak{B}'$ is the collection of sets in $\mathfrak{B}$ that are contained in $\mathcal{B}$, is itself a conditional expectation structure on $\mathcal{B}$, compatible with the conditional credence $\mu_{\mathcal{B}}$.

If $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{S})$ and $\mathcal{B} \in \mathfrak{B}$, we will abuse notation and write "$\mathbb{E}_{\mathcal{B}}[g]$" to mean $\mathbb{E}_{\mathcal{B}}[g_{\restriction \mathcal{B}}]$. We say $\mathbf{E}$ is *strictly monotonic* if, for all $\mathcal{B} \in \mathfrak{B}$ and $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, if $g(b) > 0$ for all $b \in \mathcal{B}$, then $\mathbb{E}_{\mathcal{B}}[g] > 0$. For every credence μ, there is a unique compatible conditional expectation structure $\mathbf{E}$; furthermore, if μ has full support, then $\mathbf{E}$ is monotonic (see Theorem 3.1 from Pivato and Vergopoulos (2016)).

Example 10. (a) Let $\mathcal{S} = (0,1)$, let $\mathfrak{Bas}(0,1)$ be the Boolean algebra of basic open subsets of $(0,1)$, and let μ be the credence from Example 8(a). Then the unique μ-compatible conditional expectation structure is defined as follows: for any $\mathcal{B} = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \cdots \sqcup (a_N, b_N)$ in $\mathfrak{E}$, and any $g \in \mathcal{G}_{\mathfrak{Bas}}(0,1)$,

$$\mathbb{E}_{\mathcal{B}}[g] \quad := \quad \frac{1}{\mu[\mathcal{B}]} \left(\int_{a_1}^{b_1} g(x)\ \mathrm{d}x \ + \ \int_{a_2}^{b_2} g(x)\ \mathrm{d}x \ + \ \cdots \ + \ \int_{a_N}^{b_N} g(x)\ \mathrm{d}x \right).$$

(b) Let $\mathcal{S}$ be a Baire space, let $\mathfrak{Bai}(\mathcal{S})$ be the Boolean algebra of sets with the Baire property, and let $\nu : \mathfrak{Bai}(\mathcal{S}) \longrightarrow [0,1]$ be a Baire charge, which induces a credence on $\mathfrak{R}(\mathcal{S})$ as in Example 8(b). The unique μ-compatible conditional expectation structure on $\mathfrak{R}(\mathcal{S})$ is defined by setting $\mathbb{E}_{\mathcal{R}}[f] := \dfrac{1}{\nu[\mathcal{R}]} \displaystyle\int_{\mathcal{R}} f\ \mathrm{d}\nu$ for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and $f \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$.[22] ♢

SEU representations. We will say that a function $u : \mathcal{X} \longrightarrow \mathbb{R}$ is $\mathfrak{D}$-*measurable* if $u^{-1}(\mathcal{B}) \in \mathfrak{D}$ for any $\mathcal{B} \in \mathfrak{Bas}(\mathbb{R})$. For example, if $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$, then any open, continuous real-valued function on $\mathcal{X}$ is $\mathfrak{D}$-measurable (Fremlin, 2006, Appendix 4A2B, item (f)(iii), p.453). If $u : \mathcal{X} \longrightarrow \mathbb{R}$ is $\mathfrak{D}$-measurable, and $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is $(\mathfrak{B}, \mathfrak{D})$-comeasurable, then $u \circ \alpha : \mathcal{S} \longrightarrow \mathbb{R}$ is $\mathfrak{B}$-comeasurable, and thus it can be evaluated by a conditional expectation structure (Pivato and Vergopoulos, 2016, Lemma 4.4).

Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$, and let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be a $\mathfrak{B}$-indexed conditional preference structure on $\mathcal{A}$. Let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a $\mathfrak{D}$-measurable, continuous "utility" function. Let μ be a credence on $\mathfrak{B}$. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the (unique) conditional expectation structure that is compatible with μ. The pair (u, μ) is a *subjective expected utility* (SEU) representation for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ if, for any $\mathcal{B} \in \mathfrak{B}$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, we have

$$\Big(\alpha \succeq_{\mathcal{B}} \beta\Big) \iff \Big(\mathbb{E}_{\mathcal{B}}\left[u \circ \alpha\right] \geq \mathbb{E}_{\mathcal{B}}\left[u \circ \beta\right]\Big). \tag{17}$$

[22] See Proposition 5.1 of Pivato and Vergopoulos (2016).

Thus, the agent behaves as if she uses the utility function u to attach continuous utility levels to all outcomes, the credence μ to assign probabilities to all regular events, and the conditional expectation structure $\{\mathbb{E}_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ that is generated by μ to compute the expected utility of all feasible acts. She then prefers the acts with the highest expected utility. Finally, we will say that $\mathfrak{B}$ is *nontrivial* if it contains a proper regular subset of $\mathcal{S}$. We now come to our final result.

Theorem 11 *Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces, with $\mathcal{X}$ connected. Let $\mathfrak{B}$ and $\mathfrak{D}$ be nontrivial Boolean subalgebras of $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{R}(\mathcal{X})$ respectively. Let $\mathcal{A}$ be a collection of bounded continuous and $(\mathfrak{B},\mathfrak{D})$-comeasurable functions from $\mathcal{S}$ into $\mathcal{X}$. Let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then, it further satisfies Axioms* (CEq)*,* (C)*,* (M)*,* (Dom)*,* (Sep)*,* (PC)*, and* (TC) *if and only if it has an SEU representation (u,μ) with full support. Finally, μ is unique, and u is unique up to positive affine transformation.*

The SEU representation obtained in Theorem 11 is somewhat simpler but less informative than the representations obtained so far: the agent's beliefs only assign probabilities to the observable events and do not describe the way she copes with her informational constraints. But its advantage lies in its greater generality. This SEU representation does not require any specific topological assumption on the state space and accomodates decision situations where the observable measurements only arise from a subalgebra of regular subsets. Thus, Theorem 11 provides axiomatic foundations for SEU that are adapted to decision situations where technological constraints restrict both the acts which are feasible and the information which is available to the agent. Last, note that (Rch) does not follow automatically from the representation. (Appendix C discusses necessary and sufficient conditions for (Rch).)

We will now briefly sketch of the proof of Theorem 11. The first step consists in using structural condition (Rch) to obtain a preference order on $\mathcal{X}^N$, for each $\mathfrak{B}$-partition $\{\mathcal{B}_1,\ldots,\mathcal{B}_N\}$ of $\mathcal{S}$. Then, we invoke (Sep), (C), (PC), (TC), (Dom) and (CEq) and a theorem of Wakker (1989a) to construct a continuous SEU representation for the latter order. The decision weights associated to each component of $\mathcal{X}^N$ provide the values of the credence μ on each regular subset of the partition $\{\mathcal{B}_1,\ldots,\mathcal{B}_N\}$. Moreover, axiom (M) ensures that the continuous utility function u thus constructed is $\mathfrak{D}$-measurable. The next step shows the relevance of μ and u for the representation of the primitive conditional preference structure: the expected utility of any act is shown to equal the utility of any of its certainty equivalents. The SEU representation then easily follows. Finally, Appendix B presents another result, Theorem B1, which is a simplified version of Theorem 11 that is adapted to situations where $\mathfrak{B}=\mathfrak{R}(\mathcal{S})$ and $\mathfrak{D}=\mathfrak{R}(\mathcal{X})$. All of Theorems 1, 2, 3, 5 and 6 are obtained as corollaries of this simplified version.

8 Discussion

Several previous papers have restricted the Savage universal domain of preference by introducing measurability constraints. In fact, Savage (1954) explicitly notes that his axiomatic

construction works equally well for preferences over acts that are measurable with respect to a σ-algebra, but only produces finitely additive probabilities. Arrow (1970) enriches the Savage axioms so as to further derive the σ-additivity of probability measures. See also Kopylov (2010). Epstein and Zhang (2001) construct a theory of "probabilistic sophistication" on a collection of acts that are measurable with respect to a "λ-system". Last, Kopylov (2007) provides both SEU maximization and probabilistic sophistication on the weaker structure of so-called *mosaics*, which include Boolean algebras.

Some papers aim explicitly at deriving continuous utility functions from preferences. For example, Grandmont (1972) obtained continuous utility functions in a von Neuman and Morgenstern (1947) framework. Other papers consider acts from a measurable state space into a topological outcome space, typically assumed to be connected and separable. For example, Wakker (1985, 1988) and Wakker (1989a, Chapter 5) characterized continuous, state-independent SEU representations in this setting. Wakker (1987) characterized continuous and *state-dependent* SEU over a finite state space, while Wakker and Zank (1999) characterized it over any measurable space. Wakker (1989b) characterized continuous Choquet expected utility representations over a finite state space. Finally, Casadesus-Masanell et al. (2000) characterized continuous maximin expected utility representations.

Like us, Zhou (1999) considers the case where both the state space and the outcome space are topological spaces, and acts are continuous functions. But unlike us, he restricts attention to the case where the outcomes are themselves lotteries over some finite set of consequences, so that acts correspond to "two-stage lotteries" of the kind considered by Anscombe and Aumann (1963). In this framework, Zhou proves versions of Anscombe and Aumann's SEU representation theorem, as well as the Choquet expected utility representation theorem of Schmeidler (1989), in both cases obtaining continuous utility functions. Unlike Zhou, we do not assume any special structure on the outcome space; our framework is more like the framework of Savage, rather than that of Anscombe and Aumann.

Alternative interpretations of our results. So far we have interpreted the various restrictions on acts and information in terms of *technological* feasibility. But *cognitive* feasibility provides another way to understand these restrictions. A Savage framework may contain complex acts and complex events that are difficult for the agent to fully comprehend. Can she really formulate meaningful preferences over such acts conditional on such events? If so, should these preferences be required to satisfy the Savage axioms? In contrast, *continuous* acts and *regular* events are generally easy to visualize or understand. Meanwhile, the measurement errors that are allowed by regular partitions can also be reinterpreted as the result of errors in information processing on the part of the agent. Thus, our results can be interpreted as describing a cognitively limited agent.

Continuity restrictions are also relevant in other economic applications beyond choice under uncertainty. For example, in intertemporal decisions, technological constraints may force production and consumption streams to be continuous (or differentiable) in *time*. Our main results then become axiomatic characterizations of intertemporal choice determined by a generalized discounted sum of future utilities (with "states" interpreted as moments in time, and the credence encoding the discount factors).

Another application is social choice. Suppose each person is described by a bundle of characteristics ranging over some topological space. (For a example, suppose she is described by her age, her geographic location, and her income level.) Individual outcomes (e.g. consumption bundles) range over another topological space. A social alternative assigns an outcome to each individual; however, for reasons of both feasibility and perceived fairness, we must select alternatives where the outcome varies *continuously* with the characteristics of the individuals in question. (This constraint is even more important when some characteristics of individuals are not precisely observable.) The results of this paper then become axiomatic characterizations of a weighted utilitarian social welfare function.

Conclusion

We conclude by briefly sketching potential directions for further research. This paper has presented a series of SEU representations for preferences under uncertainty. However, it is now well-established that the SEU model is often not descriptively accurate. In some cases, it may not even be normatively compelling, e.g. when the agent faces *ambiguity*, where she lacks even sufficient information to form probabilistic beliefs about the state of the world. Thus, there has been much recent interest in "non-SEU" models of decision-making under uncertainty. However, there has been little exploration of such non-SEU models in explicitly topological environments like the ones considered in this paper.[23] In that respect, the results in the present paper can be seen as benchmarks, which set the stage for future research into non-SEU representations on topological spaces.

In most of the applications of decision theory (e.g. in finance and risk management), both states and outcomes range over some Euclidean space (typically, $\mathbb{R}$), and utility functions are assumed to be continuous, while beliefs take the form of Borel probability measures. Thus, our results are particularly well-suited to such applications. However, in these environments, utility functions are typically assumed to be not only continuous, but *differentiable*. (Indeed, absolute risk aversion and absolute prudence cannot even be defined unless utility is multiply differentiable.) Furthermore, to establish the link between geometric properties of the utility function (e.g. concavity) and behavioural properties of the agent (e.g. risk aversion), one must test her utility function against a fairly rich set of acts. So this raises two questions: First, how can we axiomatically characterize *differentiable* utility functions? Second, how can we link behaviour to geometric properties of the utility function using only the acts which are feasible in some technology? A third question relates to our introduction of credences, liminal structures and/or compactifications of the state space to represent the agent's beliefs. What are the implications of such representations for her financial decisions?

A similar question arises in game theory, where mixed strategies are represented as probability distributions over strategy spaces. In many games, these strategy spaces have a natural topological structure. We could therefore introduce mixed strategies involving

[23] Wakker (1989b), Casadesus-Masanell et al. (2000), and Zhou (1999) are exceptions. But the first two only put a topology on the outcome space.

credences, liminal structures and/or compactifications of the strategy space. What sort of mixed strategy Nash equilibria result from these generalizations?

Last, in many decisions situations where postponing a decision is costless, it may be tempting for an agent to wait and acquire more information before making the decision. As is well-known, more information leads to better decisions and is therefore often valuable. But technological constraints provide an additional reason to follow such a strategy: the alternatives that would be chosen at disjoint events do not restrict each other and do not have to define jointly a feasible act. Thus, using information to design contingent plans of actions makes it possible to implement discontinuous distributions of outcomes across states of the world that would be simply unfeasible without information. From this point of view, it has been an implicit assumption of this paper that waiting for more information was not an option. But it is natural to ask if the SEU theory developed here extends to contingent planning.

Appendix: Proofs of all results.

Appendix A focuses on the proof of our main result, Theorem 11. Appendix B contains the proofs of the SEU representations from Sections 4, 5 and 6. Finally Appendix C gives some necessary and sufficient conditions for the richness condition (Rch).

A Proof of Theorem 11

The proof of Theorem 11 has two preliminary stages. First, Proposition A3 uses structural condition (Rch), as well as axioms (Sep), (C), (TC), (PC), (Dom), (CEq) and (M) to construct a credence μ on $\mathfrak{B}$ and a continuous and $\mathfrak{D}$-measurable utility function u using a theorem of Wakker (1989a) for continuous additive representations. Second, Proposition A6 shows that the expected utility of any act, with respect to μ and u, equals the utility of any certainty equivalent of this act. The rest of the appendix uses these findings to construct the SEU representations and establishes the necessity of the axioms, as well as the uniqueness of the representation.Throughout this appendix, we maintain the following standing assumptions:

> $\mathcal{S}$ and $\mathcal{X}$ are topological spaces, with $\mathcal{X}$ connected. $\mathfrak{B} \subseteq \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} \subseteq \mathfrak{R}(\mathcal{X})$ are nontrivial Boolean subalgebras, $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$, and $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ is a conditional preference structure on $\mathcal{A}$ that satisfies (Rch).

Lemma A1 *Suppose $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies axioms* (C), (Dom) *and* (CEq). *Let $\mathcal{B} \in \mathfrak{B}$. Define the function $\mathsf{K}_{\mathcal{B}} : \mathcal{X} \longrightarrow \mathcal{A}(\mathcal{B})$ by $\mathsf{K}_{\mathcal{B}}(x) := \kappa^x_{\mathcal{B}}$ for all $x \in \mathcal{X}$. Then $\mathsf{K}_{\mathcal{B}}$ is continuous relative to the $\succeq_{\mathcal{B}}$-order topology on $\mathcal{A}(\mathcal{B})$.*

Proof. For any $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$, let $(\alpha, \gamma)_{\succeq_{\mathcal{B}}} := \{\beta \in \mathcal{A}(\mathcal{B});\ \alpha \prec_{\mathcal{B}} \beta \prec_{\mathcal{B}} \gamma\}$. This collection of sets (for all $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$) forms a base for the $\succeq_{\mathcal{B}}$-order topology on $\mathcal{A}(\mathcal{B})$. So it suffices to show that $\mathsf{K}_{\mathcal{B}}^{-1}\left[(\alpha, \gamma)_{\succeq_{\mathcal{B}}}\right]$ is open in $\mathcal{X}$, for all $\alpha, \beta \in \mathcal{A}(\mathcal{B})$.

Axiom (CEq) says there exist $x, z \in \mathcal{X}$ such that $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$ and $\gamma \approx_{\mathcal{B}} \kappa^z_{\mathcal{B}}$. Define $(x,z)_{\succeq_{\text{xp}}} := \{y \in \mathcal{X};\, x \prec_{\text{xp}} y \prec_{\text{xp}} z\}$. Axiom (C) says this is an open subset of $\mathcal{X}$. Now, for any $y \in \mathcal{X}$, we have

$$\begin{aligned}
\Big(y \in \mathsf{K}_{\mathcal{B}}^{-1}\left[(\alpha,\gamma)_{\succeq_{\mathcal{B}}}\right]\Big) &\iff \Big(\mathsf{K}_{\mathcal{B}}(y) \in (\alpha,\gamma)_{\succeq_{\mathcal{B}}}\Big) \iff \Big(\alpha \prec_{\mathcal{B}} \kappa^y_{\mathcal{B}} \prec_{\mathcal{B}} \gamma\Big) \\
&\underset{(*)}{\Longleftrightarrow} \Big(\kappa^x_{\mathcal{B}} \prec_{\mathcal{B}} \kappa^y_{\mathcal{B}} \prec_{\mathcal{B}} \kappa^z_{\mathcal{B}}\Big) \underset{(\dagger)}{\Longleftrightarrow} \Big(x \prec_{\text{xp}} y \prec_{\text{xp}} z\Big) \\
&\iff \Big(y \in (x,z)_{\succeq_{\text{xp}}}\Big).
\end{aligned}$$

Here, $(*)$ is because $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$ and $\gamma \approx_{\mathcal{B}} \kappa^z_{\mathcal{B}}$, while $(\dagger)$ is by axiom (Dom) and its contrapositive. Thus, we see that $\mathsf{K}_{\mathcal{B}}^{-1}\left[(\alpha,\gamma)_{\succeq_{\mathcal{B}}}\right] = (x,z)_{\succeq_{\text{xp}}}$, which is an open subset of $\mathcal{X}$. Since this holds for all $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$, we conclude that $\mathsf{K}_{\mathcal{B}}$ is continuous. □

It will be convenient to use the following equivalent formulation of axiom (PC).

(PC′) Let $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$ as in axiom (Sep). Let $\mathcal{O} \subseteq \mathcal{A}(\mathcal{B})$ be open in the $\succeq_{\mathcal{B}}$-order topology, and let $\alpha \in \mathcal{O}$. Then there exist sets $\mathcal{O}_{\mathcal{D}} \subseteq \mathcal{A}(\mathcal{D})$ and $\mathcal{O}_{\mathcal{E}} \subseteq \mathcal{A}(\mathcal{E})$ which are open in the $\succeq_{\mathcal{D}}$-order topology and $\succeq_{\mathcal{E}}$-order topology, with $\alpha_{\restriction \mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$ and $\alpha_{\restriction \mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$, such that, for any $\beta \in \mathcal{A}(\mathcal{B})$, if $\beta_{\restriction \mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$ and $\beta_{\restriction \mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$, then $\beta \in \mathcal{O}$.

Lemma A2 *Suppose* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ *satisfies axioms* (PC), (C), (Dom) *and* (CEq). *Consider a* $\mathfrak{B}$*-partition* $\mathfrak{P} = \{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ *of* $\mathcal{S}$ *with* $N \geq 2$. *There exists a mapping* $\Phi_{\mathfrak{P}} : \mathcal{X}^N \longrightarrow \mathcal{A}$ *that is continuous with respect to the product topology on* $\mathcal{X}^N$ *and the* $\succeq_{\mathcal{S}}$*-order topology on* $\mathcal{A}$ *and satisfies* $\Phi_{\mathfrak{P}}(\mathbf{x})_{\restriction \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ *for any* $n \in [1 \ldots N]$ *and any* $\mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$.

Proof. Let $\mathbf{x} := (x_1, \ldots, x_N) \in \mathcal{X}^N$. Define $\alpha^1 := \kappa^{x_1}_{\mathcal{E}_1}$, an element of $\mathcal{A}(\mathcal{E}_1)$. Condition (Rch) yields $\alpha^2 \in \mathcal{A}(\mathcal{E}_1 \vee \mathcal{E}_2)$ such that $\alpha^2_{\restriction \mathcal{E}_1} = \kappa^{x_1}_{\mathcal{E}_1}$ and $\alpha^2_{\restriction \mathcal{E}_2} \approx_{\mathcal{E}_2} \kappa^{x_2}_{\mathcal{E}_2}$. Next, (Rch) yields $\alpha^3 \in \mathcal{A}(\mathcal{E}_1 \vee \mathcal{E}_2 \vee \mathcal{E}_3)$ such that $\alpha^3_{\restriction \mathcal{E}_1 \vee \mathcal{E}_2} = \alpha^2$ and $\alpha^3_{\restriction \mathcal{E}_3} \approx_{\mathcal{E}_3} \kappa^{x_3}_{\mathcal{E}_3}$. In particular, this means that $\alpha^3_{\restriction \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ for all $n \in \{1, 2, 3\}$.

Inductively, let $M \in [4 \ldots N]$, and suppose we have some $\alpha^{M-1} \in \mathcal{A}(\mathcal{E}_1 \vee \cdots \vee \mathcal{E}_{M-1})$ such that $\alpha^{M-1}_{\restriction \mathcal{E}_m} \approx_{\mathcal{E}_m} \kappa^{x_m}_{\mathcal{E}_m}$ for all $m \in [1 \ldots M)$. (Rch) yields $\alpha^M \in \mathcal{A}(\mathcal{E}_1 \vee \cdots \vee \mathcal{E}_M)$ such that $\alpha^M_{\restriction \mathcal{E}_1 \vee \cdots \vee \mathcal{E}_{M-1}} = \alpha^{M-1}$ and $\alpha^M_{\restriction \mathcal{E}_M} \approx_{\mathcal{E}_M} \kappa^{x_M}_{\mathcal{E}_M}$. In particular, this means that $\alpha^M_{\restriction \mathcal{E}_m} \approx_{\mathcal{E}_m} \kappa^{x_m}_{\mathcal{E}_m}$ for all $m \in [1 \ldots M]$.

Setting $M = N$ in this construction, we obtain some α^N such that $\alpha^N_{\restriction \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ for all $n \in [1 \ldots N]$. Now define $\Phi_{\mathfrak{P}}(\mathbf{x}) := \alpha^N$. To prove the continuity of $\Phi_{\mathfrak{P}}$, we need a preliminary result, which extends axiom (PC).

Claim 1: *Let* $\mathcal{O} \subseteq \mathcal{A}$ *be open in the* $\succeq_{\mathcal{S}}$*-order topology, and let* $\alpha \in \mathcal{O}$. *Then for all* $n \in [1 \ldots N]$, *there is a set* $\mathcal{O}_n \subseteq \mathcal{A}(\mathcal{E}_n)$ *which is open in the* $\succeq_{\mathcal{E}_n}$*-order topology, with* $\alpha_{\restriction \mathcal{E}_n} \in \mathcal{O}_n$, *such that, for any* $\beta \in \mathcal{A}$, *if* $\beta_{\restriction \mathcal{E}_n} \in \mathcal{O}_n$ *for all* $n \in [1 \ldots N]$, *then* $\beta \in \mathcal{O}$.

Proof. Let $\mathcal{D}_1 := \mathcal{E}_2 \vee \cdots \vee \mathcal{E}_N$. Thus, $\mathcal{B} := \mathcal{E}_1 \vee \mathcal{D}_1$. Setting $\mathcal{D} := \mathcal{D}_1$ and $\mathcal{E} := \mathcal{E}_1$ in axiom (PC′), we obtain some $\mathcal{O}_1 \subseteq \mathcal{A}(\mathcal{E}_1)$ and $\mathcal{Q}_1 \subseteq \mathcal{A}(\mathcal{D}_1)$ with $\alpha_{\restriction \mathcal{E}_1} \in \mathcal{O}_1$ and $\alpha_{\restriction \mathcal{D}_1} \in \mathcal{Q}_1$, such that, for any $\beta \in \mathcal{A}(\mathcal{B})$, if $\beta_{\restriction \mathcal{E}_1} \in \mathcal{O}_1$ and $\beta_{\restriction \mathcal{D}_1} \in \mathcal{Q}_1$ then $\beta \in \mathcal{O}$.

Now let $\mathcal{D}_2 := \mathcal{E}_3 \vee \cdots \vee \mathcal{E}_N$. Thus, $\mathcal{D}_1 := \mathcal{E}_2 \vee \mathcal{D}_2$. Setting $\mathcal{D} := \mathcal{D}_2$ and $\mathcal{E} := \mathcal{E}_2$ in axiom (PC′), we obtain some $\mathcal{O}_2 \subseteq \mathcal{A}(\mathcal{E}_2)$ and $\mathcal{Q}_2 \subseteq \mathcal{A}(\mathcal{D}_2)$ with $\alpha_{\restriction \mathcal{E}_2} \in \mathcal{O}_2$ and $\alpha_{\restriction \mathcal{D}_2} \in \mathcal{Q}_2$, such that, for any $\beta \in \mathcal{A}(\mathcal{D}_1)$, if $\beta_{\restriction \mathcal{E}_2} \in \mathcal{O}_2$ and $\beta_{\restriction \mathcal{D}_2} \in \mathcal{Q}_2$ then $\beta \in \mathcal{Q}_1$. In particular, this means that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\restriction \mathcal{E}_1} \in \mathcal{O}_1$, $\beta_{\restriction \mathcal{E}_2} \in \mathcal{O}_2$ and $\beta_{\restriction \mathcal{D}_2} \in \mathcal{Q}_2$, then $\beta \in \mathcal{O}$.

Inductively, let $M \in [3 \dots N)$, let $\mathcal{D}_{M-1} := \mathcal{E}_M \vee \cdots \vee \mathcal{E}_N$, and suppose that, for all $m \in [1 \dots M)$, we have constructed $\mathcal{O}_m \subseteq \mathcal{A}(\mathcal{E}_m)$ (open in the $\succeq_{\mathcal{E}_m}$-topology) with $\alpha_{\restriction \mathcal{E}_m} \in \mathcal{O}_m$, along with some $\mathcal{Q}_{M-1} \subseteq \mathcal{A}(\mathcal{D}_{M-1})$ (open in the $\succeq_{\mathcal{D}_{M-1}}$-topology) with $\alpha_{\restriction \mathcal{D}_{M-1}} \in \mathcal{Q}_{M-1}$, such that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\restriction \mathcal{E}_m} \in \mathcal{O}_m$ for all $m \in [1 \dots M)$ and $\beta_{\restriction \mathcal{D}_{M-1}} \in \mathcal{Q}_{M-1}$, then $\beta \in \mathcal{O}$. Now let $\mathcal{D}_M := \mathcal{E}_{M+1} \vee \cdots \vee \mathcal{E}_N$. Thus, $\mathcal{D}_{M-1} := \mathcal{E}_M \vee \mathcal{D}_M$. Setting $\mathcal{D} := \mathcal{D}_M$ and $\mathcal{E} := \mathcal{E}_M$ in axiom (PC′), we obtain some $\mathcal{O}_M \subseteq \mathcal{A}(\mathcal{E}_M)$ and $\mathcal{Q}_M \subseteq \mathcal{A}(\mathcal{D}_M)$ with $\alpha_{\restriction \mathcal{E}_M} \in \mathcal{O}_M$ and $\alpha_{\restriction \mathcal{D}_M} \in \mathcal{Q}_M$, such that, for any $\beta \in \mathcal{A}(\mathcal{D}_{M-1})$, if $\beta_{\restriction \mathcal{E}_M} \in \mathcal{O}_M$ and $\beta_{\restriction \mathcal{D}_M} \in \mathcal{Q}_M$ then $\beta \in \mathcal{Q}_{M-1}$. In particular, this means that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\restriction \mathcal{E}_m} \in \mathcal{O}_m$ for all $m \in [1 \dots M]$ and $\beta_{\restriction \mathcal{D}_M} \in \mathcal{Q}_M$, then $\beta \in \mathcal{O}$.

Suppose $M = N - 1$ in the previous paragraph. Then $\mathcal{D}_M = \mathcal{E}_N$. Thus, if we define $\mathcal{O}_N := \mathcal{Q}_{N-1}$, then we have obtained sets $\mathcal{O}_1, \dots, \mathcal{O}_N$ satisfying the claim. ◇ Claim 1

It remains to show that $\Phi_{\mathfrak{P}}$ is continuous with respect to the product topology on $\mathcal{X}^N$ and the $\succeq_{\mathcal{S}}$-order topology on $\mathcal{A}$. To see this, let $\mathcal{O} \subseteq \mathcal{A}$ be open in the $\succeq_{\mathcal{S}}$-order topology. It is sufficient to show that $\mathcal{U} := \Phi_{\mathfrak{P}}^{-1}(\mathcal{O})$ is open in the product topology on $\mathcal{X}^N$. To do this, let $\mathbf{x} \in \mathcal{U}$; we will construct an open neighbourhood around $\mathbf{x}$ inside $\mathcal{U}$.

Let $\alpha := \Phi_{\mathfrak{P}}(\mathbf{x}) \in \mathcal{A}$. Then, $\alpha \in \mathcal{O}$. For any $n \in [1 \dots N]$, let $\mathcal{O}_n \subseteq \mathcal{A}(\mathcal{E}_n)$ be the open subset in the $\succeq_{\mathcal{E}_n}$-order topology obtained in Claim 1, and define $\mathcal{V}_n := \mathsf{K}_{\mathcal{E}_n}^{-1}(\mathcal{O}_n)$. By Lemma A1, each $\mathsf{K}_{\mathcal{E}_n}$ is a continuous function from $\mathcal{X}$ to $\mathcal{A}(\mathcal{E}_n)$. So $\mathcal{V}_n$ is an open subset of $\mathcal{X}$ for any $n \in [1 \dots N]$. Define $\mathcal{V} := \mathcal{V}_1 \times \ldots \times \mathcal{V}_N$; then $\mathcal{V}$ is an open subset of $\mathcal{X}^N$ in the product topology.

Claim 2: $\mathbf{x} \in \mathcal{V}$.

Proof. Any open set in an order topology is a union of order intervals, and any order interval is a union of indifference classes (because if an order interval contains some element γ, then it also contains all other elements which are indifferent to γ). Thus, any open set is a union of indiffererence classes.

Now fix $n \in [1 \dots N]$. By Claim 1, we have $\alpha_{\restriction \mathcal{E}_n} \in \mathcal{O}_n$. Moreover, by the definition of α and the construction of $\Phi_{\mathfrak{P}}$, we have $\alpha_{\restriction \mathcal{E}_n} = \Phi_{\mathfrak{P}}(\mathbf{x})_{\restriction \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa_{\mathcal{E}_n}^{x_n}$. By the remark in the previous paragraph, and since $\mathcal{O}_n$ is open in the $\succeq_{\mathcal{E}_n}$-order topology, we obtain $\kappa_{\mathcal{E}_n}^{x_n} \in \mathcal{O}_n$. Then, $x_n \in \mathcal{V}_n$. Since this holds for any $n \in [1 \dots N]$, we obtain $\mathbf{x} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_N = \mathcal{V}$. ◇ Claim 2

Claim 3: $\mathcal{V} \subseteq \mathcal{U}$.

Proof. Let $\mathbf{y} = (y_1, \ldots, y_N) \in \mathcal{V}$ and define $\beta = \Phi_{\mathfrak{P}}(\mathbf{y}) \in \mathcal{A}$. Fix $n \in [1 \ldots N]$. Then $y_n \in \mathcal{V}_n$, so by definition of $\mathcal{V}_n$, we have $\kappa_{\mathcal{E}_n}^{y_n} \in \mathcal{O}_n$. By the construction of $\Phi_{\mathfrak{P}}$, we have $\beta_{\restriction \mathcal{E}_n} = \Phi_{\mathfrak{P}}(\mathbf{y})_{\restriction \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa_{\mathcal{E}_n}^{y_n}$. Since $\mathcal{O}_n$ is open in the $\succeq_{\mathcal{E}_n}$-order topology and $\kappa_{\mathcal{E}_n}^{y_n} \in \mathcal{O}_n$, $\mathcal{O}_n$ must also contain any act that is indifferent to $\kappa_{\mathcal{E}_n}^{y_n}$. Thus, $\beta_{\restriction \mathcal{E}_n} \in \mathcal{O}_n$. This holds for any $n \in [1 \ldots N]$. Then, by Claim 1, $\beta \in \mathcal{O}$. Finally, $y \in \Phi_{\mathfrak{P}}^{-1}(\mathcal{O}) = \mathcal{U}$. ◇ Claim 3

Thus, $\mathcal{V}$ is an open neighbourhood around $\mathbf{x}$ (in the product topology), which is contained in $\mathcal{U}$. We can construct such a neighbourhood around any $\mathbf{x} \in \mathcal{U}$. Thus, $\mathcal{U}$ is open in the product topology. Hence $\Phi_{\mathfrak{P}}$ is continuous, as claimed. □

Consider any regular partition $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ of $\mathcal{S}$ with $N \geq 2$. Let $\Phi_{\mathfrak{P}}$ be the mapping from Lemma A2. We then define a preference order $\succeq_{\mathfrak{P}}$ on $\mathcal{X}^N$ in the following way: For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \succeq_{\mathcal{S}} \Phi_{\mathfrak{P}}(\mathbf{y})\Big). \tag{A1}$$

Proposition A3 *Suppose* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ *satisfies axioms* (Sep), (C), (TC), (PC), (Dom) *and* (CEq)*. Then, there exists a credence* μ *on* $\mathfrak{B}$ *with full support and a continuous function* $u : \mathcal{X} \longrightarrow \mathbb{R}$ *such that, for any* $\mathfrak{B}$*-partition* $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ *of* $\mathcal{S}$ *with* $N \geq 2$*, we have*

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(x_n) \geq \sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(y_n)\right), \tag{A2}$$

where $\succeq_{\mathfrak{P}}$ *is defined by formula* (A1)*. Moreover,* μ *is unique, and* u *is unique up to positive affine transformation. Finally,* u *is an ordinal utility function for* $\succeq_{\text{xp}}$*. If* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ *also satisfies axiom* (M)*, then* u *is* $\mathfrak{D}$*-measurable.*

Proof. Fix a $\mathfrak{B}$-partition $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ of $\mathcal{S}$ with $N \geq 2$ (such a partition exists because $\mathfrak{B}$ is nontrivial). Define $\succeq_{\mathfrak{P}}$ on $\mathcal{X}^N$ according to formula (A1).

Claim 1: $\succeq_{\mathfrak{P}}$ *is continuous with respect to the product topology on* $\mathcal{X}^N$*.*

Proof. Fix $\mathbf{y} \in \mathcal{X}^N$ and define $\beta := \Phi_{\mathfrak{P}}(\mathbf{y}) \in \mathcal{A}$. Let $\mathcal{O} = \{\alpha \in \mathcal{A},\, \alpha \succ_{\mathcal{S}} \beta\}$; this is an open set in the $\succeq_{\mathcal{S}}$-order topology on $\mathcal{A}$. Then, by Lemma A2, $\mathcal{U} := \Phi_{\mathfrak{P}}^{-1}(\mathcal{O})$ is an open subset of $\mathcal{X}^N$ in the product topology. Moreover, for any $\mathbf{x} \in \mathcal{X}^N$, we have

$$\Big(\mathbf{x} \succ_{\mathfrak{P}} \mathbf{y}\Big) \quad \underset{(*)}{\Longleftrightarrow} \quad \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \succ_{\mathcal{S}} \beta\Big) \quad \underset{(\dagger)}{\Longleftrightarrow} \quad \Big(\mathbf{x} \in \mathcal{U}\Big),$$

where $(*)$ is by formula (A1) and $(\dagger)$ is by the definition of $\mathcal{U}$. Thus, the strict upper contour set of $\succeq_{\mathfrak{P}}$ at $\mathbf{y}$ is equal to $\mathcal{U}$ and, therefore, an open set in the product topology on $\mathcal{X}^N$. A similar proof works for strict lower contour sets. ◇ Claim 1

Claim 2: $\succeq_{\mathfrak{P}}$ *satisfies* Cardinal Coordinate Independence*: For all* $n, m \in [1 \ldots N]$*, all* $\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w} \in \mathcal{X}^N$ *and all* $a, b, c, d \in \mathcal{X}$*, if* $a_n \mathbf{x}_{-n} \preceq_{\mathfrak{P}} b_n \mathbf{y}_{-n}$*,* $c_n \mathbf{x}_{-n} \succeq_{\mathfrak{P}} d_n \mathbf{y}_{-n}$ *and* $a_m \mathbf{v}_{-m} \succeq_{\mathfrak{P}} b_m \mathbf{w}_{-m}$*, then* $c_m \mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m \mathbf{w}_{-m}$*.*

Proof. Define $(a_{\mathcal{B}_n}\alpha), (b_{\mathcal{B}_n}\beta), (c_{\mathcal{B}_n}\alpha), (d_{\mathcal{B}_n}\beta) \in \mathcal{A}$ by $(a_{\mathcal{B}_n}\alpha) := \Phi_{\mathfrak{P}}(a_n\mathbf{x}_{-n})$, $(b_{\mathcal{B}_n}\beta) := \Phi_{\mathfrak{P}}(b_n\mathbf{y}_{-n})$, $(c_{\mathcal{B}_n}\alpha) := \Phi_{\mathfrak{P}}(c_n\mathbf{x}_{-n})$ and $(d_{\mathcal{B}_n}\beta) := \Phi_{\mathfrak{P}}(d_n\mathbf{y}_{-n})$. Then, by the definition (A1) of $\succeq_{\mathfrak{P}}$, we have $(a_{\mathcal{B}_n}\alpha) \preceq_{\mathcal{S}} (b_{\mathcal{B}_n}\beta)$ and $(c_{\mathcal{B}_n}\alpha) \succeq_{\mathcal{S}} (d_{\mathcal{B}_n}\beta)$. Moreover, by the definition of $\Phi_{\mathfrak{P}}$, we have $(a_{\mathcal{B}_n}\alpha)_{\restriction\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^a_{\mathcal{B}_n}$, $(b_{\mathcal{B}_n}\beta)_{\restriction\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^b_{\mathcal{B}_n}$, $(c_{\mathcal{B}_n}\alpha)_{\restriction\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^c_{\mathcal{B}_n}$ and $(d_{\mathcal{B}_n}\beta)_{\restriction\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^d_{\mathcal{B}_n}$. Meanwhile, $(a_{\mathcal{B}_n}\alpha)_{\restriction\mathcal{B}_l} \approx_{\mathcal{B}_l} (c_{\mathcal{B}_n}\alpha)_{\restriction\mathcal{B}_l}$ and $(b_{\mathcal{B}_n}\beta)_{\restriction\mathcal{B}_l} \approx_{\mathcal{B}_l} (d_{\mathcal{B}_n}\beta)_{\restriction\mathcal{B}_l}$ for all $l \in [1\dots N]$ with $l \neq n$. So if we set $\mathcal{Q} := \neg\mathcal{B}_n$, then by (Sep) we obtain $(a_{\mathcal{B}_n}\alpha)_{\restriction\mathcal{Q}} \approx_{\mathcal{Q}} (c_{\mathcal{B}_n}\alpha)_{\restriction\mathcal{Q}}$ and $(b_{\mathcal{B}_n}\beta)_{\restriction\mathcal{Q}} \approx_{\mathcal{Q}} (d_{\mathcal{B}_n}\beta)_{\restriction\mathcal{Q}}$. This shows that $(a \overset{\mathcal{B}_n}{\rightsquigarrow} b) \succeq (c \overset{\mathcal{B}_n}{\rightsquigarrow} d)$.

Meanwhile, define $(a_{\mathcal{B}_m}\gamma), (b_{\mathcal{B}_m}\delta), (c_{\mathcal{B}_m}\gamma), (d_{\mathcal{B}_m}\delta) \in \mathcal{A}$ by $(a_{\mathcal{B}_m}\gamma) := \Phi_{\mathfrak{P}}(a_m\mathbf{v}_{-m})$, $(b_{\mathcal{B}_m}\delta) := \Phi_{\mathfrak{P}}(b_m\mathbf{w}_{-m})$, $(c_{\mathcal{B}_m}\gamma) := \Phi_{\mathfrak{P}}(c_m\mathbf{v}_{-m})$ and $(d_{\mathcal{B}_m}\delta) := \Phi_{\mathfrak{P}}(d_m\mathbf{w}_{-m})$. Proceeding as above, we obtain $(a_{\mathcal{B}_m}\gamma) \succeq_{\mathcal{S}} (b_{\mathcal{B}_m}\delta)$ with $(a_{\mathcal{B}_m}\gamma)_{\restriction\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^a_{\mathcal{B}_m}$, $(b_{\mathcal{B}_m}\delta)_{\restriction\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^b_{\mathcal{B}_m}$, $(c_{\mathcal{B}_m}\gamma)_{\restriction\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^c_{\mathcal{B}_m}$ and $(d_{\mathcal{B}_m}\delta)_{\restriction\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^d_{\mathcal{B}_m}$. Moreover, set $\mathcal{Q}' := \neg\mathcal{B}_m$. Then we also have $(a_{\mathcal{B}_m}\gamma)_{\restriction\mathcal{Q}'} \approx_{\mathcal{Q}'} (c_{\mathcal{B}_m}\gamma)_{\restriction\mathcal{Q}'}$ and $(b_{\mathcal{B}_m}\delta)_{\restriction\mathcal{Q}'} \approx_{\mathcal{Q}'} (d_{\mathcal{B}_m}\delta)_{\restriction\mathcal{Q}'}$. Now, if it is *not* the case that $c_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m\mathbf{w}_{-m}$, then (A1) implies that it is also not the case that $(c_{\mathcal{B}_m}\gamma) \succeq_{\mathcal{S}} (d_{\mathcal{B}_m}\delta)$. Thus, $(c_{\mathcal{B}_m}\gamma) \prec_{\mathcal{S}} (d_{\mathcal{B}_m}\delta)$ (because $\succeq_{\mathcal{S}}$ is a complete order). But this means that $(a \overset{\mathcal{B}_m}{\rightsquigarrow} b) \prec (c \overset{\mathcal{B}_m}{\rightsquigarrow} d)$, which contradicts (TC). Thus, we must have $c_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m\mathbf{w}_{-m}$, as claimed. ◇ Claim 2

By Claims 1 and 2, and the connectedness of $\mathcal{X}$, the assumptions of Wakker's (1988) Theorem 6.2 are satisfied. So there exist a continuous function $u_{\mathfrak{P}} : \mathcal{X} \longrightarrow \mathbb{R}$ and a probability vector $(\mu_{\mathfrak{P}}(\mathcal{B}_1), \dots, \mu_{\mathfrak{P}}(\mathcal{B}_N)) \in \Delta([1\dots N])$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot u_{\mathfrak{P}}(x_n) \geq \sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot u_{\mathfrak{P}}(y_n)\right). \tag{A3}$$

Moreover, the probability vector is unique, and the function is unique up to positive affine transformation.

By the nontriviality of $\succeq_{\mathcal{S}}$ and axiom (Dom), there exist $l, o \in \mathcal{X}$ with $l \succ_{\text{xp}} o$. Then, still by (Dom), $\kappa^l_{\mathcal{B}} \succ_{\mathcal{B}} \kappa^o_{\mathcal{B}}$ for any $\mathcal{B} \in \mathfrak{B}$. Fix $n \in [1\dots N]$, and let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$ be such that $x_n = l$, $y_n = o$, and $x_m = y_m = o$ for any $m \in [1\dots N] \setminus \{n\}$. Then, $\Phi_{\mathfrak{P}}(\mathbf{x})_{\restriction\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^o_{\mathcal{B}_m} \approx_{\mathcal{B}_m} \Phi_{\mathfrak{P}}(\mathbf{y})_{\restriction\mathcal{B}_m}$ for any $m \in [1\dots N]\backslash\{n\}$. Let $\mathcal{Q} = \neg\mathcal{B}_n$. By iterative applications of (Sep), we have $\Phi_{\mathfrak{P}}(\mathbf{x})_{\restriction\mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{P}}(\mathbf{y})_{\restriction\mathcal{Q}}$. Moreover, we have $\Phi_{\mathfrak{P}}(\mathbf{x})_{\restriction\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^l_{\mathcal{B}_n} \succ_{\mathcal{B}_n} \kappa^o_{\mathcal{B}_n} \approx_{\mathcal{B}_n} \Phi_{\mathfrak{P}}(\mathbf{y})_{\restriction\mathcal{B}_n}$ Another application of (Sep) yields $\Phi_{\mathfrak{P}}(\mathbf{x}) \succ_{\mathcal{B}_n} \Phi_{\mathfrak{P}}(\mathbf{y})$ and, by formula (A1), $\mathbf{x} \succ_{\mathfrak{P}} \mathbf{y}$. Then, from (A2) and the definition of $\mathbf{x}$ and $\mathbf{y}$, we get

$$\mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot [u_{\mathfrak{P}}(l) - u_{\mathfrak{P}}(o)] \quad > \quad 0.$$

This inequality first shows that $\mu_{\mathfrak{P}}(\mathcal{B}_n) > 0$, and this holds for any $n \in [1\dots N]$. It also shows $u_{\mathfrak{P}}(l) > u_{\mathfrak{P}}(o)$. Thus, we obtain a unique function $u_{\mathfrak{P}}$ providing a representation as in formula (A3) and satisfying $u_{\mathfrak{P}}(l) = 1$ and $u_{\mathfrak{P}}(o) = 0$. From now on, we assume that the functions $u_{\mathfrak{P}}$ are normalized in this way.

Claim 3: *For any two $\mathfrak{B}$-partitions $\mathfrak{P}$ and $\mathfrak{Q}$ of $\mathcal{S}$, each with at least two cells, $u_{\mathfrak{P}} = u_{\mathfrak{Q}}$. Moreover, if $\mathcal{B} \in \mathfrak{B}$ is a cell in each of $\mathfrak{P}$ and $\mathfrak{Q}$, then $\mu_{\mathfrak{P}}(\mathcal{B}) = \mu_{\mathfrak{Q}}(\mathcal{B})$.*

Proof. Let $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ with $\mathcal{P}_n \in \mathfrak{B}$ for all $n \in [1 \ldots N]$, and let $\mathfrak{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_M)$ with $\mathcal{Q}_m \in \mathfrak{B}$ for all $m \in [1 \ldots M]$. Consider first the case where $\mathfrak{Q}$ *refines* $\mathfrak{P}$ —that is, for all $m \in [1 \ldots M]$, there is some $n \in [1 \ldots N]$ such that $\mathcal{Q}_m \subseteq \mathcal{P}_n$. For all $n \in [1 \ldots N]$, let $\mathcal{M}_n \subseteq [1 \ldots M]$ be the set of $m \in [1 \ldots M]$ such that $\mathcal{Q}_m \subseteq \mathcal{P}_n$. Then, for all $n \in [1 \ldots N]$, the subcollection $\{\mathcal{Q}_m,\ m \in M_n\}$ is a $\mathfrak{B}$-partition of $\mathcal{P}_n$. For all $n \in [1 \ldots N]$, we define

$$p_n \ := \ \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m). \tag{A4}$$

Then, the collection $(p_1, \ldots, p_N)$ is a probability vector on $[1 \ldots N]$. Moreover, for any $\mathbf{x} \in \mathcal{X}^N$, define $\mathbf{x}' \in \mathcal{X}^M$ by setting

$$x'_m \ = \ x_n, \qquad \text{for all } m \in \mathcal{M}_n \text{ and } n \in [1 \ldots N]. \tag{A5}$$

Note that by (Sep) we have the following indifference for any $\mathbf{x} \in \mathcal{X}^N$:

$$\Phi_{\mathfrak{P}}(\mathbf{x}) \ \approx_{\mathcal{S}} \ \Phi_{\mathfrak{Q}}(\mathbf{x}') \tag{A6}$$

Thus, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\begin{aligned}
\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \ &\underset{\text{(a)}}{\iff} \ \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \ \succeq_{\mathcal{S}} \ \Phi_{\mathfrak{P}}(\mathbf{y})\Big) \ \underset{\text{(b)}}{\iff} \ \Big(\Phi_{\mathfrak{Q}}(\mathbf{x}') \ \succeq_{\mathcal{S}} \ \Phi_{\mathfrak{Q}}(\mathbf{y}')\Big) \\
&\underset{\text{(a)}}{\iff} \ \Big(\mathbf{x}' \succeq_{\mathfrak{Q}} \mathbf{y}'\Big) \ \underset{\text{(c)}}{\iff} \ \left(\sum_{m=1}^{M} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(x'_m) \geq \sum_{m=1}^{M} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(y'_m)\right) \\
&\underset{\text{(d)}}{\iff} \ \left(\sum_{n=1}^{N} \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(x_n) \geq \sum_{n=1}^{N} \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(y_n)\right) \\
&\underset{\text{(e)}}{\iff} \ \left(\sum_{n=1}^{N} p_n \cdot u_{\mathfrak{Q}}(x_n) \geq \sum_{n=1}^{N} p_n \cdot u_{\mathfrak{Q}}(y_n)\right).
\end{aligned}$$

Here, both (a) are by equation (A1), (b) is by equation (A6), (c) is by equation (A3), (d) by equation (A5), and (e) is by equation (A4). Thus, $u_{\mathfrak{Q}}$ and $(p_1, \ldots, p_N)$ provide a representation of $\succeq_{\mathfrak{P}}$ as in equation (A3). By uniqueness, we obtain $u_{\mathfrak{P}} = u_{\mathfrak{Q}}$. Moreover, for all $n \in [1 \ldots N]$,

$$\mu_{\mathfrak{P}}(\mathcal{P}_n) \ = \ \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m). \tag{A7}$$

Now, if $\mathfrak{P}$ and $\mathfrak{Q}$ have a common cell $\mathcal{B} \in \mathfrak{B}$, then $\mathcal{B} = \mathcal{P}_n = \mathcal{Q}_m$ for some $n \in [1 \ldots N]$ and $m \in [1 \ldots M]$ such that $M_n = \{m\}$. Then, equation (A7) yields $\mu_{\mathfrak{P}}(\mathcal{B}) = \mu_{\mathfrak{Q}}(\mathcal{B})$.

Now consider the general case, where neither $\mathfrak{P}$ nor $\mathfrak{Q}$ refines the other. Let $\mathfrak{P} \otimes \mathfrak{Q} := \{\mathcal{P} \cap \mathcal{Q};\ \mathcal{P} \in \mathfrak{P} \text{ and } \mathcal{Q} \in \mathfrak{Q}\}$. Then $\mathfrak{P} \otimes \mathfrak{Q}$ is a $\mathfrak{B}$-partition which refines both $\mathfrak{P}$ and $\mathfrak{Q}$. Now apply to the previous argument to $\mathfrak{P}$ and $\mathfrak{P} \otimes \mathfrak{Q}$ on the one hand, and to $\mathfrak{Q}$ and $\mathfrak{P} \otimes \mathfrak{Q}$ on the other hand to conclude. ◇ Claim 3

Now, we define a set function $\mu : \mathfrak{B} \longrightarrow [0,1]$ by setting $\mu(\mathcal{S}) = 1$, $\mu(\varnothing) = 0$ and, for any $\mathcal{B} \in \mathfrak{B}$, $\mu(\mathcal{B}) = \mu_{\mathfrak{P}}(\mathcal{B})$ where $\mathfrak{P} = \{\mathcal{B}, \neg\mathcal{B}\}$. Note that, for any nonempty $\mathcal{B} \in \mathfrak{B}$, $\mu(\mathcal{B}) > 0$ since we have already proved that $\mu_{\mathfrak{P}}(\mathcal{B}) > 0$.

Claim 4: *μ is a credence on $\mathfrak{B}$ with full support.*

Proof. Consider a collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ of pairwise disjoint regular subsets in $\mathfrak{B}$, and let $\mathcal{B}$ be its join. Consider first the case where $\mathcal{B} = \mathcal{S}$, and set $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$. Then, $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{S}$. We have

$$\sum_{n=1}^{N} \mu(\mathcal{P}_n) \underset{(*)}{=} \sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{P}_n) \underset{(\dagger)}{=} 1. \tag{A8}$$

Here $(*)$ is by Claim 3, and $(\dagger)$ is because $\mu_{\mathfrak{P}}$ is a probability distribution. Now, if $\mathcal{B} \neq \mathcal{S}$, set $\mathcal{P}_{N+1} := \neg\mathcal{B}$. Consider $\mathfrak{Q} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N, \mathcal{P}_{N+1}\}$ and $\mathfrak{Q}' = \{\mathcal{B}, \mathcal{P}_{N+1}\}$, two $\mathfrak{B}$-partitions of $\mathcal{S}$. We have

$$\sum_{n=1}^{N} \mu(\mathcal{P}_n) \underset{(a)}{=} 1 - \mu_{\mathfrak{Q}}(\mathcal{P}_{N+1}) \underset{(b)}{=} 1 - \mu_{\mathfrak{Q}'}(\mathcal{P}_{N+1}) \underset{(c)}{=} \mu_{\mathfrak{Q}'}(\mathcal{B}) \underset{(d)}{=} \mu(\mathcal{B}). \tag{A9}$$

Here, both (a) and (c) are by suitable versions of equation (A8) while both (b) and (d) are by Claim 3. Thus, μ is a credence. Finally, μ is fully supported since it has positive values at any nonempty regular subset in $\mathfrak{B}$. ◇ Claim 4

Set $u := u_{\mathfrak{Q}}$ for some $\mathfrak{B}$-partition $\mathfrak{Q}$ of $\mathcal{S}$ with at least two cells. For any $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ of $\mathcal{S}$ with $N \geq 2$, Claim 3 yields $u = u_{\mathfrak{P}}$ and $\mu_{\mathfrak{P}}(\mathcal{P}_n) = \mu(\mathcal{P}_n)$ for all $n \in [1 \ldots N]$. This, together with equation (A3), completes the proof of formula (A2).

Claim 5: *u is an ordinal utility function for $\succeq_{\mathrm{xp}}$.*

Proof. Fix $x, y \in \mathcal{X}$. Since $\mathfrak{B}$ is nondegenerate, there exists a $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{S}$ with $N \geq 2$. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$ be defined by $x_n := x$ and $y_n := y$ for any $n \in [1 \ldots N]$. Then, we have

$$\begin{aligned}
\Big(x \succeq_{\mathrm{xp}} y\Big) &\underset{(a)}{\Longleftrightarrow} \Big(\kappa^x \succeq_{\mathcal{S}} \kappa^y\Big) \underset{(b)}{\Longleftrightarrow} \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \succeq_{\mathcal{S}} \Phi_{\mathfrak{P}}(\mathbf{y})\Big) \\
&\underset{(c)}{\Longleftrightarrow} \left(\sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(x) \;\geq\; \sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(y)\right) \\
&\Longleftrightarrow \Big(u(x) \geq u(y)\Big),
\end{aligned}$$

where (a) is by the definition of $\succeq_{\mathrm{xp}}$, (b) is because by inductive applications of (Sep) and because $\Phi_{\mathfrak{P}}(\mathbf{x})_{\restriction \mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^x_{\mathcal{B}_n}$ and $\Phi_{\mathfrak{P}}(\mathbf{y})_{\restriction \mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^y_{\mathcal{B}_n}$, for all $n \in [1 \ldots N]$. Meanwhile, (c) is by formula (A2). ◇ Claim 5

Claim 6: *If $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ satisfies axiom* (M)*, then u is $\mathfrak{D}$-measurable.*

Proof. Fix an open interval $\mathcal{O} \subseteq \mathbb{R}$. We must show that $u^{-1}(\mathcal{O}) \in \mathfrak{D}$.

First, suppose $\mathcal{O} = (q, \infty)$ for some $q \in \mathbb{R}$. If $u(x) < q$ for all $x \in \mathcal{X}$, then $u^{-1}(q,\infty) = \emptyset \in \mathfrak{D}$. On the other hand, if $u(x) > q$ for all $x \in \mathcal{X}$, then $u^{-1}(q,\infty) = \mathcal{X} \in \mathfrak{D}$. If neither of these cases holds, then there must exist $x, z \in \mathcal{X}$ with $u(x) \leq q \leq u(z)$. Since u is continuous and $\mathcal{X}$ is connected, the Intermediate Value Theorem yields some $y \in \mathcal{X}$ such that $u(y) = q$. But then $u^{-1}(q,\infty) = \{z \in \mathcal{X};\ \ z \succ_{\text{xp}} y\}$, because, by Claim 5 , u is an ordinal utility representation for $\succeq_{\text{xp}}$. Thus, $u^{-1}(q,\infty)$ is an open upper contour set of $\succeq_{\text{xp}}$, so $u^{-1}(q,\infty) \in \mathfrak{D}$, by axiom (M).

The same argument works if $\mathcal{O} = (-\infty, r)$ for some $r \in \mathbb{R}$. Finally, if $\mathcal{O} = (q, r)$, then $\mathcal{O} = (-\infty, r) \cap (q, \infty)$, so $u^{-1}(\mathcal{O}) = u^{-1}(-\infty, r) \cap u^{-1}(q,\infty)$ is an intersection of two elements of $\mathfrak{D}$, and thus, an element of $\mathfrak{D}$.

Finally, let $\mathcal{B}$ be an arbitrary basic subset of $\mathbb{R}$. Then $\mathcal{B} := (a_1, b_1) \sqcup (a_2, b_2) \sqcup \cdots \sqcup (a_N, b_N)$ for some $-\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N \leq \infty$. For all $n \in [1 \dots N]$, let $\mathcal{D}_n := u^{-1}(a_n, b_n)$; then $\mathcal{D}_n \in \mathfrak{D}$ by the previous paragraph, and $u^{-1}(\mathcal{B}) = \bigsqcup_{n=1}^{N} \mathcal{D}_n$. It remains to show that this union is an element of $\mathfrak{D}$.

Claim 6A: $\bigvee_{n=1}^{N} \mathcal{D}_n \subseteq \bigsqcup_{n=1}^{N} \mathcal{D}_n$.

Proof. (by contradiction) Suppose $x \in \left(\bigvee_{n=1}^{N} \mathcal{D}_n\right) \setminus \left(\bigsqcup_{n=1}^{N} \mathcal{D}_n\right)$. Now,

$$\bigvee_{n=1}^{N} \mathcal{D}_n \quad = \quad \text{int}\left[\text{clos}\left(\bigsqcup_{n=1}^{N} \mathcal{D}_n\right)\right] \quad = \quad \text{int}\left[\bigcup_{n=1}^{N} \text{clos}(\mathcal{D}_n)\right].$$

Thus, $x \in \text{int}\left[\bigcup_{n=1}^{N} \text{clos}(\mathcal{D}_n)\right]$, but $x \notin \mathcal{D}_n = \text{int}[\text{clos}(\mathcal{D}_n)]$ for any $n \in [1 \dots N]$. Thus, if $\mathcal{U}$ is any open neighbourhood around x, then $\mathcal{U}$ overlaps $\bigcup_{n=1}^{N} \text{clos}(\mathcal{D}_n)$ but $\mathcal{U} \not\subseteq \text{clos}(\mathcal{D}_n)$ for any $n \in [1 \dots N]$; hence there must be at least two distinct $n, m \in [1 \dots N]$ such that $\mathcal{U} \cap \text{clos}(\mathcal{D}_n) \neq \emptyset$ and $\mathcal{U} \cap \text{clos}(\mathcal{D}_m) \neq \emptyset$. Define

$$\epsilon \quad := \quad \frac{1}{4} \min_{n \in [1 \dots N)} (a_{n+1} - b_n).$$

Then $\epsilon > 0$ because $b_n < a_{n+1}$ for all $n \in [1 \dots N]$, by hypothesis. Let $r := u(x)$, and let $\mathcal{V} := (r - \epsilon, r + \epsilon)$. Then $\mathcal{V}$ is an open neighbourhood around r. Let $\mathcal{U} := u^{-1}(\mathcal{V})$; then $\mathcal{U}$ is an open neighbourhood around x (because u is continuous), so by the previous paragraph there exist distinct $n < m \in [1 \dots N]$ such that $\mathcal{U} \cap \text{clos}(\mathcal{D}_n) \neq \emptyset$ and $\mathcal{U} \cap \text{clos}(\mathcal{D}_m) \neq \emptyset$. Now, $u(\mathcal{U}) = \mathcal{V}$ (by definition of $\mathcal{U}$), while $u[\text{clos}(\mathcal{D}_n)] \subseteq \text{clos}(a_n, b_n) = [a_n, b_n]$ and $u[\text{clos}(\mathcal{D}_m)] \subseteq \text{clos}(a_m, b_m) = [a_m, b_m]$ (because u is continuous). Thus, we must have $\mathcal{V} \cap [a_n, b_n] \neq \emptyset$ and $\mathcal{V} \cap [a_m, b_m] \neq \emptyset$. But this is impossible, because $\mathcal{V}$ is an interval of length $2\epsilon \leq (a_m - b_n)/2$, by construction.

To avoid the contradiction, we must have $x \in \mathcal{D}_n$ for some $n \in [1 \dots N]$. This argument applies to all $x \in \bigvee_{n=1}^{N} \mathcal{D}_n$. Thus, $\bigvee_{n=1}^{N} \mathcal{D}_n \subseteq \bigsqcup_{n=1}^{N} \mathcal{D}_n$, as claimed. $\triangledown$ Claim 6A

Clearly, $\bigsqcup_{n=1}^{N} \mathcal{D}_n \subseteq \bigvee_{n=1}^{N} \mathcal{D}_n$. Together with Claim 6A, this implies that $\bigsqcup_{n=1}^{N} \mathcal{D}_n = \bigvee_{n=1}^{N} \mathcal{D}_n$. Thus, it is an element of $\mathfrak{D}$, as desired. $\diamondsuit$ Claim 6

This completes the proof. □

Our SEU representation requires one more technical preliminary.

Proposition A4 (Pivato and Vergopoulos, 2016, Theorem 3.3) *Let $\mathcal{S}$ be any topological space, let $\mathfrak{B}$ be any Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$, and let μ be a credence with full support on $\mathfrak{B}$. There exists a unique, strictly monotonic conditional expectation system $\mathbf{E}$ that is compatible with μ.*

Lemma A5 *Suppose $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies axiom* (Sep)*. For any $\mathcal{B} \in \mathfrak{B}$, consider a $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{B}_1, \dots, \mathcal{B}_N\}$ of $\mathcal{B}$. For any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, if $\alpha_{\restriction \mathcal{B}_n} \succeq_{\mathcal{B}_n} \beta_{\restriction \mathcal{B}_n}$ for any $n \in [1 \dots N]$, then $\alpha \succeq_{\mathcal{B}} \beta$.*

Proof. We proceed by induction. Consider first a subset $\mathcal{B} \in \mathfrak{B}$ and a two-cell partition $\mathfrak{P} = \{\mathcal{B}_1, \mathcal{B}_2\}$ of $\mathcal{B}$. Let $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ be such that $\alpha_{\restriction \mathcal{B}_1} \succeq_{\mathcal{B}_1} \beta_{\restriction \mathcal{B}_1}$ and $\alpha_{\restriction \mathcal{B}_2} \succeq_{\mathcal{B}_2} \beta_{\restriction \mathcal{B}_2}$. By (Rch), there exists $\gamma \in \mathcal{A}(\mathcal{B})$ such that $\gamma_{\restriction \mathcal{B}_1} \approx_{\mathcal{B}_1} \alpha_{\restriction \mathcal{B}_1}$ and $\gamma_{\restriction \mathcal{B}_2} \approx_{\mathcal{B}_2} \beta_{\restriction \mathcal{B}_2}$. Then, we have $\alpha_{\restriction \mathcal{B}_1} \approx_{\mathcal{B}_1} \gamma_{\restriction \mathcal{B}_1}$ and $\alpha_{\restriction \mathcal{B}_2} \succeq_{\mathcal{B}_2} \gamma_{\restriction \mathcal{B}_2}$. By (Sep), we obtain $\alpha \succeq_{\mathcal{B}} \gamma$. Similarly, we have $\gamma_{\restriction \mathcal{B}_1} \succeq_{\mathcal{B}_1} \beta_{\restriction \mathcal{B}_1}$ and $\gamma_{\restriction \mathcal{B}_2} \approx_{\mathcal{B}_2} \beta_{\restriction \mathcal{B}_2}$. Still by (Sep), we obtain $\gamma \succeq_{\mathcal{B}} \beta$. Since $\alpha \succeq_{\mathcal{B}} \gamma$ and $\gamma \succeq_{\mathcal{B}} \beta$, we finally obtain $\alpha \succeq_{\mathcal{B}} \beta$ as desired.

Consider now a subset $\mathcal{B} \in \mathfrak{B}$ and an N-cell partition $\mathfrak{P} = \{\mathcal{B}_1, \dots, \mathcal{B}_N\}$ of $\mathcal{B}$ with $N \geq 2$. Let $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ be such that $\alpha_{\restriction \mathcal{B}_n} \succeq_{\mathcal{B}_n} \beta_{\restriction \mathcal{B}_n}$ for any $n \in [1 \dots N]$. Let $\mathcal{Q} = \mathcal{B}_1 \vee \dots \vee \mathcal{B}_{N_1}$. By induction, we have $\alpha_{\restriction \mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\restriction \mathcal{Q}}$. But since $\{\mathcal{Q}, \mathcal{B}_N\}$ is a two-cell partition of $\mathcal{B}$, and since we have $\alpha_{\restriction \mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\restriction \mathcal{Q}}$ and $\alpha_{\restriction \mathcal{B}_N} \succeq_{\mathcal{B}_N} \beta_{\restriction \mathcal{B}_N}$, the previous paragraph yields $\alpha \succeq_{\mathcal{B}} \beta$ as desired. □

Proposition A6 *Suppose $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies axioms* (Sep), (C), (M), (TC), (PC), (Dom) *and* (CEq)*. Let μ be the credence and u be the utility function from Proposition A3. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the unique μ-compatible conditional expectation system from Proposition A4. Then, for any $\mathcal{B} \in \mathfrak{B}$, $\alpha \in \mathcal{A}(\mathcal{B})$ and $x \in \mathcal{X}$ such that $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$, we have*

$$\mathbb{E}_{\mathcal{B}}\left[u \circ \alpha\right] \quad = \quad u(x). \tag{A10}$$

Proof. For any $\mathcal{B} \in \mathfrak{B}$, and any $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, we define $\widetilde{\mathbb{E}}_{\mathcal{B}}[g] := \mu[\mathcal{B}]\,\mathbb{E}_{\mathcal{B}}[g]$. Recall that $\mathbb{E}_{\mathcal{B}}[\mathbf{1}] = 1$. Thus, $\widetilde{\mathbb{E}}_{\mathcal{B}}[\mathbf{1}] = \mu[\mathcal{B}]$. Thus, for any $r \in \mathbb{R}$, the linearity of $\mathbb{E}_{\mathcal{B}}$ implies

$$\widetilde{\mathbb{E}}_{\mathcal{B}}[r\mathbf{1}] \quad = \quad r\,\mu[\mathcal{B}]. \tag{A11}$$

Let $\mathcal{B} \in \mathfrak{B}$ and $\alpha \in \mathcal{A}(\mathcal{B})$. Consider first the case where $u \circ \alpha$ is constant over $\mathcal{B}$. Then there exists $y \in \mathcal{X}$ such that $u \circ \alpha(s) = u(y)$ for any $s \in \mathcal{B}$. Then $\mathbb{E}_{\mathcal{B}}\,[u \circ \alpha] = u(y)$ by the linearity of $\mathbb{E}_{\mathcal{B}}$. On the other hand, by Proposition A3, u is an ordinal utility function for $\succeq_{\text{xp}}$. Therefore, $\alpha(s) \approx_{\text{xp}} y$ for any $s \in \mathcal{B}$. By (Dom), we obtain $\alpha \approx_{\mathcal{B}} \kappa^y_{\mathcal{B}}$ and thus, $\kappa^x_{\mathcal{B}} \approx_{\mathcal{B}} \kappa^y_{\mathcal{B}}$ by transitivity. Still by (Dom) we have $x \approx_{\text{xp}} y$. Thus, $u(x) = u(y)$, because u is an ordinal utility function for $\succeq_{\text{xp}}$. This shows $\mathbb{E}_{\mathcal{B}}\,[u \circ \alpha] = u(x)$, as desired.

Consider now the case where $u \circ \alpha$ takes at least two different values over $\mathcal{B}$. Let $\beta \in \mathcal{A}$ be such that $\beta_{\restriction \mathcal{B}} = \alpha$. Let $\mathcal{K} := \text{clos}[\beta(\mathcal{S})]$ (i.e. the closure of the image set $\beta(\mathcal{S})$ in $\mathcal{X}$); then $\mathcal{K}$ is compact, because $\beta \in \mathcal{C}_{\text{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}; \mathfrak{D})$. Recall that $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous, by Proposition A3. Thus, u is bounded when restricted to $\mathcal{K}$. If we define $\mathcal{U} := u \circ \beta(\mathcal{S})$, then $\mathcal{U}$ is a bounded subset of $\mathbb{R}$ —say, $\mathcal{U} \subseteq [-M, M]$ for some $M \in \mathbb{N}$.

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be large enough that $\frac{1}{N} < \epsilon$. For all $n \in [-MN \ldots MN)$, let $\mathcal{C}_n := (u \circ \beta)^{-1}[\frac{n}{N}, \frac{n+1}{N}]$. Recall that $u : \mathcal{X} \longrightarrow \mathbb{R}$ is $\mathfrak{D}$-measurable by Proposition A3, while β is $(\mathfrak{B}, \mathfrak{D})$-comeasurable, by the definition of $\mathcal{A}$. Thus, $u \circ \beta$ is $\mathfrak{B}$-comeasurable, by Lemma 4.4 of Pivato and Vergopoulos (2016). Thus if $\mathcal{B}_n := \text{int}(\mathcal{C}_n) \cap \mathcal{B}$, then $\mathcal{B}_n$ is a (possibly empty) element of $\mathfrak{B}$. Let $\mathcal{P}_{-MN} = \mathcal{B}_{-MN}$ and, for any $m \in (-MN \ldots MN)$, define $\mathcal{P}_m := \mathcal{B}_m \cap (\neg \mathcal{B}_{m-1})$. Then $\mathcal{P}_{-MN}, \ldots, \mathcal{P}_{MN-1}$ are disjoint (possibly empty) elements of $\mathfrak{B}$. Let $\mathcal{N} := \{n \in [-NM \ldots NM);\ \mathcal{P}_n \neq \emptyset\}$. Finally, define $\mathfrak{P} := \{\mathcal{P}_n\}_{n \in \mathcal{N}}$; then $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{B}$. Since $u \circ \alpha$ takes at least two different values over $\mathcal{B}$, we can take N to be large enough to make sure that $\mathfrak{P}$ has at least two cells.

Claim 1: *For any $n \in \mathcal{N}$, there exist values $x_n, y_n \in \mathcal{X}$ such that*

$$\frac{n}{N} \quad \leq \quad u(x_n) \quad \leq \quad u \circ \alpha(p) \quad \leq \quad u(y_n) \quad \leq \quad \frac{n+1}{N}, \tag{A12}$$

for all $p \in \mathcal{P}_n$.

Proof. Recall that $\mathcal{K} := \text{clos}[\beta(\mathcal{S})]$ is a compact subset of $\mathcal{X}$. Thus, its image $u(\mathcal{K})$ is a compact subset of $\mathbb{R}$, because u is continuous. Thus, the set $\mathcal{W}_n := u(\mathcal{K}) \cap [\frac{n}{N}, \frac{n+1}{N}]$ is compact. Thus, $\underline{w}_n := \min(\mathcal{W}_n)$ and $\overline{w}_n := \max(\mathcal{W}_n)$ are well-defined. Let $x_n \in u^{-1}\{\underline{w}_n\}$ and let $y_n \in u^{-1}\{\overline{w}_n\}$. Thus, $u(x_n) = \underline{w}_n \geq \frac{n}{N}$, while $u(y_n) = \overline{w}_n \leq \frac{n+1}{N}$,

For any $p \in \mathcal{P}_n$, we have $\alpha(p) = \beta(p)$ and $\beta(p) \in \mathcal{K}$, and also $u \circ \alpha(p) \in [\frac{n}{N}, \frac{n+1}{N}]$ by definition of $\mathcal{P}_n$; thus, $u \circ \alpha(p) \in \mathcal{W}_n$. Thus, $\underline{w}_n \leq u \circ \alpha(p) \leq \overline{w}_n$, i.e. $u(x_n) \leq u \circ \alpha(p) \leq u(y_n)$, as claimed. ◇ Claim 1

Now define $\underline{u}, \overline{u} \in \mathbb{R}$ in the following way:

$$\underline{u} \quad = \quad \sum_{n \in \mathcal{N}} \mu(\mathcal{P}_n) \cdot u(x_n) \quad \text{and} \quad \overline{u} = \sum_{n \in \mathcal{N}} \mu(\mathcal{P}_n) \cdot u(y_n).$$

Claim 2: $\underline{u} \quad \leq \quad \widetilde{\mathbb{E}}_{\mathcal{B}}\,[u \circ \alpha] \quad \leq \quad \overline{u}$

Proof. Fix $n \in \mathcal{N}$. For all $p \in \mathcal{P}_n$, formula (A12) says $u(x_n) \leq u \circ \alpha(p) \leq u(y_n)$, and thus,

$$\begin{aligned} \mu[\mathcal{P}_n] \cdot u(x_n) &\underset{(*)}{=} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u(x_n)\,\mathbf{1}] &&\underset{(\dagger)}{\leq} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u \circ \alpha] \\ &\underset{(\dagger)}{\leq} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u(y_n)\,\mathbf{1}] &&\underset{(*)}{=} \mu[\mathcal{P}_n] \cdot u(y_n). \end{aligned} \tag{A13}$$

Here, both $(*)$ are by equation (A11), and both $(\dagger)$ are by inequality (A12) and the weak monotonicity property of the conditional expectation operator $\mathbb{E}_{\mathcal{B}}$. Summing the versions of inequality (A13) obtained for every $n \in \mathcal{N}$, we obtain

$$\underline{u} \quad = \quad \sum_{n \in \mathcal{N}} \mu(\mathcal{P}_n) \cdot u(x_n) \quad \leq \quad \sum_{n \in \mathcal{N}} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u \circ \alpha] \quad \leq \quad \sum_{n \in \mathcal{N}} \mu(\mathcal{P}_n) \cdot u(y_n) \quad = \quad \overline{u}.$$

The result then follows by noting that the middle term in this inequality is equal to $\widetilde{\mathbb{E}}_{\mathcal{B}}[u \circ \alpha]$ thanks to equation (16) ◇ Claim 2

Claim 3: $\underline{u} \leq \mu(\mathcal{B}) \cdot u(x) \leq \overline{u}$.

Proof. Fix $o \in \mathcal{X}$. Define $\mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathcal{X}^{\mathcal{N}}$ in the following way:

$$\text{For any } n \in \mathcal{N}, \quad a'_n \quad := \quad x_n, \quad b'_n \quad := \quad x \quad \text{and} \quad c'_n \quad := \quad y_n.$$

Consider the $\mathfrak{B}$-partition $\mathfrak{Q} = \{\neg\mathcal{B}\} \cup \{\mathcal{P}_n,\, n \in \mathcal{N}\}$ made of $M := 1 + |\mathcal{N}|$ cells (so $M \geq 2$). Define $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^M$ by setting $\mathbf{a} := (\mathbf{a}', o)$, $\mathbf{b} := (\mathbf{b}', o)$ and $\mathbf{c} := (\mathbf{c}', o)$. Finally, let $\Phi_{\mathfrak{Q}}$ be the mapping constructed in Lemma A2.

Now, u is an ordinal utility function for $\succeq_{\text{xp}}$, by Proposition A3. Thus, for all $n \in \mathcal{N}$ and $p \in \mathcal{P}_n$, formula (A12) implies that $x_n \preceq_{\text{xp}} \alpha_{\restriction \mathcal{P}_n}(p) \preceq_{\text{xp}} y_n$. Thus, axiom (Dom) implies that

$$\kappa^{x_n}_{\mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \alpha_{\restriction \mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \kappa^{y_n}_{\mathcal{P}_n}. \tag{A14}$$

Given the defining properties of the mapping $\Phi_{\mathfrak{Q}}$, formula (A14) then implies that

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\restriction \mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \alpha_{\restriction \mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\restriction \mathcal{P}_n}. \tag{A15}$$

Since formula (A15) holds for every $n \in \mathcal{N}$, Lemma A5 further yields

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\restriction \mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \alpha \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\restriction \mathcal{B}}. \tag{A16}$$

Meanwhile, we have $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\restriction \mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa^x_{\mathcal{P}_n}$ for every $n \in \mathcal{N}$. By iterative applications of (Sep), we obtain $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\restriction \mathcal{B}} \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$. But by assumption $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$. Thus, $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\restriction \mathcal{B}} \approx_{\mathcal{B}} \alpha$, by transitivity. Formula (A16) then gives

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\restriction \mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{b})_{\restriction \mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\restriction \mathcal{B}}. \tag{A17}$$

Moreover, by construction, we have $\Phi_{\mathfrak{Q}}(\mathbf{a})_{\restriction \mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{Q}}(\mathbf{b})_{\restriction \mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{Q}}(\mathbf{c})_{\restriction \mathcal{Q}} \approx_{\mathcal{Q}} \kappa^o_{\mathcal{Q}}$ where $\mathcal{Q} = \neg\mathcal{B}$. Given this fact and formula (A17), axiom (Sep) implies

$$\Phi_{\mathfrak{Q}}(\mathbf{a}) \quad \preceq_{\mathcal{S}} \quad \Phi_{\mathfrak{Q}}(\mathbf{b}) \quad \preceq_{\mathcal{S}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c}). \tag{A18}$$

By the definition of $\succeq_{\mathfrak{O}}$ in formula (A1) and its representation obtained in Proposition A3, formula (A18) implies

$$\begin{aligned}\sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(a_n) \quad + \quad \mu(\mathcal{Q})\cdot u(o) \quad &\leq \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(b_n) \quad + \quad \mu(\mathcal{Q})\cdot u(o)\\ &\leq \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(c_n) \quad + \quad \mu(\mathcal{Q})\cdot u(o),\end{aligned}$$

which, given the definition of **a**, **b** and **c**, reduces to the following formula

$$\underline{u} \quad = \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(x_n) \quad \leq \quad \mu(\mathcal{B})\cdot u(x) \quad \leq \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(x_n) \quad = \quad \overline{u}.$$

This completes the proof of the claim. ◇ Claim 3

Finally, we obtain

$$\begin{aligned}|\widetilde{\mathbb{E}}_{\mathcal{B}}\left[u\circ\alpha\right]-\mu(\mathcal{B})\cdot u(x)| \quad &\underset{\text{(a)}}{\leq} \quad |\overline{u}-\underline{u}| \quad \underset{\text{(b)}}{\leq} \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot|u(y_n)-u(x_n)|\\ &\underset{\text{(c)}}{\leq} \quad \epsilon\cdot\sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n) \quad \underset{\text{(d)}}{\leq} \quad \epsilon\cdot\mu(\mathcal{B}).\end{aligned}$$

Here, (a) is by Claims 2 and 3, (b) is by the definition of $\underline{u}$ and $\overline{u}$, (c) is inequality (A12), because $1/N<\epsilon$ by definition, and (d) is because μ is a credence on $\mathfrak{B}$ and $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{B}$. This argument works for all $\epsilon>0$. Letting $\epsilon\to 0$, we conclude that $\widetilde{\mathbb{E}}_{\mathcal{B}}[u\circ\alpha]=\mu(\mathcal{B})\cdot u(x)$. Last, since μ‘ has full support, we obtain $\mathbb{E}_{\mathcal{B}}[u\circ\alpha]=u(x)$. □

Finally, we come to the proof of the main result.

Proof of Theorem 11.

SEU representation. Let $u:\mathcal{X}\longrightarrow\mathbb{R}$ be the normalized *ex post* utility function and let μ be the credence with full support from Proposition A3. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the unique μ-compatible conditional expectation system from Proposition A4.

For any $\mathcal{B}\in\mathfrak{B}$ and $\alpha,\beta\in\mathcal{A}(\mathcal{B})$, axiom (CEq) yields $x,y\in\mathcal{X}$ such that $\alpha\approx_{\mathcal{B}}\kappa^x_{\mathcal{B}}$ and $\beta\approx_{\mathcal{B}}\kappa^y_{\mathcal{B}}$. Then,

$$\begin{aligned}\Big(\alpha\succeq_{\mathcal{B}}\beta\Big) \quad &\iff \quad \Big(\kappa^x_{\mathcal{B}}\succeq_{\mathcal{B}}\kappa^y_{\mathcal{B}}\Big) \quad \underset{(*)}{\iff} \quad \Big(x\succeq_{\text{xp}}y\Big)\\ &\underset{(\dagger)}{\iff} \quad \Big(u(x)\geq u(y)\Big) \quad \underset{(\diamond)}{\iff} \quad \Big(\mathbb{E}_{\mathcal{B}}[u\circ\alpha]\geq\mathbb{E}_{\mathcal{B}}[u\circ\beta].\Big)\end{aligned}$$

Here, $(*)$ is by axiom (Dom), $(\dagger)$ is because u is an ordinal utility function for $\succeq_{\text{xp}}$ by Proposition A3, and $(\diamond)$ is by Proposition A6. This equivalence establishes the SEU representation. It remains to show that the representation is unique and demonstrate the necessity of the axioms.

Uniqueness. Let $u, u' : \mathcal{X} \longrightarrow \mathbb{R}$ be two continuous and $\mathfrak{D}$-measurable functions, and let $\mathbf{E} := \{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{R}\in\mathfrak{B}}$ and $\mathbf{E}' := \{\mathbb{E}'_{\mathcal{B}}\}_{\mathcal{R}\in\mathfrak{B}}$ be two conditional expectation systems. Let μ and μ' be two credences on $\mathfrak{B}$ with which $\mathbf{E}$ and $\mathbf{E}'$ are respectively compatible. Suppose that (u, μ) and (u', μ') are both SEU representations for the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$. We must show that $\mu = \mu'$ and u is a positive affine transformation of u'.

Let $\mathfrak{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ be a $\mathfrak{B}$-partition of $\mathcal{S}$, with $N \geq 2$ (such a partition exists because $\mathfrak{B}$ is nontrivial). For any $\mathbf{x} \in \mathcal{X}^N$, Lemma A2 yields an act $\alpha^{\mathbf{x}} \in \mathcal{A}$ such that $\alpha^{\mathbf{x}}_{\restriction \mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^{x_n}_{\mathcal{E}_n}$ for all $n \in [1 \ldots N]$. Then we have

$$\begin{aligned}\mathbb{E}_{\mathcal{S}}[u \circ \alpha^{\mathbf{x}}] &\underset{(*)}{=} \sum_{n=1}^{N} \mu(\mathcal{E}_n)\ \mathbb{E}_{\mathcal{E}_n}[u \circ \alpha^{\mathbf{x}}] \\ &\underset{(\dagger)}{=} \sum_{n=1}^{N} \mu(\mathcal{E}_n)\, \mathbb{E}_{\mathcal{E}_n}[u \circ \kappa^{x_n}_{\mathcal{E}_n}] \quad = \quad \sum_{n=1}^{N} \mu(\mathcal{E}_n)\ u(x_n). \end{aligned} \tag{A19}$$

Here, $(*)$ is by equation (16), while $(\dagger)$ is by the SEU representation and the fact that $\alpha^{\mathbf{x}}_{\restriction \mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^{x_n}_{\mathcal{B}_n}$ for all $n \in [1 \ldots N]$. Define $\succeq_{\mathfrak{E}}$ as in equation (A1). By equation (A19), we have for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$

$$\Big(\mathbf{x} \succeq_{\mathfrak{E}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu(\mathcal{E}_n)\ u(x_n) \geq \sum_{n=1}^{N} \mu(\mathcal{E}_n)\ u(y_n)\right).$$

The SEU representation (u', μ') provides similarly the following representation: for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$

$$\Big(\mathbf{x} \succeq_{\mathfrak{E}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu'(\mathcal{E}_n)\ u'(x_n) \geq \sum_{n=1}^{N} \mu'(\mathcal{E}_n)\ u'(y_n)\right).$$

Now, by the uniqueness part of Proposition A3, we obtain that μ and μ' are equal to each other, and that u and u' are positive affine transformation of each other.

Necessity of the axioms. Assume that $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies condition (Rch) and has an SEU representation in the sense of Theorem 11 with respect to a continuous and $\mathfrak{D}$-measurable utility function u and a credence μ with full support. Let $\mathbf{E} := \{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the unique, strictly monotonic conditional expectation system defined by μ via Proposition A4. Axiom (Dom) is a simple consequence of the strict monotonicity of each expectation functional in $\mathbf{E}$. Axiom (Sep) follows from the fact that $\mathbf{E}$ satisfies Equation (16). The proofs of the other axioms are somewhat more involved.

Axiom (TC): Fix two disjoint subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, and let $\mathcal{Q}_1 = \neg\mathcal{B}_1$ and $\mathcal{Q}_2 = \neg\mathcal{B}_2$. Fix $x, y, v, w \in \mathcal{X}$. By contradiction, assume that $(x \overset{\mathcal{B}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}_1}{\rightsquigarrow} w)$ but $(x \overset{\mathcal{B}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}_2}{\rightsquigarrow} w)$. Then, since $(x \overset{\mathcal{B}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}_1}{\rightsquigarrow} w)$, there exist $\alpha, \beta \in \mathcal{A}(\mathcal{Q}_1)$, an (x, α)-bet $(x_{\mathcal{B}_1}\alpha) \in \mathcal{A}$, a (y, β)-bet $(y_{\mathcal{B}_1}\beta) \in \mathcal{A}$, a (v, α)-bet $(v_{\mathcal{B}_1}\alpha) \in \mathcal{A}$ and a (w, β)-bet $(w_{\mathcal{B}_1}\beta) \in \mathcal{A}$ such that

$(x_{\mathcal{B}_1}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}_1}\beta)$ while $(v_{\mathcal{B}_1}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}_1}\beta)$. We now show that $u(x)-u(y) \leq u(v)-u(w)$. Indeed, first we have

$$\begin{aligned}
&\mu(\mathcal{B}_1)\Big(u(x)-u(y)\Big)\\
&\underset{(*)}{=} \mathbb{E}_{\mathcal{S}}[u\circ(x_{\mathcal{B}_1}\alpha)] - \mu(\mathcal{Q}_1)\mathbb{E}_{\mathcal{Q}_1}[u\circ(x_{\mathcal{B}_1}\alpha)] - \mathbb{E}_{\mathcal{S}}[u\circ(y_{\mathcal{B}_1}\beta)] + \mu(\mathcal{Q}_1)\mathbb{E}_{\mathcal{Q}_1}[u\circ(y_{\mathcal{B}_1}\beta)]\\
&\underset{(\dagger)}{\leq} \mu(\mathcal{Q}_1)\Big(\mathbb{E}_{\mathcal{Q}_1}[u\circ(y_{\mathcal{B}_1}\beta)] - \mathbb{E}_{\mathcal{Q}_1}[u\circ(x_{\mathcal{B}_1}\alpha)]\Big). \qquad \text{(A20)}
\end{aligned}$$

Here, $(*)$ is by formula (B1) in the definition of bets, equation (16) and the SEU representation. Meanwhile, $(\dagger)$ is because $\mathbb{E}_{\mathcal{S}}[u\circ(x_{\mathcal{B}_1}\alpha)] \leq \mathbb{E}_{\mathcal{S}}[u\circ(y_{\mathcal{B}_1}\beta)]$ because $(x_{\mathcal{B}_1}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}_1}\beta)$. Proceeding similarly for $(v_{\mathcal{B}_1}\alpha)$ and $(w_{\mathcal{B}_1}\beta)$, we obtain

$$\mu(\mathcal{B}_1)\Big(u(v)-u(w)\Big) \quad \geq \quad \mu(\mathcal{Q}_1)\Big(\mathbb{E}_{\mathcal{Q}_1}[u\circ(w_{\mathcal{B}_1}\beta)] - \mathbb{E}_{\mathcal{Q}_1}[u\circ(v_{\mathcal{B}_1}\alpha)]\Big). \qquad \text{(A21)}$$

Meanwhile, by formula (B2) in the definition of bets and the SEU representation, we have $\mathbb{E}_{\mathcal{Q}_1}[u\circ(x_{\mathcal{B}_1}\alpha)] = \mathbb{E}_{\mathcal{Q}_1}[u\circ(v_{\mathcal{B}_1}\alpha)]$ and $\mathbb{E}_{\mathcal{Q}_1}[u\circ(y_{\mathcal{B}_1}\beta)] = \mathbb{E}_{\mathcal{Q}_1}[u\circ(w_{\mathcal{B}_1}\beta)]$. Combining inequalities (A20) and (A21) and using the fact that μ has full support, we obtain

$$u(x)-u(y) \quad \leq \quad u(v)-u(w). \qquad \text{(A22)}$$

Now, since $(x \overset{\mathcal{B}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}_2}{\rightsquigarrow} w)$, there exist $\gamma,\delta \in \mathcal{A}(\mathcal{Q}_2)$, an (x,γ)-bet $(x_{\mathcal{B}_2}\gamma) \in \mathcal{A}$, a (y,δ)-bet $(y_{\mathcal{B}_2}\delta) \in \mathcal{A}$, a (v,γ)-bet $(v_{\mathcal{B}_2}\gamma) \in \mathcal{A}$ and a (w,δ)-bet $(w_{\mathcal{B}_2}\delta) \in \mathcal{A}$ such that $(x_{\mathcal{B}_2}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{B}_2}\delta)$ while $(v_{\mathcal{B}_2}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{B}_2}\delta)$. Thus,

$$\begin{aligned}
\mu(\mathcal{B}_2)\Big(u(x)-u(y)\Big) &\underset{(*)}{\geq} \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u\circ(y_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u\circ(x_{\mathcal{B}_2}\gamma)]\Big)\\
&\underset{(\dagger)}{=} \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u\circ(w_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u\circ(v_{\mathcal{B}_2}\gamma)]\Big), \qquad \text{(A23)}
\end{aligned}$$

where $(*)$ is obtained like inequality (A20), while $(\dagger)$ is by formula (B2) in the definition of bets and the SEU representation. Combining inequalities (A22) and (A23), we get

$$\mu(\mathcal{B}_2)\Big(u(v)-u(w)\Big) \quad \geq \quad \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u\circ(w_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u\circ(v_{\mathcal{B}_2}\gamma)]\Big). \qquad \text{(A24)}$$

Finally, applying equation (16) and the SEU representation to inequality (A24), we obtain $(v_{\mathcal{B}_2}\gamma) \succeq_{\mathcal{S}} (w_{\mathcal{B}_2}\delta)$. But this contradicts the fact that $(v_{\mathcal{B}_2}\gamma) \prec_{S} (w_{\mathcal{B}_2}\delta)$.

Axioms (C) and (M): Let $x \in \mathcal{X}$. Let $(x,\rightarrow)_{\succeq_{\text{xp}}} := \{y \in \mathcal{X};\ \ y \succ_{\text{xp}} x\}$ and let $(\leftarrow,x)_{\succeq_{\text{xp}}} := \{y \in \mathcal{X};\ \ y \prec_{\text{xp}} x\}$. To verify axiom (C), we must show that these sets are open in $\mathcal{X}$. To verify axiom (M), we must show that they are elements of $\mathfrak{D}$. To verify both, let $r := u(x)$, and observe that $(x,\rightarrow)_{\succeq_{\text{xp}}} = u^{-1}(r,\infty)$ and $(\leftarrow,x)_{\succeq_{\text{xp}}} = u^{-1}(-\infty,r)$, because u is an ordinal utility representation for $\succeq_{\text{xp}}$. Since u is continuous, these preimage sets are open in $\mathcal{X}$. Since u is $\mathfrak{D}$-measurable, these preimage sets are elements of $\mathfrak{D}$.

Axiom (CEq): Let $\mathcal{B} \in \mathfrak{B}$ and let $\alpha \in \mathcal{A}(\mathcal{B})$. Then $\alpha = \alpha'_{\restriction \mathcal{B}}$ for some $\alpha' \in \mathcal{A}$.

Claim 1: *$\mathrm{clos}[\alpha'(\mathcal{S})]$ has a $\succeq_{\mathrm{xp}}$-maximal element and a $\succeq_{\mathrm{xp}}$-minimal element.*[24]

Proof. (By contradiction) Suppose $\mathrm{clos}[\alpha'(\mathcal{S})]$ had no $\succeq_{\mathrm{xp}}$-maximal element. Thus, for any $x \in \mathrm{clos}[\alpha'(\mathcal{S})]$, there exists some $y \in \mathrm{clos}[\alpha'(\mathcal{S})]$ with $y \succ_{\mathrm{xp}} x$. In other words, $x \in (\leftarrow, y)_{\succeq_{\mathrm{xp}}}$. Thus, the collection $\{(\leftarrow, y)_{\succeq_{\mathrm{xp}}};\ y \in \mathrm{clos}[\alpha'(\mathcal{S})]\}$ is an open cover for $\mathrm{clos}[\alpha'(\mathcal{S})]$.

However, $\alpha' \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, so its image $\alpha'(\mathcal{S})$ is relatively compact; hence $\mathrm{clos}[\alpha'(\mathcal{S})]$ is compact. Thus, this open cover has a finite subcover; in other words, there exists some $y_1, \ldots, y_N \in \mathrm{clos}[\alpha'(\mathcal{S})]$ such that $\mathrm{clos}[\alpha'(\mathcal{S})]$ is covered by the collection $\{(\leftarrow, y_n)_{\succeq_{\mathrm{xp}}}\}_{n=1}^{N}$. Now, let $\overline{y} := \max_{\succeq_{\mathrm{xp}}}\{y_1, \ldots, y_N\}$ (this maximum exists because the set is finite). Then $\overline{y} \in \mathrm{clos}[\alpha'(\mathcal{S})]$, and $(\leftarrow, y_n)_{\succeq_{\mathrm{xp}}} \subseteq (\leftarrow, \overline{y})_{\succeq_{\mathrm{xp}}}$ for all $n \in [1 \ldots N]$. Thus, $\mathrm{clos}[\alpha'(\mathcal{S})] \subseteq (\leftarrow, \overline{y})_{\succeq_{\mathrm{xp}}}$. But clearly, $\overline{y} \notin (\leftarrow, \overline{y})_{\succeq_{\mathrm{xp}}}$, whereas $\overline{y} \in \mathrm{clos}[\alpha'(\mathcal{S})]$. Contradiction.

To avoid the contradiction, $\mathrm{clos}[\alpha'(\mathcal{S})]$ must have a $\succeq_{\mathrm{xp}}$-maximal element. The proof for $\succeq_{\mathrm{xp}}$-minimal elements is analogous. ◇ Claim 1

Let x be a $\succeq_{\mathrm{xp}}$-minimal element of $\mathrm{clos}[\alpha'(\mathcal{S})]$, and let z be a $\succeq_{\mathrm{xp}}$-maximal element of $\mathrm{clos}[\alpha'(\mathcal{S})]$; these exist by Claim 1. Then $x \preceq_{\mathrm{xp}} \alpha(b) \preceq_{\mathrm{xp}} z$ for all $b \in \mathcal{B}$. Thus, axiom (Dom) implies that $\kappa^x_{\mathcal{B}} \preceq_{\mathcal{B}} \alpha \preceq_{\mathcal{B}} \kappa^z_{\mathcal{B}}$. Thus,

$$u(x) \quad = \quad \mathbb{E}_{\mathcal{B}}[u \circ \kappa^x_{\mathcal{B}}] \quad \underset{(*)}{\leq} \quad \mathbb{E}_{\mathcal{B}}[u \circ \alpha] \quad \underset{(*)}{\leq} \quad \mathbb{E}_{\mathcal{B}}[u \circ \kappa^z_{\mathcal{B}}] \quad = \quad u(z),$$

where both $(*)$ are because of the assumed SEU representation. However, $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous, and $\mathcal{X}$ is connected. Thus, the Intermediate Value Theorem yields some $y \in \mathcal{X}$ such that $u(y) = \mathbb{E}_{\mathcal{B}}[u \circ \alpha]$. Thus, $\mathbb{E}_{\mathcal{B}}[u \circ \kappa^y_{\mathcal{B}}] = \mathbb{E}_{\mathcal{B}}[u \circ \alpha]$. But then the assumed SEU representation yields $\kappa^y_{\mathcal{B}} \approx_{\mathcal{B}} \alpha$, as desired.

Axiom (PC): Let $\mathcal{D} \in \mathfrak{B}$ and $\mathcal{E} \in \mathfrak{B}$ be disjoint, and let $\mathcal{G} := \mathcal{D} \vee \mathcal{E}$. Let $\mathcal{O} \subseteq \mathcal{A}(\mathcal{G})$ be open in the $\succeq_{\mathcal{G}}$-order topology, and let $\beta \in \mathcal{O}$. Thus, there exist some $\alpha, \gamma \in \mathcal{A}(\mathcal{G})$ such that $\alpha \prec_{\mathcal{G}} \beta \prec_{\mathcal{G}} \gamma$, and $\mathcal{O}$ contains the order-interval $(\alpha, \gamma)_{\succeq_{\mathcal{G}}}$. Let $a := \mathbb{E}_{\mathcal{G}}[u \circ \alpha]$, $b := \mathbb{E}_{\mathcal{G}}[u \circ \beta]$, and $c := \mathbb{E}_{\mathcal{G}}[u \circ \gamma]$; then $a < b < c$. Let $\epsilon := \min\{b - a, c - b\}$. Then $\epsilon > 0$.

Claim 2: *There exist a subset $\mathcal{O}_{\mathcal{D}} \subseteq \mathcal{A}(\mathcal{D})$, open in the $\succeq_{\mathcal{D}}$-order topology, such that $\beta_{\restriction \mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$, and such that $|\mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] - \mathbb{E}_{\mathcal{D}}[u \circ \beta_{\restriction \mathcal{D}}]| < \epsilon$ for all $\omega_{\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$.*

Proof. (*Case 1*) First, suppose that $\beta_{\restriction \mathcal{D}}$ is neither $\succeq_{\mathcal{D}}$-maximal nor $\succeq_{\mathcal{D}}$-minimal in $\mathcal{A}(\mathcal{D})$. Then there exists some $\phi_{\mathcal{D}}, \psi_{\mathcal{D}} \in \mathcal{A}(\mathcal{D})$ such that $\phi_{\mathcal{D}} \prec_{\mathcal{D}} \beta_{\restriction \mathcal{D}} \prec_{\mathcal{D}} \psi_{\mathcal{D}}$. Now, $\phi_{\mathcal{D}} := \phi'_{\restriction \mathcal{D}}$ and $\psi_{\mathcal{D}} := \psi'_{\restriction \mathcal{D}}$ for some $\phi, \psi \in \mathcal{A}$. Let w be a $\succeq_{\mathrm{xp}}$-minimal element of $\mathrm{clos}[\phi'(\mathcal{S})]$, and let z be a $\succeq_{\mathrm{xp}}$-maximal element of $\mathrm{clos}[\psi'(\mathcal{S})]$; these exist by Claim 1. Then $w \preceq_{\mathrm{xp}} \phi_{\mathcal{D}}(d)$ and $\psi_{\mathcal{D}}(d) \preceq_{\mathrm{xp}} z$ for all $d \in \mathcal{D}$. Thus,

$$\kappa^w_{\mathcal{D}} \quad \preceq_{\mathcal{D}} \quad \phi_{\mathcal{D}} \quad \prec_{\mathcal{D}} \quad \beta_{\mathcal{D}} \quad \prec_{\mathcal{D}} \quad \psi_{\mathcal{D}} \quad \preceq_{\mathcal{D}} \quad \kappa^z_{\mathcal{D}},$$

[24] Actually, we only need to obtain an upper and lower bound for $\mathrm{clos}[\alpha'(\mathcal{S})]$ in $\mathcal{X}$. But constructing a maximum and minimum is no more difficult.

where the "$\preceq_{\mathcal{D}}$" comparisons are by axiom (Dom), and the "$\prec_{\mathcal{D}}$" comparisons are by the definitions of $\phi_{\mathcal{D}}$ and $\psi_{\mathcal{D}}$. Thus,

$$u(w) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{w}] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{z}] \quad = \quad u(z),$$

where both $(*)$ are because of the assumed SEU representation. Thus, $u(w) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] < u(z)$. Now, u is continuous, and $\mathcal{X}$ is connected. Thus, the Intermediate Value Theorem yields $x, y \in \mathcal{X}$ such that $\mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon < u(x) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] < u(y) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon$. (It is even possible that w and z themselves already satisfy these inequalities). Thus,

$$\mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{x}] \quad = \quad u(x) \quad < \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] \quad < \quad u(y) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{y}],$$

so $\kappa_{\mathcal{D}}^{x} \prec_{\mathcal{D}} \beta_{|\mathcal{D}} \prec_{\mathcal{D}} \kappa_{\mathcal{D}}^{y}$, by the assumed SEU representation. Thus, if we define $\mathcal{O}_{\mathcal{D}} := (\kappa_{\mathcal{D}}^{x}, \kappa_{\mathcal{D}}^{y})_{\succeq_{\mathcal{D}}}$, then $\mathcal{O}_{\mathcal{D}}$ is open in the $\succeq_{\mathcal{D}}$-order topology, and $\beta_{|\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$. Furthermore, for any $\omega_{\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$, we have $\kappa_{\mathcal{D}}^{x} \prec_{\mathcal{D}} \omega_{\mathcal{D}} \prec_{\mathcal{D}} \kappa_{\mathcal{D}}^{y}$, and thus,

$$\begin{aligned} \mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon \quad \underset{(\diamond)}{<} \quad u(x) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{x}] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] \\ \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{y}] \quad = \quad u(y) \quad \underset{(\diamond)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon, \end{aligned}$$

so that $|\mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] - \mathbb{E}_{\mathcal{D}}[u \circ \beta_{|\mathcal{D}}]| < \epsilon$, as desired. Here, the $(*)$ inequalities are by the assumed SEU representation, and $(\diamond)$ inequalities are by the definitions of x and y.

(*Case 2*) Suppose $\beta_{|\mathcal{D}}$ is $\succeq_{\mathcal{D}}$-maximal in $\mathcal{A}(\mathcal{D})$, but not $\succeq_{\mathcal{D}}$-minimal. The logic is similar to *Case 1*, so we will be more cursory. There exists some $\phi_{\mathcal{D}} \in \mathcal{A}(\mathcal{D})$ such that $\phi_{\mathcal{D}} \prec_{\mathcal{D}} \beta_{|\mathcal{D}}$. As in Case 1, use Claim 1 to obtain some $w \in \mathcal{X}$ such that $w \preceq_{\text{xp}} \phi_{\mathcal{D}}(d)$ for all $d \in \mathcal{D}$. Thus, $\kappa_{\mathcal{D}}^{w} \preceq_{\mathcal{D}} \phi_{\mathcal{D}} \prec_{\mathcal{D}} \beta_{|\mathcal{D}}$, and thus, $u(w) = \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{w}] < \mathbb{E}_{\mathcal{D}}[u \circ \beta]$.

Now, u is continuous, and $\mathcal{X}$ is connected, so the Intermediate Value Theorem yields $x \in \mathcal{X}$ such that $\mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon < u(x) < \mathbb{E}_{\mathcal{D}}[u \circ \beta]$. Thus, $\mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{x}] = u(x) < \mathbb{E}_{\mathcal{D}}[u \circ \beta]$, so $\kappa_{\mathcal{D}}^{x} \prec_{\mathcal{D}} \beta_{|\mathcal{D}}$. Thus, if we define $\mathcal{O}_{\mathcal{D}} := (\kappa_{\mathcal{D}}^{x}, \rightarrow)_{\succeq_{\mathcal{D}}}$, then $\mathcal{O}_{\mathcal{D}}$ is open in the $\succeq_{\mathcal{D}}$-order topology, and $\beta_{|\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$; in fact, $\beta_{|\mathcal{D}}$ is a $\succeq_{\mathcal{D}}$-maximal element of $\mathcal{O}_{\mathcal{D}}$. Thus, for any $\omega_{\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$, we have $\kappa_{\mathcal{D}}^{x} \prec_{\mathcal{D}} \omega_{\mathcal{D}} \preceq_{\mathcal{D}} \beta_{|\mathcal{D}}$, and thus,

$$\mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon \quad \underset{(\diamond)}{<} \quad u(x) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{x}] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] \quad \underset{(*)}{\leq} \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] \quad < \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon,$$

as desired. Here, the $(*)$ inequalities are by the assumed SEU representation, and the $(\diamond)$ inequality is by the definition of x.

(*Case 3*) Suppose $\beta_{\mathcal{D}}$ is $\succeq_{\mathcal{D}}$-minimal in $\mathcal{A}(\mathcal{D})$, but not $\succeq_{\mathcal{D}}$-maximal. The logic is exactly the same as *Case 2*, but with all the preferences and inequalities reversed.

(*Case 4*) Suppose $\beta_{\mathcal{D}}$ is both $\succeq_{\mathcal{D}}$-minimal and $\succeq_{\mathcal{D}}$-maximal in $\mathcal{A}(\mathcal{D})$. In this case, $\mathbb{E}_{\mathcal{D}}[u \circ \omega] = \mathbb{E}_{\mathcal{D}}[u \circ \beta]$ for all $\omega \in \mathcal{A}(\mathcal{D})$. Thus, if we define $\mathcal{O}_{\mathcal{D}} := \mathcal{A}(\mathcal{D})$, then the claim is trivially satisfied. $\diamond$ `Claim 2`

Claim 3: *There exist a subset $\mathcal{O}_{\mathcal{E}} \subseteq \mathcal{A}(\mathcal{E})$, open in the $\succeq_{\mathcal{E}}$-order topology, such that $\beta_{\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$, and such that $|\mathbb{E}_{\mathcal{E}}[u \circ \omega_{\mathcal{E}}] - \mathbb{E}_{\mathcal{E}}[u \circ \beta_{\mathcal{E}}]| < \epsilon$ for all $\omega_{\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$.*

Proof. The argument is identical to Claim 2. ◇ Claim 3

Now let $\omega \in \mathcal{A}(\mathcal{G})$, and suppose $\omega_{|\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$ and $\omega_{|\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$. Then

$$\begin{aligned} \mu[\mathcal{G}]\,\mathbb{E}_{\mathcal{G}}[u \circ \omega] &\underset{(*)}{=} \mu[\mathcal{D}]\,\mathbb{E}_{\mathcal{D}}[u \circ \omega] + \mu[\mathcal{E}]\,\mathbb{E}_{\mathcal{E}}[u \circ \omega] \\ &\underset{(\dagger)}{<} \mu[\mathcal{D}]\left(\mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon\right) + \mu[\mathcal{E}]\left(\mathbb{E}_{\mathcal{E}}[u \circ \beta] + \epsilon\right) \\ &\underset{(*)}{=} \mu[\mathcal{G}]\ (\mathbb{E}_{\mathcal{G}}[u \circ \beta] + \epsilon) \quad \underset{(\diamond)}{\leq} \quad \mu[\mathcal{G}]\,\mathbb{E}_{\mathcal{G}}[u \circ \gamma]. \end{aligned}$$

Here, both $(*)$ are by equation (16), $(\dagger)$ is by the inequalities in Claims 2 and 3, and the full support of μ while $(\diamond)$ is by the definition of ϵ. Thus, $\mathbb{E}_{\mathcal{G}}[u \circ \omega] < \mathbb{E}_{\mathcal{G}}[u \circ \gamma]$. Thus, by the presumed SEU representation $\omega \prec_{\mathcal{G}} \gamma$. By an identical argument, $\mathbb{E}_{\mathcal{G}}[u \circ \omega] > \mathbb{E}_{\mathcal{G}}[u \circ \alpha]$, and thus, $\omega \succ_{\mathcal{G}} \alpha$. Thus, $\omega \in (\alpha, \gamma)_{\succeq_{\mathcal{G}}}$, and thus, $\omega \in \mathcal{O}$, as desired. □

B Proofs of results from Sections 4, 5 and 6

The proofs in this appendix draw heavily on results from a companion paper, which studies credences and their representations by classical probability measures (Pivato and Vergopoulos, 2016). We will refer to results in the companion paper with the prefix "PV". Thus, "Theorem PV-3.3" should be read as, "Theorem 3.3 from Pivato and Vergopoulos (2016)." Theorems 1, 2, 3, 5 and 6 are all obtained as corollaries of the following alternate version of Theorem 11 in the particular case where $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$.

Theorem B1 *Let $\mathcal{S}$ be a nondegenerate topological space, let $\mathcal{X}$ be a connected topological space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then, it further satisfies Axioms* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC)*, and* (TC) *if and only if it has an SEU representation (u, μ), where u is a continuous function and μ is a credence on $\mathfrak{R}(\mathcal{S})$ with full support. Finally, μ is unique, and u is unique up to positive affine transformation.*

Proof of Theorem B1. In Theorem 11, the $\mathfrak{D}$-measurability of the utility function u only serves to obtain axiom (M), and not the other axioms. To prove the necessity of the axioms, we can therefore proceed here exactly as in Theorem 11. Conversely, in Theorem 11, axiom (M) only serves to obtain the $\mathfrak{D}$-measurability of u. (See Claim 6 in the proof of Proposition A3.) This is only used to make sure that $u \circ \alpha$ is $\mathfrak{B}$-comeasurable for any $\alpha \in \mathcal{A}$. Here, since u and α are continuous, $u \circ \alpha$ is also continuous, and therefore automatically $\mathfrak{R}(\mathcal{S})$-measurable for any $\alpha \in \mathcal{A}$, by Lemma PV-4.4. Thus, we do not need axiom (M) in showing the sufficiency of the axioms for the representation. Finally, the uniqueness of the representation can be obtained exactly as in Theorem 11 since the argument invoked there uses neither axiom (M) nor $\mathfrak{D}$-measurability. □

Proof of Theorem 1. "$\Longleftarrow$" Let ν be a Baire charge on $\mathcal{S}$ with full support, and $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a continuous function that together provide an SEU representation of $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ as in equation (2). Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained by restricting ν to $\mathfrak{R}(\mathcal{S})$ as in Example 8(b). As explained in Example 10(b), $\mathbb{E}^{\mu}_{\mathcal{B}}[u \circ \alpha] = \int_{\mathcal{B}} u \circ \alpha \, \mathrm{d}\nu$; thus, (u, μ) provides an SEU representation as in Theorem B1. Moreover, since ν has full support, μ also has full support. Thus, $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ must satisfy all of axioms (CEq), (C), (Dom), (Sep), (PC) and (TC) by Theorem B1.

"$\Longrightarrow$" If $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies Axioms (CEq), (C), (Dom), (Sep), (PC) and (TC), then Theorem B1 says it has an SEU representation given by a credence μ on $\mathfrak{R}(\mathcal{S})$ with full support, and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$. Let $\mathbf{E}$ be the μ-compatible conditional expectation structure from Proposition A4. Since $\mathcal{S}$ is a Baire space, Proposition PV-5.1 yields a Baire charge on $\mathcal{S}$ representing μ and $\mathbf{E}$ as in Examples 8(b) and 10(b). Combining this and the credence SEU representation from Theorem B1, we obtain an SEU representation as in formula (2). Since μ has full support, ν has also full support.

Finally, suppose that both (u, ν) and (u', ν') provide a representation of $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}}$. Let μ and μ' be the credences obtained by restricting respectively ν and ν' to $\mathfrak{R}(\mathcal{S})$. Then, by the assumed representations and Example 10(b), (u, μ) and (u', μ') both provide SEU representations as in formula (17). By uniqueness in Theorem B1, u and u' are positive affine transformations of each other. Moreover, μ and μ' are equal to each other. Thus, Proposition PV-2.7 implies that $\nu = \nu'$. □

The proofs of Theorems 2, 3, 5 and 6 are very similar to the proof of Theorem 1, and we only briefly sketch them. They require the following lemma.

Lemma B2 **(a)** *Any nonsingleton Hausdorff space is nondegenerate.*

(b) *If $\mathcal{S}$ is a nondegenerate space, then $\mathfrak{R}(\mathcal{S})$ is nontrivial.*

(c) *Suppose $\mathcal{S}$ is either locally compact or normal Hausdorff. For any nonempty open $\mathcal{O} \subseteq \mathcal{S}$, there is a nonempty $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ with* $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{O}$.

Proof. (a) Since $|\mathcal{S}| \geq 2$, there exist $s_1, s_2 \in \mathcal{S}$ with $s_1 \neq s_2$. Since $\mathcal{S}$ is Hausdorff, there are disjoint open neighbourhoods $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{S}$ around s_1 and s_2. Thus, $\mathcal{O}_1$ and $\mathcal{O}_2$ are both nonempty open subsets which are not dense in $\mathcal{S}$.

(b) Let $\mathcal{O} \subset \mathcal{S}$ be a nonempty, non-dense open subset. Let $\mathcal{C} := \mathcal{S} \setminus \mathcal{O}$. Then $\mathcal{C}$ is a proper closed subset of $\mathcal{S}$ with a nonempty interior (because $\mathcal{O}$ is not dense). Let $\mathcal{R} := \mathrm{int}(\mathcal{C})$. Then $\emptyset \neq \mathcal{R} \neq \mathcal{S}$, and $\mathcal{R}$ is regular, because it is the interior of a closed set. Thus, $\mathfrak{R}(\mathcal{S})$ is nontrivial.

(c) First suppose $\mathcal{S}$ is locally compact. Let $s \in \mathcal{O}$. By local compactness, $\mathcal{O}$ contains a compact subset $\mathcal{K}$ which is also a neighbourhood of s. Let $\mathcal{R} := \mathrm{int}(\mathcal{K})$. Then

$\text{clos}(\mathcal{R}) \subseteq \mathcal{K} \subseteq \mathcal{O}$, and $\mathcal{R} \neq \emptyset$, because $s \in \mathcal{R}$. Finally, $\mathcal{R}$ is regular, because it is the interior of the closed set $\mathcal{K}$.

Now suppose $\mathcal{S}$ is normal. If $\mathcal{O} = \mathcal{S}$, the statement is trivial. So assume $\mathcal{O} \neq \mathcal{S}$. Let $s \in \mathcal{O}$ and let $\mathcal{C} := \mathcal{S} \setminus \mathcal{O}$. Then $\mathcal{C}$ is closed, and $\{s\}$ is also closed, because $\mathcal{S}$ is Hausdorff. By normality, there exist disjoint open sets $\mathcal{U}_1$,$\mathcal{U}_2$ containing $\{s\}$ and $\mathcal{C}$. Let $\mathcal{K} = \text{clos}(\mathcal{U}_1)$; the $\mathcal{K}$ is closed and disjoint from $\mathcal{U}_2$. Thus, $\mathcal{K}$ is disjoint from $\mathcal{C}$, so $\mathcal{K} \subseteq \mathcal{O}$. Let $\mathcal{R} := \text{int}(\mathcal{K})$. Then $\mathcal{R}$ is regular (being the interior of a closed set), $\mathcal{R}$ is nonempty (it contains $\mathcal{U}_1$) and $\text{clos}(\mathcal{R}) \subseteq \mathcal{K} \subseteq \mathcal{O}$. □

Proof of Theorem 2. By Lemma B2(a,b), $\mathfrak{R}(\mathcal{S})$ is nontrivial. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 1, with equations (PV-5D) and (PV-5E) and Theorem PV-5.4 playing respectively the same role as Example 8(b), Example 10(b) and Proposition PV-5.1. Finally, there is a minor difference in the proof of the full support property of the probability charge in the sufficiency of the axioms. Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained by applying Theorem B1. Let $\mathbf{E}$ be the μ-compatible conditional expectation structure from Proposition A4. Let ν be the normal probability charge obtained by applying Theorem PV-5.4 to $\mathbf{E}$. Now, suppose $\nu[\mathcal{O}] = 0$ for some nonempty open subset $\mathcal{O}$ of $\mathcal{S}$. By Lemma B2(c), there exists a nonempty $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ such that $\text{clos}(\mathcal{R}) \subseteq \mathcal{O}$. Thus,

$$\begin{aligned} \mu(\mathcal{R}) &\underset{\text{(a)}}{=} \nu(\mathcal{R}) + \rho_{\mathcal{R}}(\partial\mathcal{R}) \underset{\text{(b)}}{\leq} \nu(\mathcal{R}) + \nu(\partial\mathcal{R}) \\ &= \nu(\text{clos}(\mathcal{R})) \underset{\text{(c)}}{\leq} \nu(\mathcal{O}) = 0, \end{aligned}$$

where (a) is by Theorem PV-5.4, (b) is by equation (3) and (c) is because $\text{clos}(\mathcal{R}) \subseteq \mathcal{O}$. But this contradicts the fact that μ has full support. Thus, $\nu[\mathcal{O}] > 0$ for every nonempty open subset $\mathcal{O}$ of $\mathcal{S}$, so ν has full support. □

Proof of Theorem 3. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 2, with equations (PV-5T) and (PV-5U) and Corollary PV-5.7 playing respectively the same role as equations (PV-5D) and (PV-5E) and Proposition PV-5.4. The proof of full support is as in Theorem 2. □

Proof of Theorem 5. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 2, with equations (PV-5V) and (PV-5W), Theorem PV-5.10, and Example PV-5.11(a) playing respectively the same role as equations (PV-5D) and (PV-5E) and Proposition PV-5.4. However, in the sufficiency of the axioms, the proof that the Borel probability measure ν^* on $\mathcal{S}^*$ has full support is slightly different. Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained from Theorem B1. Let $\mathbf{E}$ be the μ-compatible conditional expectation structure from Proposition A4. Let

ν^* be the Borel probability measure on $\mathcal{S}^*$ obtained by applying Theorem PV-6.1 to $\mathbf{E}$. Now, suppose $\nu^*[\mathcal{O}^*] = 0$ for some nonempty open subset $\mathcal{O}^*$ of $\mathcal{S}^*$. Since $\mathcal{S}^*$ is compact Hausdorff, Lemma B2(c) gives a nonempty regular subset $\mathcal{R}^* \in \mathfrak{R}(\mathcal{S}^*)$ such that $\text{clos}_{\mathcal{S}^*}(\mathcal{R}^*) \subseteq \mathcal{O}^*$. Define $\mathcal{R} := \mathcal{R}^* \cap \mathcal{S}$. By Lemma PV-6.2(b), $\mathcal{R}$ is a regular subset of $\mathcal{S}$. By equation (PV-6A), we have $\mu(\mathcal{R}) = \check{\nu}(\mathcal{R}^*) + \int_{\partial\mathcal{R}^*} \check{\phi}^*_{\mathcal{R}^*} \, \mathrm{d}\nu^*_{\mathcal{R}^*}$. At this point, we can proceed as in Theorem 2 to obtain $\mu(\mathcal{R}) = 0$, which contradicts the fact that μ has full support. Hence the full support of ν^*. □

Proof of Theorem 4. The proof is very similar to Theorem 5, but using Example PV-5.11(b) instead of Example PV-5.11(a). □

Proof of Theorem 6. The proof is very similar to Theorem 5, but using Example PV-5.11(c) instead of Example PV-5.11(a). □

C Necessary and sufficient conditions for (Rch)

In this appendix, we will show that condition (Rch) places an important restriction on the utility function u. We say that a function $u : \mathcal{X} \longrightarrow \mathbb{R}$ is *nonsatiating* if, for any $x \in \mathcal{X}$, there exist $\underline{x}, \overline{x} \in \mathcal{X}$ such that $u(\underline{x}) < u(x) < u(\overline{x})$. Equivalently: u never obtains a global maximum or global minimum on $\mathcal{X}$. (For example, this is the case if $\sup_{x\in\mathcal{X}} u(x) = \infty$ and $\inf_{x\in\mathcal{X}} u(x) = -\infty$.) In particular, if u is continuous and nonsatiating, then $\mathcal{X}$ *cannot* be compact. Our next result says that, if $\mathcal{S}$ and $\mathcal{X}$ satisfy certain conditions, then condition (Rch) is more or less equivalent to a nonsatiating utility function. To state this result, we need two topological preliminaries.

A topological space $\mathcal{X}$ is *contractible* if there is a function $f : [0,1] \times \mathcal{X} \longrightarrow \mathcal{X}$ and a point $z \in \mathcal{X}$ such that $f(0, x) = x$ for all $x \in \mathcal{X}$, whereas $f(1, x) = z$ for all $x \in \mathcal{X}$. (In other words: the identity map on $\mathcal{X}$ is *null-homotopic*.) Heuristically, this mean that $\mathcal{X}$ is not only path-connected, but it has no "holes" of any dimension.[25] For example, any convex subset of $\mathbb{R}^N$ is contractible. More generally, any star-shaped subset of any topological vector space is contractible.[26] However, there exist other, less obvious contractible spaces (even contractible manifolds).

A topological space is *extremally disconnected* if the closure of every open set is open. Clearly, these are rather exotic spaces: any extremally disconnected Hausdorff space is totally disconnected, and any extremally disconnected metrizable space is discrete. It is safe to say that most of the topological spaces which would arise in economic applications are *not* extremally disconnected.

A credence μ is *nonatomic* if, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, if $0 < \epsilon < \mu[\mathcal{R}]$, then there exists some $\mathcal{R}' \in \mathfrak{R}(\mathcal{S})$ with $\mathcal{R}' \subset \mathcal{R}$ and $\mu[\mathcal{R}'] = \epsilon$.

[25] To be precise: all the homotopy groups of $\mathcal{X}$ are trivial.

[26] A subset $\mathcal{X}$ of a topological vector space is *star-shaped* if $r\,x \in \mathcal{X}$ for all $x \in \mathcal{X}$ and all $r \in [0,1]$.

Proposition C1 *Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces, let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ with an SEU representation given by a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$ and a credence μ with full support.*

(a) *Suppose $\mathcal{S}$ is a normal Hausdorff space, $\mathcal{X}$ is contractible, and $\mathcal{A} = \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. If u is nonsatiating and μ is nonatomic, then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies axiom* (Rch).

(b) *If $\mathcal{S}$ is* not *extremally disconnected, and $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies axiom* (Rch)*, then u is nonsatiating.*

(c) *If $\mathcal{S}$ is extremally disconnected and $\mathcal{A} = \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ always satisfies axiom* (Rch).

A first consequence of Proposition C1 is the following restriction on the primitive topologies: if the state space is not extremally disconnected, then the outcome space cannot be compact. (For most applications, this is only a mild loss of generality: let $\mathcal{X}'$ be some large but non-compact subset of $\mathcal{X}$ where $\succeq_{\mathrm{xp}}$ has neither a maximal nor a minimal element, and let $\mathcal{A}'$ be the set of elements in $\mathcal{A}$ which range over $\mathcal{X}'$; then we can construct an SEU representation of the agent's preferences on $\mathcal{A}'$ instead.) Proposition C1 also provides a characterization of condition (Rch). For example, if $\mathcal{S}$ is any non-discrete metric space (e.g. a non-discrete subset of $\mathbb{R}^N$) and $\mathcal{X}$ is a convex subset of $\mathbb{R}^M$, while $\mathcal{A} = \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$ and μ is nonatomic, then Proposition C1(a,b) implies that $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies (Rch) if and only if u is nonsatiating.

At the opposite extreme, suppose $\mathcal{S}$ is the Stone space of an abstract, complete Boolean algebra $\mathfrak{A}$; then $\mathcal{S}$ is a compact and extremally disconnected topological space, and there is a canonical isomorphism from $\mathfrak{A}$ to $\mathfrak{R}(\mathcal{S})$ (Fremlin, 2004, 314S-314T). In this case, Proposition C1(c) says that *any* conditional preference on $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$ satisfies (Rch).

The rest of this appendix is the the proof of Proposition C1. First, we need three topological lemmas.

Lemma C2 *Let $\mathcal{S}$ be any topological space. The interior of any closed subset of $\mathcal{S}$ is a regular set.*

Proof. Suppose $\mathcal{C} \subseteq \mathcal{S}$ is closed. Let $\mathcal{D} := \mathrm{int}(\mathcal{C})$. We want to show that $\mathcal{D}$ is regular. Let $\mathcal{C}' := \mathrm{clos}(\mathcal{D})$. Then $\mathcal{C}' \subseteq \mathcal{C}$. But $\mathcal{D}$ is (by definition) the largest open subset of $\mathcal{C}$, and since $\mathcal{D} \subseteq \mathcal{C}' \subseteq \mathcal{C}$, that means that $\mathcal{D}$ is also the largest open subset of $\mathcal{C}'$ —hence $\mathcal{D} = \mathrm{int}(\mathcal{C}')$, as desired. □

Lemma C3 (*$\mathcal{S}$ is extremally disconnected*) $\iff$ (*Every regular subset of $\mathcal{S}$ is clopen*).

Proof. "$\Longrightarrow$" Let $\mathcal{R}$ be regular. Then $\mathcal{R} = \mathrm{int}[\mathrm{clos}(\mathcal{R})]$. But $\mathrm{clos}(\mathcal{R})$ is open (because $\mathcal{S}$ is extremally disconnected), so $\mathrm{int}[\mathrm{clos}(\mathcal{R})] = \mathrm{clos}(\mathcal{R})$. Thus, $\mathcal{R} = \mathrm{clos}(\mathcal{R})$, so $\mathcal{R}$ is closed —hence clopen.

"$\Longleftarrow$" Let $\mathcal{O} \subseteq \mathcal{S}$ be open. Let $\mathcal{R} := \text{int}[\text{clos}(\mathcal{O})]$; then $\mathcal{R}$ is a regular set (by Lemma C2) which contains $\mathcal{O}$ (because it contains any open subset of $\text{clos}(\mathcal{O})$). But by hypothesis, $\mathcal{R}$ is clopen. Thus, $\text{clos}(\mathcal{R}) = \mathcal{R}$. Thus, $\mathcal{R}$ is a closed set containing $\mathcal{O}$, so $\mathcal{R} \supseteq \text{clos}(\mathcal{O})$. But $\mathcal{R} = \text{int}[\text{clos}(\mathcal{O})]$, so $\mathcal{R} \subseteq \text{clos}(\mathcal{O})$. Thus, $\mathcal{R} = \text{clos}(\mathcal{O})$. So the closure of $\mathcal{O}$ is open. □

Lemma C4 *Suppose $\mathcal{S}$ is a normal Hausdorff space, and $\mathcal{X}$ is contractible. Let $\mathcal{A} := \mathcal{C}_{\text{b}}(\mathcal{S}, \mathcal{X})$. Let $\mathcal{R}$ and $\mathcal{Q}$ be regular open subsets of $\mathcal{S}$, such that* $\text{clos}(\mathcal{R})$ *and* $\text{clos}(\mathcal{Q})$ *are disjoint. For any acts α, ξ^0, and ξ^1 in $\mathcal{A}$, there is a bounded continuous function $\Gamma : [0,1] \times \mathcal{S} \longrightarrow \mathcal{X}$ with the following properties:*

(a) *For all $t \in [0,1]$, we get a function $\gamma^t \in \mathcal{A}$ by setting $\gamma^t(s) := \Gamma(t,s)$ for all $s \in \mathcal{S}$.*

(b) *For all $t \in [0,1]$, $\gamma^t_{\restriction \mathcal{R}} = \alpha_{\restriction \mathcal{R}}$.*

(c) *$\gamma^0_{\restriction \mathcal{Q}} = \xi^0_{\restriction \mathcal{Q}}$ and $\gamma^1_{\restriction \mathcal{Q}} = \xi^1_{\restriction \mathcal{Q}}$.*

(d) *There is a compact subset $\mathcal{K} \subseteq \mathcal{X}$ such that $\Gamma(t,s) \in \mathcal{K}$ for all $t \in [0,1]$ and $s \in \mathcal{S}$. Furthermore, the set $\mathcal{K}$ depends only on α, ξ^0, and ξ^1, and is independent of the choice of $\mathcal{Q}$ and $\mathcal{R}$.*

(e) *For any continuous function $u : \mathcal{X} \longrightarrow \mathbb{R}$, and any $T \in [0,1]$, we have*

$$\lim_{t \to T} \left\| u \circ \gamma^t - u \circ \gamma^T \right\|_\infty \quad = \quad 0.$$

Given two acts α and ξ^0, Lemma C4 says that we can construct an act γ^0 which "interpolates" between them, in the sense that it agrees with α on $\mathcal{R}$ and agrees with ξ^0 on $\mathcal{Q}$. Likewise, we can construct an act γ^1 which agrees with α on $\mathcal{R}$ and agrees with ξ^1 on $\mathcal{Q}$. Finally, we can *continuously deform* γ^0 into γ^1 via a continuously varying family of maps $\{\gamma^t\}_{t \in [0,1]}$, such that *every one* of the functions γ^t agrees with α on $\mathcal{R}$. In the jargon of algebraic topology, the function Γ is a *homotopy* from γ^0 to γ^1 *relative to* $\mathcal{R}$. Furthermore, this relative homotopy can be constructed such that it never leaves some compact region $\mathcal{K}$, and when composed with any real-valued function u, the functions $\{u \circ \gamma^t\}_{t \in [0,1]}$ trace a continuous "path" through the space $\mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$ in the uniform norm topology.

Proof of Lemma C4. The space $\mathcal{S}$ is normal and Hausdorff, so it is Tychonoff (Willard, 2004, Corollary 15.7). So, let $\overline{\mathcal{S}}$ be the Stone-Čech compactification of $\mathcal{S}$, and let $\iota : \mathcal{S} \hookrightarrow \overline{\mathcal{S}}$ be the canonical embedding. Let $\mathcal{K}_\alpha := \text{clos}[\alpha(\mathcal{S})]$, $\mathcal{K}_0 := \text{clos}[\xi^0(\mathcal{S})]$, and $\mathcal{K}_1 := \text{clos}[\xi^1(\mathcal{S})]$. These are compact subsets of $\mathcal{X}$, because α, ξ^0, and ξ^1 are bounded continuous functions. Let $\mathcal{K}_* := \mathcal{K}_\alpha \cup \mathcal{K}_0 \cup \mathcal{K}_1$; then $\mathcal{K}_*$ is also compact, and α, ξ^0, and ξ^1 are continuous functions from $\mathcal{S}$ into $\mathcal{K}_*$. Thus, by the universal property of the Stone-Čech compactification, we can extend them to continuous functions $\overline{\alpha}$, $\overline{\xi}^0$, and $\overline{\xi}^1$ from $\overline{\mathcal{S}}$ into $\mathcal{K}_*$, such that $\overline{\alpha} \circ \iota = \alpha$, $\overline{\xi}^0 \circ \iota = \xi^0$, and $\overline{\xi}^1 \circ \iota = \xi^1$ (Willard, 2004, Thm. 19.5; Aliprantis & Border, 2006, Thm. 2.79).

The space $\mathcal{X}$ is contractible, so there is a continuous function $\Xi : [-1,1]\times\overline{\mathcal{S}}\longrightarrow\mathcal{X}$ such that $\Xi(-1,s) = \overline{\alpha}(s)$, $\Xi(0,s) = \overline{\xi}^0(s)$ and $\Xi(1,s) = \overline{\xi}^1(s)$ for all $s \in \overline{\mathcal{S}}$ (Willard, 2004, Theorem 32.7). Let $\mathcal{K} := \Xi\big([-1,1]\times\overline{\mathcal{S}}\big)$. Then $\mathcal{K}$ is a compact subset of $\mathcal{X}$, because $[-1,1]\times\overline{\mathcal{S}}$ is compact and Ξ is continuous. Note that the construction of $\mathcal{K}$ depends only on α, ξ^0 and ξ^1; we have not yet refered to $\mathcal{R}$ and $\mathcal{Q}$.

Let $\overline{\mathcal{Q}} := \mathrm{clos}(\mathcal{Q})$ and $\overline{\mathcal{R}} := \mathrm{clos}(\mathcal{R})$; by hypotheses, these sets are disjoint. Since $\mathcal{S}$ is normal, Urysohn's Lemma yields a function $\phi : \mathcal{S}\longrightarrow[0,1]$ such that $\phi(\overline{\mathcal{R}}) = \{0\}$ and $\phi(\overline{\mathcal{Q}}) = \{1\}$ (Willard, 2004, Thm. 15.6; Aliprantis & Border, 2006, Thm. 2.46; Royden, 1988, Thm 8.7). Now define $\Gamma : [0,1]\times\mathcal{S}\longrightarrow\mathcal{K}$ by setting $\Gamma(t,s) := \Xi[\phi(s)\cdot(t+1)-1,\ \iota(s)]$, for all $t \in [0,1]$ and all $s \in \mathcal{S}$. Thus, Γ is a continuous function (because it is a composition of continuous functions). Furthermore:

- For all $t \in [0,1]$, if we define the function $\gamma^t : \mathcal{S}\longrightarrow\mathcal{X}$ by setting $\gamma^t(s) := \Gamma(t,s)$ for all $s \in \mathcal{S}$, then γ^t is continuous (because Γ is continuous) and bounded (because its image is contained in the compact set $\mathcal{K}$); thus, $\gamma^t \in \mathcal{A}$.
- For all $t \in [0,1]$, and all $r \in \mathcal{R}$, we have $\gamma^t(r) = \Gamma(t,r) = \Xi[\phi(r)\cdot(t+1)-1, \iota(r)] = \Xi[0\cdot(t+1)-1, \iota(r)] = \Xi[-1,\iota(r)] = \overline{\alpha}[\iota(r)] = \alpha(r)$. Thus, $\gamma^t_{\restriction\mathcal{R}} = \alpha_{\restriction\mathcal{R}}$.
- For all $q \in \mathcal{Q}$, we have $\gamma^0(q) = \Gamma(0,q) = \Xi[\phi(q)\cdot(0+1)-1, \iota(q)] = \Xi[1\cdot 1-1, \iota(q)] = \Xi[0,\iota(q)] = \overline{\xi}^0[\iota(q)] = \xi^0(q)$. Thus, $\gamma^0_{\restriction\mathcal{Q}} = \xi^0_{\restriction\mathcal{Q}}$
- For all $q \in \mathcal{Q}$, we have $\gamma^1(q) = \Gamma(1,q) = \Xi[\phi(q)\cdot(1+1)-1, \iota(q)] = \Xi[1\cdot 2-1, \iota(q)] = \Xi[1,\iota(q)] = \overline{\xi}^1[\iota(q)] = \xi^1(t)$. Thus, $\gamma^1_{\restriction\mathcal{Q}} = \xi^1_{\restriction\mathcal{Q}}$.

Thus, Γ verifies all of properties (a)-(d).

To check property (e), let $\overline{\phi} : \overline{\mathcal{S}}\longrightarrow[0,1]$ be the Stone-Čech extension of ϕ, and define the function $\overline{\Gamma} : [0,1]\times\overline{\mathcal{S}}\longrightarrow\mathcal{K}$ by setting $\overline{\Gamma}(t,\overline{s}) := \Xi[\phi(\overline{s})\cdot(t+1)-1,\ \overline{s}]$, for all $t \in [0,1]$ and all $\overline{s} \in \overline{\mathcal{S}}$. For all $t \in [0,1]$, define the function $\overline{\gamma}^t : \overline{\mathcal{S}}\longrightarrow\mathcal{K}$ by setting $\overline{\gamma}^t(\overline{s}) := \overline{\Gamma}(t,\overline{s})$ for all $\overline{s} \in \overline{\mathcal{S}}$. Clearly, for all $t \in [0,1]$ and $s \in \mathcal{S}$, we have $\Gamma(t,s) = \overline{\Gamma}[t,\iota(s)]$ and hence $u\circ\Gamma(t,s) = u\circ\overline{\Gamma}[t,\iota(s)]$; in other words, $u\circ\gamma^t(s) = u\circ\overline{\gamma}^t[\iota(s)]$. Thus, for any $t, T \in [0,1]$, $\big\|u\circ\gamma^t - u\circ\gamma^T\big\|_\infty \leq \big\|u\circ\overline{\gamma}^t - u\circ\overline{\gamma}^T\big\|_\infty$. Thus, it suffices to show that $\lim_{t\to T}\big\|u\circ\overline{\gamma}^t - u\circ\overline{\gamma}^T\big\|_\infty = 0$.

To see this, let $\epsilon > 0$. Since $u\circ\overline{\Gamma}$ is continuous, for every $\overline{s} \in \overline{\mathcal{S}}$, there is some $\delta_{\overline{s}} > 0$ and some open neighbourhood $\mathcal{Z}_{\overline{s}}$ around $\overline{s}$ in $\overline{\mathcal{S}}$ such that, if we define $\mathcal{O}_{\overline{s}} := (T-\delta_{\overline{s}}, T+\delta_{\overline{s}})\times\mathcal{Z}_{\overline{s}}$ (an open neighbourhood of $(T,\overline{s})$), then $|u\circ\overline{\Gamma}(t,\overline{z}) - u\circ\overline{\Gamma}(T,\overline{s})| < \epsilon/2$ for all $(t,\overline{z}) \in \mathcal{O}_{\overline{s}}$. The collection $\{\mathcal{O}_{\overline{s}}\}_{\overline{s}\in\overline{\mathcal{S}}}$ is an open cover of the set $\{T\}\times\overline{\mathcal{S}}$. But $\{T\}\times\overline{\mathcal{S}}$ is compact (because $\{T\}$ and $\overline{\mathcal{S}}$ are compact). Thus, this collection has a finite subcover, say, $\{\mathcal{O}_{\overline{s}_n}\}_{n=1}^N$ for some finite collection $\{\overline{s}_n\}_{n=1}^N$ of points in $\overline{\mathcal{S}}$. Let $\delta_\epsilon := \min\{\delta_{\overline{s}_n}\}_{n=1}^N$.

Claim 1: *Suppose $|t-T| < \delta_\epsilon$. Then $|u\circ\overline{\Gamma}(t,\overline{z}) - u\circ\overline{\Gamma}(T,\overline{z})| < \epsilon$, for all $\overline{z} \in \overline{\mathcal{S}}$.*

Proof. Let $\overline{z} \in \overline{\mathcal{S}}$. Since $\{\mathcal{O}_{\overline{s}_n}\}_{n=1}^N$ is an open cover of $\{T\}\times\overline{\mathcal{S}}$, there is some $n \in [1\ldots N]$ such that $(T,\overline{z}) \in \mathcal{O}_{\overline{s}_n}$, which means that $\overline{z} \in \mathcal{Z}_{\overline{s}_n}$. Since $|t-T| < \delta_\epsilon \leq \delta_{\overline{s}_n}$, it follows

that $(t, \overline{z}) \in \mathcal{O}_{\overline{s}_n}$ also. Thus,

$$\begin{aligned} |u \circ \overline{\Gamma}(t, \overline{z}) - u \circ \overline{\Gamma}(T, \overline{z})| &\underset{(\triangle)}{\leq} |u \circ \overline{\Gamma}(t, \overline{z}) - u \circ \overline{\Gamma}(T, \overline{s}_n)| + |u \circ \overline{\Gamma}(T, \overline{s}_n) - u \circ \overline{\Gamma}(T, \overline{z})| \\ &\underset{(\dagger)}{\leq} \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad = \quad \epsilon, \end{aligned}$$

as desired. Here $(\triangle)$ is the triangle inequality, and $(\dagger)$ is two applications of the defining property of $\mathcal{O}_{\overline{s}_n}$. ◇ Claim 1

Since Claim 1 holds for all $y \in \overline{\mathcal{S}}$, we conclude that $\left\|u \circ \overline{\gamma}^t - u \circ \overline{\gamma}^T\right\|_\infty < \epsilon$. We can construct such a δ_ϵ for any $\epsilon > 0$. We conclude that $\lim_{t \to T} \left\|u \circ \overline{\gamma}^t - u \circ \overline{\gamma}^T\right\|_\infty = 0$. □

Proof of Proposition C1. (a) Suppose u is nonsatiating. Let $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathcal{S}$, let $\alpha_1 \in \mathcal{A}(\mathcal{R}_1)$, and let $\alpha_2 \in \mathcal{A}(\mathcal{R}_2)$. We must find some act $\beta \in \mathcal{A}(\mathcal{R}_2)$ compatible with α_1 such that $\beta \approx_{\mathcal{R}_2} \alpha_2$.

Claim 1: *Let $E := \mathbb{E}_{\mathcal{R}_2}[u \circ \alpha_2]$. There exist some some $x_0, x_1 \in \mathcal{X}$ such that $u(x_0) < E < u(x_1)$.*

Proof. (by contradiction) Suppose there was no $x_0 \in \mathcal{X}$ such that $u(x_0) < E$. Thus, $u(x) \geq E$ for all $x \in \mathcal{X}$. In particular, this means that $u \circ \alpha(r) \geq E$ for all $r \in \mathcal{R}$. There are now two cases.

Case 1. If $u \circ \alpha(r) = E$ for some $r \in \mathcal{R}$, then there exists $x \in \mathcal{X}$ such that $u(x) = E$, but by hypothesis, there does not exist any $x \in \mathcal{X}$ such that $u(x) < E$. Thus, u obtains a minimal value of E at x, which contradicts the hypothesis that u is nonsatiating.

Case 2. Proposition A4 says that $\mathbb{E}_{\mathcal{R}}$ is strictly monotonic, because μ has full support. Thus, if $u \circ \alpha(r) > E$ for all $r \in \mathcal{R}$, then $\mathbb{E}_{\mathcal{R}}[u \circ \alpha] > E$. But this contradicts the definition of E.

Either way, we have a contradiction. To avoid this, there must be some $x_0 \in \mathcal{X}$ with $u(x_0) < E$. An identical argument yields some $x_1 \in \mathcal{X}$ with $u(x_1) > E$. ◇ Claim 1

Let $\overline{\mathcal{Q}}$ be a closed subset of $\mathcal{R}_2$ with nonempty interior. Since it is a subset of the open set $\mathcal{R}_2$, it is disjoint from the closed set $\text{clos}(\mathcal{R}_1)$. Let $\mathcal{Q} := \text{int}(\overline{\mathcal{Q}})$. This is a nonempty regular subset of $\mathcal{R}_2$.

By definition of $\mathcal{A}(\mathcal{R}_1)$, there is some function $\overline{\alpha}^1 \in \mathcal{A}$ such that $\alpha_1 := \overline{\alpha}^1_{\restriction \mathcal{R}_1}$. Let x_0 and x_1 be the values identified by Claim 1, and let $\xi^0, \xi^1 : \mathcal{S} \longrightarrow \mathcal{X}$ be the constant functions with value x^0 and x^1, respectively. Applying Lemma C4 to $\overline{\alpha}^1$, ξ^0 and ξ^1, we obtain a continuous function $\Gamma : [0,1] \times \mathcal{S} \longrightarrow \mathcal{X}$ with the following properties:

- For all $t \in [0,1]$, we get a function $\gamma^t \in \mathcal{A}$ by setting $\gamma^t(s) := \Gamma(t, s)$ for all $s \in \mathcal{S}$.
- For all $t \in [0,1]$, we have $\gamma^t_{\restriction \mathcal{R}_1} = \alpha_1$; thus, $\gamma^t_{\restriction \mathcal{R}_2}$ is compatible with α_1.
- $\gamma^0(q) = x_0$ and $\gamma^1(q) = x_1$ for all $q \in \mathcal{Q}$.

Furthermore, there is a compact subset $\mathcal{K} \subseteq \mathcal{X}$ such that $\Gamma(t,s) \in \mathcal{K}$ for all $t \in [0,1]$ and $s \in \mathcal{S}$. This set depends only on $\overline{\alpha}_1$, x_0, and x_1, and is independent of the choice of $\mathcal{Q}$. Let $M := \max\{|u(k)|;\ \ k \in \mathcal{K}\}$; this maximum is well-defined because the function $|u|$ is continuous and $\mathcal{K}$ is compact. Let $\epsilon := \frac{1}{2}\min\{|E - u(x_0)|, |E - u(x_1)|\}$; then $\epsilon > 0$ by Claim 1. Let $\mathcal{Q}^c := \mathcal{R}_2 \cap (\neg\mathcal{Q})$. Since μ is nonatomic, we can make $\mathcal{Q}$ big enough that $\mu[\mathcal{Q}^c] \leq \epsilon\,\mu[\mathcal{R}_2]/M$.

Finally, define $v(t) := \mathbb{E}_{\mathcal{R}_2}[u \circ \gamma^t_{\restriction \mathcal{R}_2}]$ for all $t \in [0,1]$; this yields a function $v : [0,1] \longrightarrow \mathbb{R}$.

Claim 2: $v(0) < E < v(1)$.

Proof.

$$\begin{aligned}
\mu[\mathcal{R}_2] \cdot \mathbb{E}_{\mathcal{R}_2}[u \circ \gamma^0_{\restriction \mathcal{R}_2}] &\underset{(*)}{=} \mu[\mathcal{Q}] \cdot \mathbb{E}_{\mathcal{Q}}[u \circ \gamma^0_{\restriction \mathcal{Q}}] + \mu[\mathcal{Q}^c] \cdot \mathbb{E}_{\mathcal{Q}^c}[u \circ \gamma^0_{\restriction \mathcal{Q}^c}] \\
&\underset{(\dagger)}{\leq} \mu[\mathcal{Q}] \cdot u(x_0) + \mu[\mathcal{Q}^c] \cdot M \\
&\leq \mu[\mathcal{R}_2] \cdot u(x_0) + M \cdot \epsilon\,\mu[\mathcal{R}_2]/M \\
&= \mu[\mathcal{R}_2] \cdot u(x_0) + \epsilon \cdot \mu[\mathcal{R}_2] \quad = \quad \mu[\mathcal{R}_2] \cdot (u(x_0) + \epsilon).
\end{aligned}$$

Here, $(*)$ is by equation (16), because $\mathcal{R}_2 = \mathcal{Q} \vee \mathcal{Q}^c$. Next, $(\dagger)$ is because $\gamma^0(q) = x_0$ for all $q \in \mathcal{Q}$, while $u(x) \leq M$ for all $x \in \mathcal{X}$.

Dividing both sides by $\mu[\mathcal{R}_2]$, we get $v(0) = \mathbb{E}_{\mathcal{R}_2}[u \circ \gamma^0_{\restriction \mathcal{R}_2}] \leq u(x_0) + \epsilon < E$. Through an identical proof, we establish that $v(1) = \mathbb{E}_{\mathcal{R}_2}[u \circ \gamma^1_{\restriction \mathcal{R}_2}] \geq u(x_1) - \epsilon > E$. $\diamondsuit$ `Claim 2`

Claim 3: $v : [0,1] \longrightarrow \mathbb{R}$ *is a continuous function.*

Proof. Fix $T \in [0,1]$. Lemma C4(e) says that $\lim_{t \to T} \left\| u \circ \gamma^t - u \circ \gamma^T \right\|_\infty = 0$. *A fortiori*, $\lim_{t \to T} \left\| u \circ \gamma^t_{\restriction \mathcal{R}_2} - u \circ \gamma^T_{\restriction \mathcal{R}_2} \right\|_\infty = 0$. Thus, *Unimodularity* property of $\mathbb{E}_{\mathcal{R}_2}$ implies that $\lim_{t \to T} \left| \mathbb{E}_{\mathcal{R}_2}[u \circ \gamma^t_{\restriction \mathcal{R}_2}] - \mathbb{E}_{\mathcal{R}_2}[u \circ \gamma^T_{\restriction \mathcal{R}_2}] \right| = 0$. In other words, $\lim_{t \to T} |v(t) - v(T)| = 0$, meaning that v is continuous at T. This argument works for all $T \in [0,1]$. $\diamondsuit$ `Claim 3`

Combining the Intermediate Value Theorem with Claims 2 and 3, we obtain some $t \in [0,1]$ such that $\mathbb{E}_{\mathcal{R}_2}[u \circ \gamma^t_{\restriction \mathcal{R}_2}] = E$. But then since (u, μ) is an SEU representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$, we conclude that $\gamma^t_{\restriction \mathcal{R}_2} \approx_{\mathcal{R}_2} \alpha_2$. We have already observed that $\gamma^t_{\restriction \mathcal{R}_2}$ is compatible with α_1. Thus, set $\beta := \gamma^t_{\restriction \mathcal{R}_2}$ to prove the theorem.

(b) By contradiction, suppose u was satiating. In particular, suppose u had a global maximizer $x \in \mathcal{X}$. (The argument for a global minimizer is analogous).

Since $\mathcal{S}$ is *not* extremally disconnected, Lemma C3 says there is some $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ which is *not* a clopen set. (It follows that $\emptyset \neq \mathcal{R} \neq \mathcal{S}$.) Let $\mathcal{Q} = \neg\mathcal{R}$; then $\mathcal{Q} \in \mathfrak{R}(\mathcal{S})$ also. Let $o \in \mathcal{X}$ be such that $u(o) < u(x)$. Without loss of generality, suppose $u(o) = 0$ and $u(x) = 1$. Let κ^x be the constant x-valued act, and let κ^o be the constant o-valued act; by hypotheses, both of these are elements of $\mathcal{A}$. Structural condition (Rch) says that we can find some $\alpha^x_{\mathcal{R}} \in \mathcal{A}$ such that $(\alpha^x_{\mathcal{R}})_{\restriction \mathcal{R}} \approx_{\mathcal{R}} \kappa^x_{\mathcal{R}}$, while $(\alpha^x_{\mathcal{R}})_{\restriction \mathcal{Q}} = \kappa^o_{\restriction \mathcal{Q}}$.

Claim 4: $\mathcal{O}_1 := (u \circ \alpha^x_{\mathcal{R}})^{-1}(0, \frac{1}{2})$ *is a nonempty open subset of* $\mathcal{R}$.

Proof. $\mathcal{O}_1$ is the preimage of the open set $(0, \frac{1}{2})$ under the continuous function $u \circ \alpha^x_{\mathcal{R}}$, so $\mathcal{O}_1$ is open. Furthermore, $u \circ \alpha^x_{\mathcal{R}}(q) = u(o) = 0$ for all $q \in \mathcal{Q}$. Thus, by continuity, $u \circ \alpha^x_{\mathcal{R}}(s) = 0$ for all $s \in \mathrm{clos}(\mathcal{Q}) = \mathcal{S} \setminus \mathcal{R}$. Thus, if $\mathcal{O}_1$ is nonempty, it must be a subset of $\mathcal{R}$. It remains to show that $\mathcal{O}_1$ is nonempty.

Now, $\mathcal{R}$ is *not* closed (by construction), so there exists a net $\{r_\lambda\}_{\lambda\in\Lambda}$ (for some directed set Λ) which converges to some point $s \in \mathcal{S} \setminus \mathcal{R}$. Thus, since $u \circ \alpha^x_{\mathcal{R}}$ is continuous, we have $\lim_{\lambda\in\Lambda} u \circ \alpha^x_{\mathcal{R}}(r_\lambda) = u(s) = 0$. In particular, this means that $r_\lambda \in \mathcal{O}_1$ for some $\lambda \in \Lambda$. Thus, $\mathcal{O}_1 \neq \emptyset$. ◇ Claim 4

Now, let $\mathcal{P}_1 := \mathrm{int}[\mathrm{clos}(\mathcal{O}_1)]$; then $\mathcal{P}_1$ is a regular subset of $\mathcal{R}$ (by Lemma C2) which is nonempty by Claim 4 (because it contains $\mathcal{O}_1$). Furthermore, $0 \leq u \circ \alpha^x_{\mathcal{R}}(p) \leq 1/2$ for all $p \in \mathcal{P}_1$. Let $\mathcal{P}_2 := \mathcal{R} \cap (\neg\mathcal{P}_1)$, then $\{\mathcal{P}_1, \mathcal{P}_2\}$ is a regular partition of $\mathcal{R}$, so that

$$\begin{aligned}\mu[\mathcal{R}]\,\mathbb{E}_{\mathcal{R}}[u \circ \alpha^x_{\mathcal{R}}] &\underset{(*)}{=} \mu[\mathcal{P}_1]\,\mathbb{E}_{\mathcal{P}_1}[u \circ \alpha^x_{\mathcal{R}}] + \mu[\mathcal{P}_2]\,\mathbb{E}_{\mathcal{P}_2}[u \circ \alpha^x_{\mathcal{R}}] \underset{(\dagger)}{\leq} \frac{1}{2}\cdot\mu[\mathcal{P}_1] + 1\cdot\mu[\mathcal{P}_2] \\ &\underset{(\diamond)}{<} \mu[\mathcal{P}_1] + \mu[\mathcal{P}_2] = \mu[\mathcal{R}] \underset{(\ddagger)}{=} \mu[\mathcal{R}]\,\mathbb{E}_{\mathcal{R}}[u \circ \kappa^x]. \end{aligned} \tag{C1}$$

Here, $(*)$ is by equation (16), while $(\dagger)$ is by the monotonicity of the expectation operator $\mathbb{E}_{\mathcal{R}}$, because $(u \circ \alpha^x_{\mathcal{R}})(p) \leq \frac{1}{2}$ for all $p \in \mathcal{P}_1$, while $(u \circ \alpha^x_{\mathcal{R}})(p) \leq u(x) = 1$ for all $p \in \mathcal{P}_2$. Next $(\diamond)$ is because $\mu[\mathcal{P}_1] > 0$, because μ has full support. Finally, $(\ddagger)$ is because for all $r \in \mathcal{R}$, we have $\kappa^x(r) = x$, and thus, $u \circ \kappa^x(r) = u(x) = 1$.

Dividing both sides of inequality (C1) by $\mu[\mathcal{R}]$, we obtain:

$$\mathbb{E}_{\mathcal{R}}[u \circ \alpha^x_{\mathcal{R}}] < \mathbb{E}_{\mathcal{R}}[u \circ \kappa^x].$$

Thus, by the assumed SEU representation, we must have $\alpha^x_{\mathcal{R}} \prec_{\mathcal{R}} \kappa^x_{\mathcal{R}}$. But this contradicts the defining property of $\alpha^x_{\mathcal{R}}$.

To avoid the contradiction, x *cannot* be a global maximizer for u.

(c) Fix two disjoint regular subsets $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}(\mathcal{S})$, and two acts $\alpha_1 \in \mathcal{A}(\mathcal{R}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{R}_2)$. There exist $\tilde{\alpha}_1 \in \mathcal{A}$ and $\tilde{\alpha}_2 \in \mathcal{A}$ such that $\alpha_1 = \tilde{\alpha}_{1|\mathcal{R}_1}$ and $\alpha_2 = \tilde{\alpha}_{2|\mathcal{R}_2}$. Define $\beta \in \mathcal{A}$ by $\beta(s) = \tilde{\alpha}_2(s)$ if $s \in \mathcal{R}_2$ and $\beta(s) = \tilde{\alpha}_1(s)$ otherwise.

Let us show that $\beta \in \mathcal{A}$. For any open subset $\mathcal{V}$ of $\mathcal{X}$, we have by construction that $\beta^{-1}(\mathcal{V}) = (\mathcal{R}_2 \cap \tilde{\alpha}_2^{-1}(\mathcal{V})) \cup (\mathcal{R}_2^{\complement} \cap \tilde{\alpha}_1^{-1}(\mathcal{V}))$. Now, $\mathcal{S}$ is extremally disconnected, so Lemma C3 says that $\mathcal{R}_2$ is clopen. Thus, $\mathcal{R}_2^{\complement}$ is open, so $\mathcal{R}_2^{\complement} = \neg\mathcal{R}_2$. Thus, $\beta^{-1}(\mathcal{V}) = (\mathcal{R}_2 \cap \tilde{\alpha}_2^{-1}(\mathcal{V})) \cup (\neg\mathcal{R}_2 \cap \tilde{\alpha}_1^{-1}(\mathcal{V}))$. Since $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are both continuous, and $\mathcal{R}_2$ and $\neg\mathcal{R}_2$ are both open, we finally obtain that $\beta^{-1}(\mathcal{V})$ is open in $\mathcal{S}$. Hence the continuity of β. To see that β is also bounded, note that $\beta(\mathcal{S}) = (\mathcal{R}_2 \cap \tilde{\alpha}_2(\mathcal{S})) \cup (\mathcal{R}_2^{\complement} \cap \tilde{\alpha}_1(\mathcal{S}))$. Therefore, $\beta(\mathcal{S}) \subseteq \mathrm{clos}(\tilde{\alpha}_1(\mathcal{S})) \cup \mathrm{clos}(\tilde{\alpha}_2(\mathcal{S}))$ and thus $\mathrm{clos}(\beta(\mathcal{S})) \subseteq \mathrm{clos}(\tilde{\alpha}_1(\mathcal{S})) \cup \mathrm{clos}(\tilde{\alpha}_2(\mathcal{S}))$. By assumption, each of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ is bounded, and both $\mathrm{clos}(\tilde{\alpha}_1(\mathcal{S}))$ and $\mathrm{clos}(\tilde{\alpha}_2(\mathcal{S}))$ are compact. Then, $\mathrm{clos}(\tilde{\alpha}_1(\mathcal{S})) \cup \mathrm{clos}(\tilde{\alpha}_2(\mathcal{S}))$ is also compact. Finally, $\mathrm{clos}(\beta(\mathcal{S}))$ is closed in a compact set and, therefore, compact. Hence boundedness. So $\beta \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X}) = \mathcal{A}$.

Now, since $\beta_{|\mathcal{R}_1} = \tilde{\alpha}_{1|\mathcal{R}_1} = \alpha_1$ and $\beta_{|\mathcal{R}_2} = \tilde{\alpha}_{1|\mathcal{R}_2} = \alpha_2$, we have that α_1 and α_2 are compatible. Last, we naturally have $\alpha_2 \approx_{\mathcal{R}_2} \alpha_2$, which completes the proof of (Rch). □

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