Sovereign Debt and Structural Reforms*

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Abstract

Motivated by the European debt crisis, we construct a tractable theory of sovereign debt and structural reforms under limited commitment. A sovereign country which has fallen into a recession of an uncertain duration issues one-period debt to smooth consumption. It can also implement structural policy reforms to speed up recovery from the recession. The sovereign can renege on its obligations by suffering an observable stochastic default cost, in which case creditors offer a take-it-or-leave-it debt haircut to avert default. We characterize the competitive Markov equilibrium and compare it to the constrained efficient allocation. The competitive equilibrium features large fluctuations in consumption, debt, and reform effort. During the recession, consumption falls whenever debt is honored. A large debt destroys incentives for structural reforms since some of the gains from reform accrue to the lenders. The constrained optimum yields step-wise increasing consumption and step-wise decreasing reform effort. Markets for state-contingent debt alone cannot restore efficiency. The optimum can be interpreted as a flexible assistance program providing budget support during recession followed by a debt increase after recovery. Restrictions are imposed on debt issuance and reform effort. The terms of the program are improved each time the country can credibly threaten to leave.

JEL Codes: E62, F33, F34, F53, H12, H63

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1 Introduction

The recent European debt crisis has revamped the interest in understanding the dynamics of sovereign debt during economic downturns. Economic theory offers two simple policy prescriptions for countries suffering a temporary decline in output. First, they should borrow on international markets to smooth consumption. Second, they should undertake reforms – possibly painful ones – to speed up economic recovery. However, the fact that there is limited enforcement of sovereign debt may have major effects on the simple policy prescriptions. On the one hand, risk sharing may be hampered by rising default premia associated with the risk of debt renegotiations.[1] On the other hand, a large debt can reduce the borrower's incentive to undertake economic reforms to boost economic growth since some of the gains from growth would accrue to the lenders.

This paper proposes a theory of sovereign debt to address the following questions. First, what does the optimal dynamic contract between a planner and a sovereign, temporarily impoverished, country prescribe in an environment where the country cannot commit to honoring its debt and costly reforms can speed up economic recovery? Second, does the market equilibrium attain the efficient level of structural reforms and consumption smoothing? If not, what policies and institutional arrangements can improve efficiency?

The theory rests on three building blocks. The first is that sovereign debt is subject to limited enforcement, and that countries can renege on their obligations subject to real costs, as in e.g. Aguiar and Gopinath (2006), Arellano (2008) and Yue (2010). Different from this literature, we assume the size of default costs to be stochastic, reflecting exogenous changes in the domestic and international situation. The second building block is that whenever creditors face a credible default threat, they can make a take-it-or-leave-it renegotiation offer to the indebted country. This approach conforms with the empirical observations that unordered defaults are rare events, and that there is great heterogeneity in the terms at which debt is renegotiated, as documented by Tomz and Wright (2007) and Sturzenegger and Zettelmeyer (2008). The third building block is the possibility for the government of the indebted country to make "structural" policy reforms that speed up recovery from an existing recession. Examples of such reforms include labor and product market deregulation, and the establishment of fiscal capacity that allows the government to raise tax revenue efficiently (see, e.g., Ilzkovitz and Dierx 2011). While these reforms are beneficial in the long run, they may entail short-run costs for citizens at large, governments or special-interest groups (see, e.g., Blanchard and Giavazzi 2003, and Boeri 2005).

More formally, we construct a dynamic model of sovereign debt. Productivity is subject to aggregate shocks following a two-state Markov process. A benevolent local government can issue sovereign debt to smooth consumption.[2] The sovereign country starts in a recession of an unknown duration. The probability that the recession ends is endogenous, and hinges on the government's reform effort. Debt issuance is subject to a limited-commitment problem: the government can, *ex-post,* repudiate its debt, based on the publicly observable realization of a stochastic default cost. When this realization is sufficiently low relative to the outstanding debt, the default threat is credible. In this case, a

[1] Greece, the hardest hit country in Europe, saw its debt-GDP ratio soar from 107% in 2008 to 170% in 2011, at which point creditors had to agree to a debt haircut implying a 53% loss on its face value. Meanwhile, international organizations stepped in to provide financial assistance and access to new loans, asking in exchange fiscal austerity and a commitment to economic reforms. A new crisis with a request from the Greek government of a haircut on the outstanding debt was barely averted in the summer of 2015.

[2] In the benchmark model, sovereign debt is a non-state-contingent bond. Our results do not hinge on this assumption, and we show in an extension that if the government could issue state-contingent debt, i.e., securities whose pay-off is contingent to the realization of the aggregate productivity, the main results would remain unchanged.

syndicate of creditors makes a take-it-or-leave-it debt haircut offer, as in Bulow and Rogoff (1989). In equilibrium, there is no outright default, but debt renegotiations. Haircuts are more frequent during recession, and more frequent the larger is the outstanding sovereign debt. Consumption increases after a renegotiation, in line with the empirical evidence that economic conditions improve in the aftermath of debt relief, as documented in Reinhart and Trebesch (2016).

We first characterize the laissez-faire equilibrium. During recessions, the government would like to issue debt in order to smooth consumption. However, as debt accumulates, the probability of renegotiation increases, implying a rising risk premium and consumption volatility. The reform effort exhibits a non-monotonic pattern: it is increasing with debt at low levels of debt because of the disciplining effect of recession. However, for sufficiently high debt levels the relationship is flipped because of a debt overhang problem: very high debt levels deter useful reforms because most of the gains from the reforms accrue to foreign lenders in the form of capital gains on the outstanding debt. The moral hazard problem exacerbates the country's inability to achieve consumption smoothing: at high debt levels, creditors expect little reform effort, are pessimistic about the economic outlook, and request an even higher risk premium.

Next, to establish a normative benchmark, we characterize the optimal dynamic contract between a planner without enforcement power and a country that has fallen into recession. In contrast to the laissez-faire equilibrium, the constrained optimal allocation features non-decreasing consumption and non-increasing reform effort during the recession. More precisely, consumption and effort remain constant whenever the country's participation constraint is not binding. However, when the constraint is binding the planner increases the country's promised utility and consumption, and reduces the required reform effort.

Having characterized the constrained efficient allocation, we consider its implementation in a decentralized environment. We first show that, in the absence of aggregate productivity shocks, the laissez-faire equilibrium attains the constrained-efficient allocation. However, the equilibrium is not constrained-efficient when the economy is in a recession. In the laissez-faire equilibrium, consumption and reform effort fluctuate too much, and reform effort is inefficient. The inefficiency cannot be overcome by allowing the government to issue state-contingent debt. In standard models, state-contingent debt would provide insurance against the continuation of a recession – i.e., Arrow securities paying off conditional on the aggregate state, recession or normal times. However, the better insured the country is, the more severe the moral hazard problem becomes. For instance, full insurance would remove incentives to exert reform effort. Since this is priced into the debt, the social value of state-contingent debt is limited. In a calibrated version of the model, we show that state-contingent debt yields very small welfare gains relative to the equilibrium with non-state contingent debt.

The constrained optimal allocation can be attained through the intervention of an independent institution (e.g., the IMF) that monitors the country's fiscal policy and the structural reform program. During the recession the optimal program entails a persistent budget support through extending loans on favorable terms, combined with a larger reform effort than the borrower would choose on its own. Upon recovery from the recession, the sovereign is settled with a (large) debt on market terms. A common objection to schemes implying deferred repayment is that the country may refuse to repay its loans when the economy recovers. In our theory, this risks is factored in as part of the contract. The optimal program has the interesting feature that, whenever the country can credibly threaten to default, the international institution should relent and improve the terms of the agreement for the debtor by granting her higher consumption and a lower reform effort. In other words, the austerity program is relaxed over time, whenever necessary to avert the breakdown of the program.

We provide a quantitative evaluation of the theory with the aid of a calibrated version of the model.

The model matches realistic debt-to-GDP ratios, as well as default premia, renegotiation frequencies, and recovery rates. We regard this as a contribution in itself. In the existing quantitative literature, it is difficult to sustain high debt levels, contrary both to the observation that many countries have managed to finance debt-GDP ratios above 100%, and to the estimates of a recent study by Collard, Habib, and Rochet (2015) showing that OECD countries can sustain debt-GDP ratios even in excess of 200%. We find that an assistance program implementing the constrained optimum yields large welfare gains, equivalent to a transfer of 45% of the initial GDP with a zero expected cost for the institution running the assistance program.

1.1 Literature review

Our paper relates to several streams of the literature on sovereign debt. By focusing on Markov equilibria, we abstract from reputational mechanisms, being close in the spirit to the direct-punishment approach proposed by Bulow and Rogoff (1989).[3] Our work is related to the more recent quantitative models of sovereign default such as Aguiar and Gopinath (2006) and Arellano (2008). Yue (2010) considers, as we do, the possibility of renegotiation, although in her model renegotiation is costly and is determined by Nash bargaining between creditors and debtors - with no stochastic shocks to outside options. In her model, ex-post inefficient restructuring helps ex-ante discipline and provides incentives to honor the debt.[4] This literature does not consider the efficient allocation nor economic reforms. Moreover, we pursue an analytical characterization of the properties of the model, whereas the main focus of this literature has been quantitative.

In terms of reform effort, our paper is related to Krugman (1988), Atkeson (1991) and Jeanne (2009). Krugman (1988) showed that when a borrower has a large debt, productive investments might not be undertaken (the "debt overhang"). His is a static model where debt is exogenous. Atkeson (1991) studies an environment in which an infinitely-lived borrower faces a sequence of two-period lived lenders. The borrower can use funds to invest in productive future capacity or to consume the funds. However, the lenders cannot observe the allocation to investment or consumption. Our paper differs from Atkeson's in three key respects. First, in our model we focus on Markov equilibria where the borrower cannot commit the reform effort, but the lender can observe it. This seems a plausible abstraction in the context of, for example, the European debt crisis. Second, in our theory structural reforms affect the future stochastic process of income, while his model investments only affect next period's income. Third, in our model all agents (and the planner) have an infinite horizon. The results are different. Atkeson (1991) shows that the optimal contract involves capital outflow from the borrower during the worst aggregate state. Our model predicts instead that in a recession the borrower keeps accumulating debt and renegotiates it periodically. Moreover, in our model the constrained optimal allocation (though not necessarily the market equilibrium) has non-decreasing consumption. Jeanne (2009) studies an economy where the government takes a policy action that affects the return to foreign investors (e.g., the enforcement of creditor's right) but this can be reversed within a time horizon that is shorter than that at which investors must commit their resources.

Hopenhayn and Werning (2008) study the optimal corporate debt contract between a bank and a

[3] The distinction between the reputation approach and the punishment approach as the two main conceptual frameworks in the literature on sovereign debt crisis has been introduced recently by Bulow and Rogoff (2015).

Pioneer contributions to the analysis of debt repudiation based on reputational mechanisms such as the threat of future exclusion from credit markets include Eaton and Gersovitz (1981), Grossman and Van Huyck (1988), and Fernandez and Rosenthal (1989).

[4] Other papers focusing on the restructuring of sovereign debt include Asonuma and Trebesch (2016), Benjamin and Wright (2009), Bolton and Jeanne (2007), Dovis (2014), and Mendoza and Yue (2012).

risk-neutral borrowing firm. As we do, they assume that the borrower has a stochastic default cost. Different from us, they focus on the case when this outside option is not observable to the lender and show that this implies that default can occur in equilibrium. They do not study reform effort nor do they analyze the case of sovereign debt issued by a country in recession.

Another recent paper complementary to ours is Conesa and Kehoe (2015). In their theory, under some circumstances, the government of the indebted country may opt to "gamble for redemption." Namely, it runs an irresponsible fiscal policy that sends the economy into the default zone if the recovery does not happen soon enough. The source and the mechanism of the crisis are different from ours. Their model is based on the framework of Cole and Kehoe (1996, 2000) inducing multiple equilibria and sunspots.

Our paper is related also to the literature on endogenous incomplete markets due to limited enforcement or limited political commitment. This includes Alvarez and Jermann (2000) and Kehoe and Perri (2002). The analysis of constrained efficiency is related to the literature on competitive risk sharing contracts with limited commitment, including Thomas and Worrall (1988), Kocherlakota (1996), and Krueger and Uhlig (2006). An application of this methodology to the optimal design of a Financial Stability Fund is provided by Abraham, Carceles-Poveda, and Marimon (2014). In our model all debt is held by foreign lenders. Recent papers by Broner, Martin, and Ventura (2010), Broner and Ventura (2011), and Brutti and Sauré (2016) study the implications for the incentives to default of having part of the government debt held by domestic residents. Song *et al.* (2012) and Müller *et al.* (2016) focus, as we do, on Markov equilibria to study the politico-economic determination of debt in open economies where governments are committed to honor their debt. An excellent review of the sovereign debt literature is provided by Aguiar and Amador (2014).

In the large empirical literature, our paper is related to the finding of Tomz and Wright (2007). Using a dataset for the period 1820–2004, they find a negative but weak relationship between economic output in the borrowing country and default on loans from private foreign creditors. While countries default more often during recessions, there are many cases of default in good times and many instances in which countries have maintained debt service during times of very bad macroeconomic conditions. They argue that these findings are at odds with the existing theories of international debt. Our theory is consistent with the pattern they document. In our model, due to the stochastic default cost, countries may default during booms (though this is less likely, consistent with the data) and can conversely fail to renegotiate their debt during very bad times. Their findings are reinforced by Sturzenegger and Zettelmeyer (2008) who document that even within a relatively short period (1998-2005) there are very large differences between average investor losses across different episodes of debt restructuring. The observation of such a large variability in outcomes is in line with our theory, insofar as the bargaining outcome hinges on an outside option that is subject to stochastic shocks. Borensztein and Panizza (2009) evaluate empirically the costs that may result from an international sovereign default, including reputation costs, international trade exclusion, costs to the domestic economy through the financial system, and political costs to the authorities. They find that the economic costs are generally short-lived. Finally, the relationship between consumption and renegotiations is in line with the evidence documented by Reinhart and Trebesch (2016), as discussed above. For a thorough review of the evidence, see also Panizza *et al.* (2009).

2 The model environment

The model economy is a small open endowment economy populated by an infinitely-lived representative agent. The endowments follow a two-state Markov switching process, with realizations $w \in \{\underline{w}, \bar{w}\}$, where $0 < \underline{w} < \bar{w}$. We label the two endowment states, respectively, *recession* and *normal times*. Normal times is assumed to be an absorbing state. If the economy starts in a recession, it switches to normal times with probability p and remains in the recession with probability $1-p$. The assumption that normal times is an absorbing state aids tractability and allows us to focus sharply on the salient aspects of a recovery from a deep economic crisis. A benevolent government can implement a costly reform policy to increase the probability of a recovery. In our notation, p is both the *reform effort* and the probability that the recession ends.

The preferences of the representative agent are described by the following expected utility function:

$$E_0 \sum \beta^t \left[u(c_t) - \phi I_{\{\text{default in } t\}} - X(p_t) \right].$$

The utility function u is twice continuously differentiable and satisfies $\lim_{c \to 0} u(c) = -\infty$, $u'(c) > 0$, and $u''(c) < 0$. $I \in \{0,1\}$ is an indicator switching on when the economy is in a default state and ϕ is a stochastic default cost assumed to be *i.i.d.* over time and to be drawn from the p.d.f. $f(\phi)$ with an associated c.d.f. $F(\phi)$. We assume that $F(\phi)$ is continuously differentiable everywhere, and denote its support by $\aleph \equiv [0, \phi_{\max}] \subseteq \mathbb{R}^+$, where $\phi_{\max} < \infty$. The assumption that shocks are independent is inessential, but aids tractability. X is the cost of reform, assumed to be an increasing convex function of the probability of exiting recession, $p \in [\underline{p}, \bar{p}] \subseteq [0,1]$. X is assumed to be twice continuously differentiable, with the properties that $X(\underline{p}) = 0$, $X'(p) > 0$ and $X''(p) > 0$. In normal times, $X = 0$.

To establish a benchmark, we characterize the optimal allocation under full insurance and full enforcement. The economy is assumed to start with an outstanding obligation of b given an implicit gross rate of return of $R = \beta^{-1}$. The first best entails perfect insurance: the country enjoys a constant stream of consumption and exerts a constant reform effort during recession (during normal times, there is no effort). The level of b lowers consumption and increases reform effort in recession.

Proposition 1 *Let $W^{FB}(b,w)$, $c^{FB}(b,w)$ and $p^{FB}(b)$ denote, respectively, the discounted utility, consumption and effort as a function of the outstanding obligation b, with $w \in \{\underline{w}, \bar{w}\}$ denoting the initial state of productivity. Then, for an economy starting in normal state:*

$$c^{FB}(b,\bar{w}) = \bar{w} - (1-\beta) b, \qquad W^{FB}(b,\bar{w}) = \frac{u\left(c^{FB}(b,\bar{w})\right)}{1-\beta}.$$

For an economy starting in recession:

$$\begin{aligned} c^{FB}(b,\underline{w}) &= \frac{(1-\beta)\underline{w} + \beta p^{FB}(b)\bar{w}}{1-\beta(1-p^{FB}(b))} - (1-\beta) b, \\ W^{FB}(b,\underline{w}) &= \frac{u\left(c^{FB}(b,\underline{w})\right)}{1-\beta} - \frac{X\left(p^{FB}(b)\right)}{1-\beta(1-p^{FB}(b))} \end{aligned}$$

where $p^{FB}(b)$ is the reform effort exerted for as long as the economy stays in recession. $p^{FB}(b)$ is the unique solution for p^{FB} satisfying the following condition:

$$\frac{\beta}{1-\beta(1-p^{FB})} \left(\underbrace{(\bar{w}-\underline{w}) \times u'\left(c^{FB}(b,\underline{w})\right)}_{\textit{increase in output if econ. recovers}} + \underbrace{X\left(p^{FB}\right)}_{\textit{saved effort cost if econ. recovers}} \right) = X'\left(p^{FB}\right). \qquad (1)$$

Moreover, when effort is interior, $c^{FB}(b, \underline{w})$ and $p^{FB}(b)$ are, respectively, decreasing and increasing functions of b.

3 Laissez-faire equilibrium

In the laissez-faire equilibrium, the benevolent government can issue a one-period bond (sovereign debt) to smooth consumption. The bond, b, is a claim to one unit of the next-period consumption good, which sells today at the price $Q(b, w)$. Bonds are purchased by a representative risk neutral foreign creditor who has access to an international risk-free portfolio paying the world interest rate R. For simplicity, we focus on the case in which $\beta R = 1$, although our main insights would carry over to the case in which $\beta R < 1$. After issuing debt, the country decides its reform effort.

The key assumptions are that (i) the country cannot commit to repay its sovereign debt, and (ii) the reform effort is not contractible. At the beginning of each period, the government observes the realization of the default cost ϕ, and decides whether to repay the debt that reaches maturity or to announce default on all its debt. The cost ϕ is publicly observed, and captures in a reduced form a variety of shocks including both taste shocks (e.g., the sentiments of the public opinion about defaulting on foreign debt) and institutional shocks (e.g., the election of a new prime minister, a new central bank governor taking office, the attitude of foreign governments, etc.).[5] If a country defaults, no debt is reimbursed.[6]

When the government announces its intention to default, a syndicate of creditors can make a take-it-or-leave-it renegotiation offer that we assume to be binding for all creditors. By accepting the renegotiation offer, the government averts the default cost. In equilibrium, a haircut is offered only if the default threat is credible, i.e., if the realization of ϕ is sufficiently low to make the country prefer default to full repayment. When they offer renegotiation, creditors make the debtor indifferent between an outright default and the proposed haircut.

In summary, the timing is as follows: The government enters the period with the pledged debt b, observes the realization of w and ϕ, and then decides whether to announce default. If the threat is credible, the creditors offer a haircut. Next, the country decides whether to accept or decline the offer. Then, the government issues new debt subject to the period budget constraint $Q \times b' = b + c - w$. For technical reasons we also impose that debt is bounded, $b \in [\underline{b}, \tilde{b}]$ where $\underline{b} \in (-\infty, 0]$ and $\tilde{b} = \bar{w}/(R-1)$ is the natural borrowing constraint. In equilibrium, these bounds will never be binding. If the country could commit to honor its debt, it would sell bonds at the price $Q = 1/R$. However, due to the risk of default or renegotiation, it sells at a discount, $Q \leq 1/R$. Next, consumption is realized, and finally the government decides its reform effort.

[5] Alternatively, ϕ could be given a politico-economic interpretation, as reflecting special interests of the groups in power. For instance, the government may care about the cost of default to its constituency rather than to the population at large. In the welfare analysis, we stick to the interpretation of a benevolent government and abstract from politico-economic factors, although the model could be extended in this direction.

[6] For simplicity, we assume that ϕ captures all costs associated with default. In an earlier version of this paper, we assumed that the government could not issue new debt in the default period, but were allowed to start issuing bonds already in the following period. The results are unchanged. One could even consider richer post-default dynamics, such as prolonged or stochastic exclusion from debt markets. Since outright default does not occur in equilibrium, the details of the post-default dynamics are immaterial.

3.1 Definition of Markov equilibrium

In the characterization of the laissez-faire equilibrium, we restrict attention to Markov-perfect equilibria where the set of equilibrium functions only depend on the pay-off relevant state variables, b and ϕ. This rules out that the government's decisions can be affected by the desire to establish or maintain a reputation.

Definition 1 *A Markov-perfect equilibrium is a set of value functions $\{V, W\}$, a threshold renegotiation function Φ, an equilibrium debt price function Q, a set of optimal decision rules $\{\mathbb{B}, B, C, \Psi\}$, equilibrium law of motions of debt B and of the aggregate state w, such that, conditional on the state vector $(b, \phi, w) \in \left([\underline{b}, \tilde{b}] \times [0, \phi_{\max}] \times \{\underline{w}, \bar{w}\}\right)$, the sovereign and the international creditors maximize utility, and markets clear. More formally:*

- *The value function V satisfies*

$$V(b, \phi, w) = \max\{W(b, w), W(0, w) - \phi\}, \tag{2}$$

where $W(b, w)$ is the value function conditional on the debt level b being honored,

$$W(b, w) = \max_{b' \in [\underline{b}, \tilde{b}]} u\left(Q\left(b', w\right) \times b' + w - b\right) + Z\left(b', w\right),$$

and where Z is defined as

$$Z\left(b', \underline{w}\right) = \max_{p \in [\underline{p}, \bar{p}]} \left\{-X(p) + \beta\left(p \times E\left[V\left(b', \phi', \bar{w}\right)\right] + (1-p) \times E\left[V\left(b', \phi', \underline{w}\right)\right]\right)\right\}, \tag{3}$$

$$Z\left(b', \bar{w}\right) = \beta E\left[V\left(b', \phi', \bar{w}\right)\right] \tag{4}$$

and $E\left[V\left(x, \phi', w\right)\right] = \int_{\aleph} V(x, \phi, w)\, dF(\phi)$.

- *The threshold renegotiation function Φ satisfies*

$$\Phi(b, w) = W(0, w) - W(b, w). \tag{5}$$

- *The debt price function satisfies the following arbitrage conditions:*

$$Q(b, \bar{w}) = \hat{Q}(b, \bar{w}) \tag{6}$$

$$Q(b, \underline{w}) = \Psi(b) \times \hat{Q}(b, \bar{w}) + [1 - \Psi(b)] \times \hat{Q}(b, \underline{w}) \tag{7}$$

where $\hat{Q}(b, w)$ is the bond price conditional on next period being in state w,

$$\hat{Q}(b, w) \equiv \frac{1}{R}\left(1 - F(\Phi(b, w))\right) + \frac{1}{R}\frac{1}{b}\int_0^{\Phi(b,w)} \hat{b}(\phi, w) \times f(\phi)\ d\phi, \tag{8}$$

and where $\hat{b}(\phi, w)$ is the new debt after a renegotiation given a realization ϕ. $\hat{b}$ is implicitly defined by the condition $W\left(\hat{b}(\phi, w), w\right) = W(0, w) - \phi$.

- *The set of optimal decision rules comprises:*

1. *A take-it-or-leave-it debt renegotiation offer:*

$$\mathbb{B}(b,\phi,w)=\begin{cases}\hat{b}(\phi,w) & \text{if} \quad \phi\leq\Phi(b,w),\\ b & \text{if} \quad \phi>\Phi(b,w).\end{cases} \tag{9}$$

2. *An optimal debt accumulation and an associated consumption decision rule:*

$$B(\mathbb{B}(b,\phi,w),w)=\arg\max_{b'\in[\underline{b},\tilde{b}]}\left\{u\left(Q\left(b',w\right)\times b'+w-\mathbb{B}(b,\phi,w)\right)+Z\left(b',w\right)\right\}, \tag{10}$$

$$C(\mathbb{B}(b,\phi,w),w)=Q(B(\mathbb{B}(b,\phi,w),w),w)\times B(\mathbb{B}(b,\phi,w),w)+w-\mathbb{B}(b,\phi,w). \tag{11}$$

3. *An optimal effort decision rule:*

$$\Psi\left(b'\right)=\arg\max_{p\in[\underline{p},\bar{p}]}\left\{-X(p)+\beta\left(p\times E\left[V\left(b',\phi',\bar{w}\right)\right]+(1-p)\times E\left[V\left(b',\phi',\underline{w}\right)\right]\right)\right\}. \tag{12}$$

- *The equilibrium law of motion of debt is* $b'=B(\mathbb{B}(b,\phi,w),w)$.
- *The probability that the recession ends is* $p=\Psi(b')$.

V and W denote the value functions of the benevolent government. Equation (2) states that when $\phi>\Phi(b,w)$ there is no renegotiation, otherwise the government renegotiates its debt. Since, *ex-post*, creditors have all the bargaining power, the discounted utility accruing to the sovereign equals the value that she would get under outright default. Thus,

$$V(b,\phi,w)=\begin{cases}W(b,w) & \text{if} \quad b\leq\hat{b}(\phi,w),\\ W(0,w)-\phi & \text{if} \quad b>\hat{b}(\phi,w).\end{cases}$$

Consider, next, the equilibrium debt price function. Since creditors are risk neutral, the expected rate of return on the sovereign debt must equal the risk-free rate of return. Then, the arbitrage conditions (6)–(7) ensure market clearing in the bond market and pin down the equilibrium bond price in normal times and recession, respectively. The function $\hat{Q}$ defined in equation (8) yields the bond price after the state w has realized but before knowing ϕ. With probability $1-F(\Phi(b,w))$ debt is honored, where $\Phi(b,w)$ denotes the threshold default shock realization such that, conditional on the debt b, the government cannot credibly threaten to default for all $\phi\geq\Phi(b,w)$. With probability $F(\Phi(b,w))$, debt is renegotiated to a level that depends on the realization of ϕ. This level is given by $\hat{b}(\phi,w)$ which, recall, denotes the renegotiated debt level that keeps the government indifferent between accepting the creditors' offer and defaulting. In the rest of the paper, we use the more compact notation $EV(b,w)\equiv E[V(b,\phi,w)]$ and $EV(b',w)\equiv E\left[V\left(b',\phi',w\right)\right]$.

Consider, finally, the set of decision rules. (9) stipulates that creditors always extract the entire surplus at the renegotiation stage. Equations (10)-(11) yield the optimal consumption-saving decisions subject to a resource constraint. Equation (12) yields the optimal effort decision. Note that the effort exerted depends on b', since effort is chosen after the new debt is issued.

3.2 Existence of a laissez-faire Markov equilibrium

The crux for proving the existence of a Markov equilibrium lies in establishing the existence of the value function W. Once this is done, all the equilibrium functions $(V,\Phi,\hat{b},Q,Z,\mathbb{B},B,\Psi)$ can be derived from the set of definitions above. Before proving the existence of W, we establish an intuitive property linking $\hat{b}$ and Φ:

Lemma 1 *Suppose a value function $W(b,w)$ exists and is strictly decreasing in b. Then, $\hat{b}(\Phi(b,w),w) = b$. Moreover, $\Phi(b,w)$ is strictly increasing in b, hence, $\hat{b}(\phi,\bar{w}) = \bar{\Phi}^{-1}(\phi)$ and $\hat{b}(\phi,\underline{w}) = \underline{\Phi}^{-1}(\phi)$, where $\bar{\Phi}(b) \equiv \Phi(b,\bar{w})$, and $\underline{\Phi}(b) \equiv \Phi(b,\underline{w})$.*

The lemma follows from the definitions of $\hat{b}$ and Φ. On the one hand, $\hat{b}(\phi,w)$ is the debt level that, conditional on ϕ, makes the debtor indifferent between honoring and defaulting. On the other hand, $\Phi(b,w)$ is the realization of ϕ that, conditional on b, makes the debtor indifferent between honoring and defaulting.

The next proposition establishes the existence of a Markov equilibrium. The proof establishes that the value function W is a fixed-point of a monotone mapping following Theorem 17.7 in Stokey and Lucas (1989).

Proposition 2 *There exists a Markov equilibrium, i.e., a set of equilibrium functions $(V,W,\Phi,Q,\mathbb{B},B,C,\Psi)$ satisfying Definition 1. The value functions V and W are continuous and non-increasing in b. The equilibrium functions Φ, Q and Ψ are also continuous in b. The bond revenue, $Q(x,w)x$, is non-decreasing in x. The policy function $B(b,w)$ is non-decreasing in b.*

Proposition 2 establishes the existence but not the uniqueness of the Markov equilibrium. The corollary of Theorem 17.7 in Stokey and Lucas (1989) provides a strategy to verify numerically whether an equilibrium is unique. We state this as Corollary 1 in the appendix.

The next proposition establishes local differentiability properties of the value function W, and that the first-order conditions are necessary. Due to the possibility that debt is renegotiated, the value functions are not necessarily concave. Nevertheless, we establish that the equilibrium functions are differentiable at all debt levels that can result from an optimal choice given some initial debt level. We define formally the set of such debt levels as $\mathcal{B}(w) = \{x \in [\underline{b},\tilde{b}] \mid B(b,w) = x, \text{ for some } b \in [\underline{b},\tilde{b}]\}$.[7]

Proposition 3 *The equilibrium functions $W(b,w)$, $Z(b,w)$, $\Phi(b,w)$, $Q(b,w)$, $\ominus(w)$ are differentiable for all $b \in \mathcal{B}(w)$. Moreover, for any $b' \in \mathcal{B}(w)$, the first-order condition $(\partial/\partial b')\, u(Q(b',w)b' + w - b) + (\partial/\partial b')\, Z(b',w) = 0$ and the envelope condition $\partial W(b',w)/\partial b' = -u'(C(b',w))$ hold true.*

The proof is an application of the envelope theorem of Clausen and Strub (2013) that applies to problems including endogenous functions such as default probabilities and interest rates (see also Arellano *et al.* 2014).

3.3 Equilibrium in normal times

In this section, we characterize the equilibrium when the economy is in normal times ($w = \bar{w}$), in which case there is no aggregate uncertainty. The next lemma establishes properties of the threshold and of the debt revenue function, $Q(x,\bar{w})\,x$ at the optimal interior debt choice.

Lemma 2 *The debt revenue function $Q(b',\bar{w})b'$ is concave in $b' \in [\underline{b},\tilde{b}]$ and strictly increasing and differentiable for all $b' \in \mathcal{B}(w)$ such that $b < \bar{b}$. Moreover, $\partial(Q(b',\bar{w})\,b')/\partial b' = R^{-1}(1 - F(\Phi(b',\bar{w})))$.*

[7] We prove later that during normal times the equilibrium functions are differentiable everywhere. However, this is not true in recession. In this case Proposition 3 shows that differentiability still holds for all b that can be attained as an optimal choice.

An immediate implication of the lemma is that if we define $\bar{b}$ to be the lowest debt inducing renegotiation almost surely (i.e., such that $\lim_{b' \to \bar{b}} F(\Phi(b', \bar{w})) = 1$), then, $\bar{b}$ is also the top of the Laffer curve, i.e., the endogenous debt limit. More formally, $\bar{b} = \arg\max_{b \in [\underline{b}, \tilde{b}]} \{Q(b, \bar{w}) b\} < \tilde{b}$. Although the borrower could issue debt exceeding $\bar{b}$, the marginal debt revenue would be zero for $b > \bar{b}$ since this debt would never be honored.

We now study consumption and debt dynamics. We introduce a definition that will be useful throughout the paper.

Definition 2 *A Conditional Euler Equation (CEE) describes the (expected) marginal rate of substitution between current and next-period consumption in all states of nature ϕ' that induce the government to honor its debt next period.*

Next, we characterize formally the CEE. The government solves the consumption-saving problem given by (10). The first-order condition and the envelope theorem yield the following proposition.

Proposition 4 *If the realization of ϕ' induces no renegotiation, then the following CEE holds true:*

$$\beta R \frac{u'(c'|_{H,\bar{w}})}{u'(c)} = 1, \tag{13}$$

where $c = C(\mathbb{B}(b, \phi, \bar{w}), \bar{w})$ is current consumption and $c'|_{H,\bar{w}} = C(b', \bar{w}) = C(B(\mathbb{B}(b, \phi, \bar{w}), \bar{w}), \bar{w})$ is next-period consumption. Since $\beta R = 1$, then $b' = B(b, \bar{w}) = b$, and consumption remains constant. Moreover, for all $b < \bar{b}$, the value function $W(b, \bar{w})$ is strictly decreasing, strictly concave and twice continuously differentiable in b, and consumption $C(b, \bar{w})$ is strictly falling in b.

Although the CEE (13) resembles a standard Euler equation under full commitment, the similarity is deceiving: R is *not* the ex-post interest rate when debt is fully honored; the realized interest rate is in fact higher due to the default premium.

When debt is renegotiated, consumption increases discretely, hence $u'(c_t)/u'(c_{t+1}) > \beta R$. This is not surprising, since the country benefits from a reduction in debt repayment.[8] Thus, consumption and debt are, respectively, increasing and decreasing step functions over time: they remain constant in every period in which the country honors its debt, while changing discretely upon every episode of renegotiation. Figure 1 illustrates a simulation of the consumption and debt dynamics. Note that the sequence of renegotiations eventually brings the debt to a sufficiently low level where the risk of renegotiation vanishes. This consumption path is different from the first-best allocation where consumption and debt are constant for ever. Interestingly, in the long run, consumption is higher in the market equilibrium with the risk of repudiation than in the first best allocation.

It is straightforward to generalize the results to the case of $\beta R < 1$ under the assumption that utility features constant relative risk aversion. In this case, when the debt is honored debt would increase and consumption would fall. After each episode of renegotiation the economy would start again accumulating debt. In a world comprising economies with different β, e.g., some with $\beta R = 1$ and some with $\beta R < 1$, economies with low β would experience recurrent debt crises.

[8] The prediction that whenever debt is renegotiated consumption increases permanently is extreme, and hinges on the assumptions that $\beta R = 1$ and that ϕ is i.i.d. with a known distribution. In Section 6 we extend the model to a setting where there is uncertainty about the true distribution of ϕ and the market learns about this distribution by observing the sequence of ϕ's. In this case, a low realization of ϕ has two opposing effects on consumption: on the one hand, a low ϕ triggers debt renegotiation which on its own would increase consumption; on the other hand, a low ϕ affects the beliefs about the distribution of ϕ, inducing the market to regard the country as less creditworthy (namely, the country draws from a distribution where low ϕ is more likely). This tends to increase the default premium on bonds and to lower consumption.

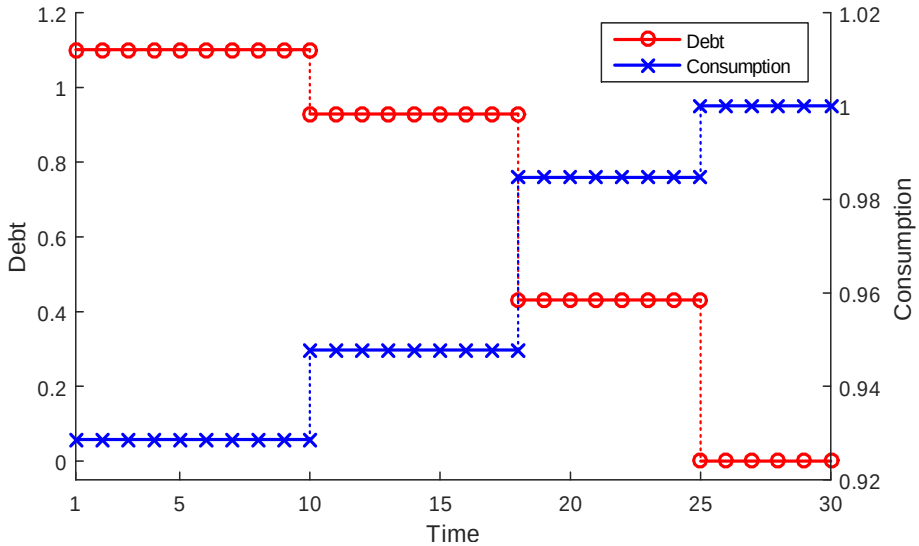

Figure 1: Simulation of debt and consumption for a particular sequence of ϕ's during normal times.

3.4 Equilibrium under recession

When the economy is in recession the government chooses, sequentially, whether to honor the current debt, how much new debt to issue, and how much reform effort to exert. In this section, we assume that the government cannot issue state-contingent debt, i.e., securities whose payment is contingent on the aggregate state of the economy. In Section 5.1 below we relax this restriction.

A natural property of the laissez-faire equilibrium allocation is that $C(b, \underline{w}) < C(b, \bar{w})$ for all $b \leq \bar{b}$: conditional on honoring a giving debt level, consumption is higher in normal times than in a recession. Although we could find no numerical counterexample to this property, it is difficult to prove it in general because the equilibrium functions for consumption, effort and debt price are determined simultaneously. However, we can provide a sufficient condition in terms of the primitive parameters of the model.

Proposition 5 *The following conditions are sufficient to ensure that $C(b, \underline{w}) < C(b, \bar{w})$ for all $b \in [0, \bar{b}]$: (i) $\bar{w} - \underline{w} > \frac{\beta}{1-\beta}\bar{w}$, and (ii) $F[(u(\bar{w}) - u((1-\beta)(\bar{w} - \underline{w})))/(1-\beta)] = 0$.*

Conditional on the consumption ranking, it is straightforward to show, using Definition 1, that $\Phi(b, \underline{w}) > \Phi(b, \bar{w})$, $\hat{b}(\phi, \underline{w}) < \hat{b}(\phi, \bar{w})$, $Q(b, \underline{w}) < Q(b, \bar{w})$ and $W(b, \underline{w}) < W(b, \bar{w})$. Note, in particular, that the price of the bond increases if the recession ends. The reason is that the probability of renegotiation decreases as the economy switches into normal times. The property that $\Phi(b, \underline{w}) > \Phi(b, \bar{w})$ implies that one can partition the state space into three regions:

- if $b < b^-$, the country honors the debt with a positive probability, irrespective of the aggregate state (the probability of renegotiation being higher if the recession continues than if it ends);[9]
- if $b \in [b^-, \bar{b}]$, the country renegotiates with probability one if the recession continues, while it honors the debt with a positive probability if the recession ends;

[9] b^- is implicitly determined by the equation $W(b^-, \underline{w}) = W(0, \underline{w}) - \phi_{\max}$.

- if $b > \bar{b}$, the country renegotiates its debt with probability one, irrespective of the aggregate state.

Note that the risk of repudiation introduces some elements of state contingency, since debt is repaid with different probabilities under recession and normal times.

3.4.1 Reform effort in equilibrium

We denote by $\Psi(b')$ the equilibrium policy function for effort, i.e., the probability that the recession ends next period, as a function of the newly-issued debt. More formally, the first-order condition from (12) yields:

$$X'\left(\Psi\left(b'\right)\right) = \beta\left[\int_0^\infty V\left(b', \phi', \bar{w}\right) dF(\phi) - \int_0^\infty V\left(b', \phi', \underline{w}\right) dF(\phi)\right]. \tag{14}$$

The continuity and monotonicity of X, together with the continuity of the value functions and the continuity of the c.d.f. F ensure that Ψ is a continuous function. Intuitively, the government has an incentive to do some reform effort because a recovery will increase expected utility. The gain is the difference in expected utility between recovery and recession, and the larger this difference, the larger will the reform effort be. However, effort is not provided efficiently. To see why, recall that the bond price increases upon economic recovery. Thus, the creditors reap part of the gain from economic recovery, whereas the country bears the full burden of the effort cost.

We can prove that effort is inefficiently provided with the aid of a simple one-period deviation argument. Consider an equilibrium effort choice path consistent with (14) – corresponding to the case of non-contractible effort. Next, suppose that, only in the initial period, the country could contract effort before issuing new debt. As it turns out, the country would choose a higher reform effort in the first period than in the laissez-faire equilibrium. We state this result as a lemma.

Lemma 3 *Suppose that $b' > 0$ and that the borrower could, in the initial period, commit to an effort level upon issuing new debt, then the reform effort would be strictly larger than in the case in which effort is never contractible.*

If the government could commit to reform, its reform effort would be monotone increasing in the debt level, since a high debt increases the hardship of a recession. However, under moral hazard, the equilibrium reform effort exhibits a non-monotonic behavior. More precisely, $\Psi(b)$ is increasing at low levels of debt, and decreasing in a range of high debt levels, including the entire region $\left[b^-, \bar{b}\right]$. Proposition 6 establishes this result more formally.

Proposition 6 *There exist three ranges, $[0, b_1] \subseteq [0, b^-]$, $[b_2, \bar{b}] \supseteq [b^-, \bar{b}]$, and $\left[\bar{b}, \infty\right)$ such that:*

1. *If $b \in [0, b_1)$, $\Psi'(b) > 0$;*
2. *If $b \in \left(b_2, \bar{b}\right)$, $\Psi'(b) < 0$;*
3. *If $b \in \left[\bar{b}, \infty\right)$, $\Psi'(b) = 0$.*

The following argument establishes the result. Consider a low (possibly negative) debt range where the probability of renegotiation is zero. In this range, there is no moral hazard. Thus, a higher debt level has a disciplining effect, i.e., it strengthens the incentive for economic reforms: due to the

concavity of the utility function, the discounted gain of leaving the recession is an increasing function of debt.

As one moves to a larger initial debt, however, moral hazard becomes more severe, since the reform effort decreases the probability of default, and shifts some of the gains to the creditors. This is reminiscent of the *debt overhang* effect in Krugman (1988). The debt overhang dominates over the disciplining effect in the region $[b^{-}, \bar{b}]$. In this range, debt has a stark state contingency. If the economy remains in recession, debt is renegotiated for sure, so the continuation utility $\int_0^\infty V\left(b', \phi', \underline{w}\right) dF(\phi)$ is independent of b. If the recession ends, the continuation utility is decreasing in b. Therefore, in this region the value of reform effort necessarily declines in b. By continuity, the same argument applies to a range of debt below b^{-}. Finally, when $b > \bar{b}$, debt is always renegotiated, and thus the gain from leaving the recession is independent of b.[10]

Note, finally, that the debt-overhang effect hinges on the presence of some renegotiation risk and an associated premium on debt. If the borrower instead could commit to repay the debt, the price of debt would be $1/R$ regardless of the aggregate state, so an economic recovery would not yield any benefits to the lenders. Consequently, there would be no moral hazard in the effort choice and the effort function would be monotone increasing in debt.

3.4.2 Debt issuance and consumption dynamics

In this section we characterize the dynamics of consumption and debt. We proceed in two steps. First, we derive the properties of the CEE. Then we summarize its characterization in a formal proposition.

The government solves the problem (10) for $w = \underline{w}$. Using the first-order condition together with the envelope theorem yields the following CEE:

$$\begin{aligned} & E\left\{\frac{MU_{t+1}}{MU_t}|\text{debt is honored at } t+1\right\} \\ = \; & 1 + \frac{\Psi'(b_{t+1})}{\Pr(\text{debt is honored at } t+1)} R\left[Q(b_{t+1}, \bar{w}) - \hat{Q}(b_{t+1}, \underline{w})\right] b_{t+1}. \end{aligned} \tag{15}$$

Equation (15) is the analogue of (13). There are two differences. First, the left-hand side has the *expected* ratio between the marginal utilities, due to the uncertainty about the future aggregate state. Second, there is a new term on the right-hand side capturing the effect of debt on reform effort.

For expositional purposes, consider first the case in which the probability that the recession ends is exogenous, i.e., $\Psi'(.) = 0$. In this case, the CEE requires that the expected marginal utility be constant. For this to be true, conditional on debt being honored, consumption growth must be positive if the recession ends, and negative if the recession continues. The lack of consumption insurance stems from the incompleteness of financial markets, and would disappear if the government could issue state-contingent bonds. However, this conclusion does not carry over to the economy with moral hazard, as we show in Section 5.1 below.

Consider, next, the general case. Moral hazard introduces a new strategic motive. By changing the level of newly-issued debt, the government manipulates its own *ex-post* incentive to make reforms. The sign of this strategic effect is ambiguous, and hinges on the sign of Ψ' (see Proposition 6). When the outstanding debt is low, $\Psi' > 0$. Then, more debt strengthens the *ex-post* incentive to reform, thereby increasing the price of the newly-issued debt. The right-hand side of (15) is in this case larger

[10] In a variety of numerical simulations, we have always found Ψ to be hump-shaped with a unique peak (see Figure 3), although in general this depends on the distribution $F(\phi)$.

than unity, and the CEE implies a lower consumption fall (hence, higher debt accumulation) than in the absence of moral hazard. In contrast, in the region of high initial debt, $\Psi' < 0$. In this case, it is optimal to issue less debt than in the absence of moral hazard. The reason is that the market anticipates that a larger debt reduces the reform effort. In response, the government restrains its debt issuance strategically in order to mitigate the ensuing fall in the debt price. Thus, when the recession continues, a highly indebted country will obtain less consumption insurance when the reform is endogenous than when p is exogenous.

We summarize the results in a formal proposition.

Proposition 7 *If the economy starts in a recession and the realization of ϕ' induces no renegotiation, the optimal debt level, $b' = B\left(\mathbb{B}\left(b, \phi, \underline{w}\right), \underline{w}\right)$, induces a consumption sequence that satisfies the following CEE:*

$$\beta R \left(\frac{\underbrace{\left[1-\Psi\left(b'\right)\right] \times \left(1-F\left(\underline{\Phi}\left(b'\right)\right)\right)}_{\text{prob. of repayment and continuing recession}}}{\underbrace{\Pr\left(H|b'\right)}_{\text{unconditional prob. of repayment}}} \times \underbrace{\frac{u'\left(c'|_{H,\underline{w}}\right)}{u\left(c\right)}}_{\text{MRS if rec. cont.}} + \frac{\underbrace{\Psi\left(b'\right) \times \left(1-F\left(\bar{\Phi}\left(b'\right)\right)\right)}_{\text{prob. of repayment and end of recession}}}{\underbrace{\Pr\left(H|b'\right)}_{\text{unconditional prob. of repayment}}} \times \underbrace{\frac{u'\left(c'|_{H,\bar{w}}\right)}{u\left(c\right)}}_{\text{MRS if rec. ends}} \right) = 1 + \frac{\Psi'\left(b'\right)}{\Pr\left(H|b'\right)} \times \underbrace{R\left(Q\left(b', \bar{w}\right) - \hat{Q}\left(b', \underline{w}\right)\right) b'}_{\text{gain to lenders if recession ends}} \tag{16}$$

where $c = C\left(\mathbb{B}\left(b, \phi, \underline{w}\right), \underline{w}\right)$ is current consumption, $c'|_{H,w} = C\left(b', w\right) = C\left(B\left(\mathbb{B}\left(b, \phi, w\right), w\right), w\right)$ is next-period consumption conditional on w and no renegotiation, and $\Pr\left(H|b'\right)$ is the unconditional probability that the debt b' be honored, i.e., $\Pr\left(H|b'\right) = \left[1-\Psi\left(b'\right)\right] \times \left(1-F\left(\bar{\Phi}\left(b'\right)\right)\right) + \Psi\left(b'\right) \times \left(1-F\left(\underline{\Phi}\left(b'\right)\right)\right)$.

We end this section by noting that the top of the Laffer curve of debt corresponds to a lower debt level in recession than during normal times.

Lemma 4 *Let $\bar{b} = \arg\max_b \left\{Q\left(b, \bar{w}\right) b\right\}$ and $\bar{b}^R = \arg\max_b \left\{Q\left(b, \underline{w}\right) b\right\}$. Then, $\bar{b}^R \leq \bar{b}$, with equality holding only if $\Psi\left(b\right) = p$ (i.e., if the probability of staying in a recession is exogenous).*

The reason why the top of the Laffer curve under recession is located strictly to the left of $\bar{b}$ is that the reform effort is decreasing in debt (i.e., $\Psi' < 0$) for b close to $\bar{b}$, as established in Proposition 6. This implies that for b close to but smaller than $\bar{b}$, bond revenue is strictly decreasing in b. By reducing the newly-issued debt, the borrower increases the subsequent reform effort, which in turn increases the current bond price and debt revenue.

3.4.3 Taking stock

The previous sections have established the main properties of the laissez-faire equilibrium. The first property is that moral hazard induces an inefficient provision of reform effort in equilibrium, especially for high debt levels. Figure 2 shows the effort function $\Psi\left(b\right)$ in a calibrated economy. Note that the reform effort plunges for high debt levels.[11] The hump-shaped effort function contrasts sharply with

[11] This prediction is consistent with the casual observation that in the recent European debt crisis structural reforms have met stronger opposition in highly indebted countries. Countries with moderate initial debt levels, such as for instance Spain, have arguably been more prone to enact structural reforms than has Greece.

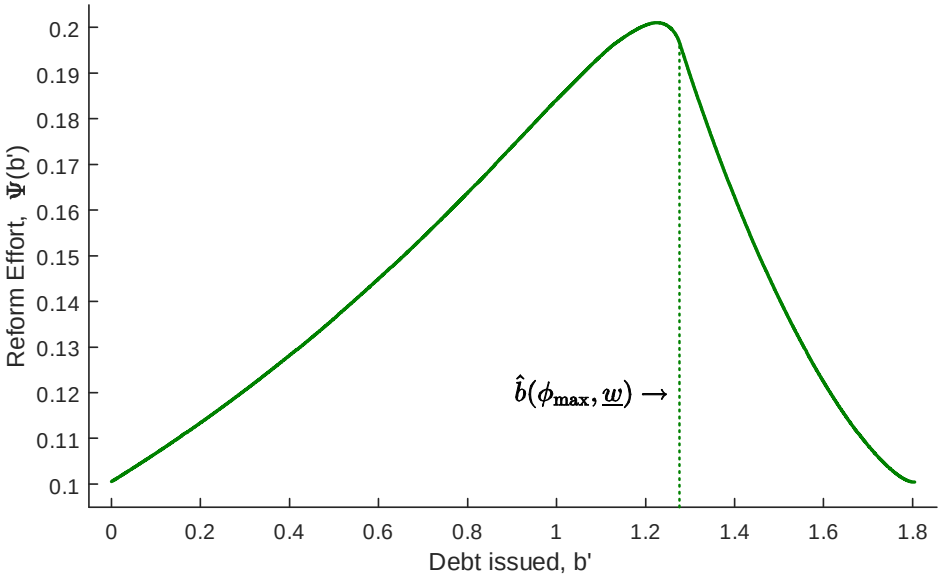

Figure 2: Reform effort function $\Psi(b)$. The parameter values correspond to the calibration in Section 7 with η fixed to 0.5.

the optimum effort in Proposition 1. In the first best, reform effort is monotone increasing in the initial debt level, and remains constant over time. The second property is that the possibility of renegotiating a non-state-contingent debt may improve risk sharing. This is *per se* welfare-enhancing but it exacerbates the moral hazard in reform effort.

The third property is that in periods when debt is fully honored, the equilibrium features positive debt accumulation if the economy remains in recession, and constant debt when the economy returns to normal times. An implication of the first and third property is that, as the recession persists, the reform effort initially increases, but then, for high debt levels, it declines over time. Figure 3 illustrates a time path for debt and consumption (left panel) and of the corresponding reform effort (right panel) for a particular simulated sequence of ϕ's. The volatility in consumption and effort contrast sharply with the optimal allocation of Proposition 1 where consumption and reform effort are constant over time.

The fourth property concerns post-renegotiation debt dynamics. Debt accumulation resumes immediately after the haircut, while consumption increases upon debt relief and start falling again thereafter. This prediction is broadly consistent with the empirical evidence that economic conditions of debtors improve following a debt relief, as documented in Reinhart and Trebesch (2016). It is also consistent with the recent debt dynamics of Greece – after the 2011 debt relief, the debt-GDP ratio fell from 171% to 157%, but subsequently it increased back to 177%. Interestingly, the theory predicts that for highly indebted countries a large haircut may enhance the reform effort, contrary to the common view that pardoning debt would have perverse effects on incentives.

For simplicity we have assumed that the government can only issue one-period non-state-contingent debt. Issuing debt at multiple maturities could in principle allow the borrower to obtain some additional insurance. In a world without moral hazard, this could complete the markets (cf. Angeletos 2002). However, as we show in Section 5.1 below, in our model even an economy with a full set

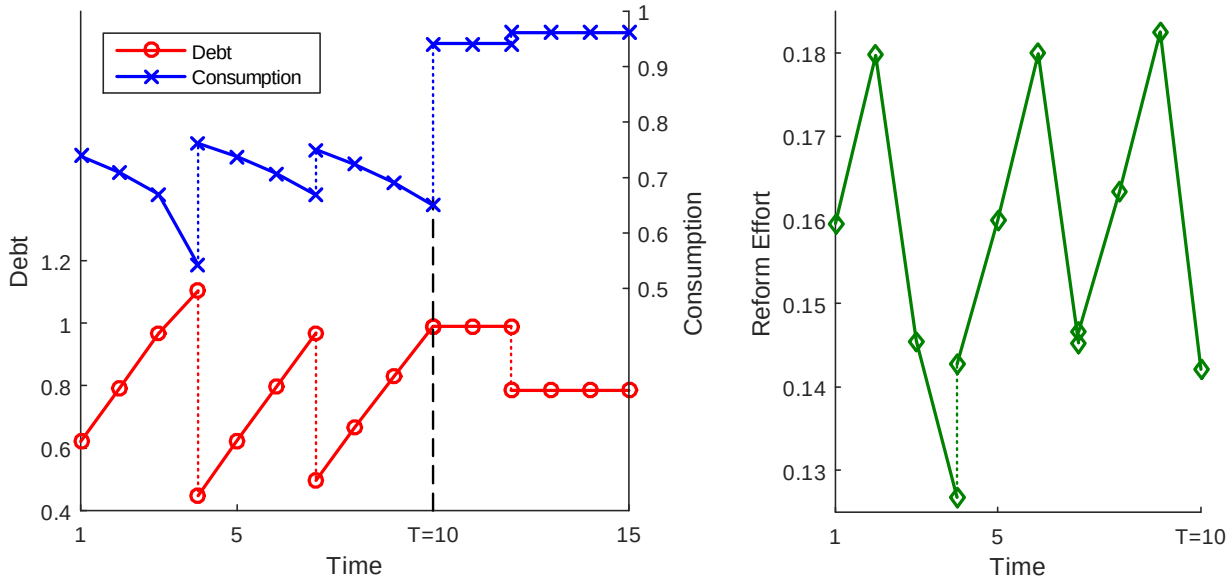

Figure 3: Simulation of debt, consumption and effort for a particular sequence of ϕ's in the competitive equilibrium. In this particular siumulation the recession ends at time $T = 10$.

of state-contingent assets would fail to attain efficiency due to the moral hazard problem associated with structural reforms. This mitigates the concern about the loss of generality associated with the assumption that there is only one-period debt.[12]

4 Constrained Pareto optimum

In this section, we study the constrained efficient allocation in an environment where the planner cannot overrule the limited commitment constraint. We characterize the optimal dynamic contract, subject to limited commitment: the country can quit the contract, suffer the default cost, and resort to market financing. The problem is formulated as a one-sided commitment with lack of enforcement, following Ljungqvist and Sargent (2012) and based on a promised-utility approach in the vein of Spear and Srivastava (1987), Thomas and Worrall (1988 and 1990) and Kocherlakota (1996). Allowing two-sided limited commitment, where even the planner could terminate the contract, would yield identical results, since in our environment the planner has never an incentive to quit.

We assume that the planner has full information: she observes the realization of ϕ (as does the market) and can control effort. This is a useful benchmark for two reasons. First, it simplifies the analysis and makes for sharper results. It is well-known that solving models involving both incomplete information and limited commitment is involved. Second, from an applied perspective, an international agency might observe reforms but have limited instruments to prevent sovereign debt renegotiation. While in this section we empower the planner with the ability to actually *enforce* the desired effort

[12] In addition, we conjecture that if we extened the model to allow the borrower to issue debt at multiple maturities, it would only issue one-period debt in steady state in order to limit the extent of moral hazard. Aguiar and Amador (2013) reach a similar conclusion in a different model. From an empirical standpoint, Broner, Lorenzoni, and Schmukler (2013) document that in emerging markets governments issue mostly short term debt.

level, in Section 4.2 below we relax this assumption and consider an environment where effort must be self-enforcing.

We denote by ν the utility promised to the risk-averse agent in the beginning of the period, before the realization of ϕ. ν is the key state variable of the problem. We denote by $\bar{\omega}_\phi$ and $\underline{\omega}_\phi$ the promised continuation utilities conditional on the realization ϕ and on the aggregate state $\bar{w}$ and $\underline{w}$, respectively. $\underline{P}(\nu)$ and $\bar{P}(\nu)$ denote the expected present value of profits accruing to the principal conditional on delivering the promised utility ν in the most cost-effective way in recession and in normal times, respectively. The planning problem is evaluated after the uncertainty about the aggregate state has been resolved (i.e., the economy is either in recession or in normal times in the current period), but before the realization of ϕ is known.

In Proposition 14 in the online appendix we prove, following the strategy in Thomas and Worrall (1990), that the functional equations defined in equations (17) and (22) below are contraction mappings, that the profit functions $\underline{P}(\nu)$ and $\bar{P}(\nu)$ are decreasing, strictly concave and continuously differentiable, and that the associated maximands are unique.

4.1 Constrained efficiency

In normal times, the optimal value $\bar{P}(\nu)$ satisfies the following functional equation:

$$\bar{P}(\nu) = \max_{\{c_\phi, \bar{\omega}_\phi\}_{\phi \in \aleph}} \int_{\aleph} \left[\bar{w} - c_\phi + \beta \bar{P}(\bar{\omega}_\phi)\right] dF(\phi), \tag{17}$$

where the maximization is subject to the constraints

$$\begin{aligned} \int_{\aleph} \left[u(c_\phi) + \beta \bar{\omega}_\phi\right] dF(\phi) &\geq \nu, && (18) \\ u(c_\phi) + \beta \bar{\omega}_\phi &\geq \bar{\nu} - \phi, \quad \phi \in \aleph, && (19) \\ c_\phi \in [0, \bar{w}],\ \nu, \bar{\omega}_\phi &\in [\bar{\nu} - E[\phi], \bar{\nu}]. \end{aligned}$$

Here, $\bar{\nu}$ is the value of the outside option for the agent. We return to the interpretation of $\bar{\nu}$ below. (18) is a promise-keeping constraint, whereas (19) is a participation constraint (PC).

The application of recursive methods allows us to establish the following proposition.

Proposition 8 *Assume the economy is in normal times. (I) For all realizations ϕ such that the PC of the agent, (19), is binding, $\bar{\omega}_\phi > \nu$ and the solution for $(c_\phi, \bar{\omega}_\phi)$ is determined by the following conditions:*

$$u'(c_\phi) = -\frac{1}{\bar{P}'(\bar{\omega}_\phi)}, \tag{20}$$

$$u(c_\phi) + \beta \bar{\omega}_\phi = \bar{\nu} - \phi. \tag{21}$$

The solution is not history-dependent, i.e., the initial promise, ν, does not matter. (II) For all realizations ϕ such that the PC of the agent, (19), is not binding, $\bar{\omega}_\phi = \nu$ and $c_\phi = c(\nu)$. The solution is history-dependent.

The efficient allocation has standard properties. Whenever the agent's PC is not binding, consumption and promised utility remain constant over time. Whenever the PC binds, the planner increases the agent's consumption and promised utility in order to meet her PC.

Next, we consider an economy in recession. The principal's profit obeys the following functional equation:

$$\underline{P}(\nu) = \max_{\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi \in \aleph}} \int_{\aleph} \left[\underline{w} - c_\phi + \beta\left((1-p_\phi)\,\underline{P}\left(\underline{\omega}_\phi\right) + p_\phi \bar{P}\left(\bar{\omega}_\phi\right)\right)\right] dF(\phi), \tag{22}$$

where the maximization is subject to the constraints

$$\begin{aligned} \int_{\aleph} \left(u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right)\right) dF(\phi) &\geq \nu, && (23)\\ u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right) &\geq \underline{\nu} - \phi, \quad \phi \in \aleph, && (24)\\ c_\phi \in [0, \bar{w}],\, p_\phi \in [\underline{p}, \bar{p}],\, \nu, \underline{\omega}_\phi \in [\underline{\nu} - E[\phi], \underline{\nu}],\, \bar{\omega}_\phi &\in [\bar{\nu} - E[\phi], \bar{\nu}], \end{aligned}$$

and where $\underline{\nu}$ is the outside option for the agent of breaking the contract when the economy is in recession. Note that there are two separate promised utilities, $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$, associated with the two possible realizations of the aggregate state in the next period. The following proposition can be established.

Proposition 9 *Assume the economy is in recession. (I) For all realizations ϕ such that the PC of the agent, (19), is binding, the solution for $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ is determined by the following conditions:*

$$\begin{aligned} u'(c_\phi) &= -\frac{1}{\underline{P}'\left(\underline{\omega}_\phi\right)}, && (25)\\ \underline{\nu} - \phi &= u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right), && (26)\\ \underline{P}'\left(\underline{\omega}_\phi\right) &= \underline{P}'\left(\bar{\omega}_\phi\right), && (27)\\ X'(p_\phi) &= \beta\left(u'(c_\phi)\left(\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right) + \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right). && (28) \end{aligned}$$

The solution is not history-dependent, i.e., the promised utility ν does not matter.
(II) For all realizations ϕ such that the PC of the agent, (19), is not binding, $\underline{\omega}_\phi = \nu$, $\bar{\omega}_\phi = \bar{\omega}(\nu)$, $c_\phi = \underline{c}(\nu)$, and $p_\phi = p(\nu)$. The solution is history-dependent. The reform effort is decreasing in the promised utility level. (III) For all $\phi \in \aleph$, $\bar{\omega}_\phi > \underline{\omega}_\phi$.

When the agent's PC is slack, consumption, reform effort, and promised utilities remain constant. Every time the PC binds, the planner increases the promised utilities, and grants the agent an increase in consumption and a reduction in the reform effort. The agent will, when entering the contract, be offered lower consumption and be required to exercise higher effort, compared to the first best allocation. The conditions the agent faces improve subsequently every time the PC binds. Note that, if we compare two countries entering the contract with different initial promised utilities, the country with a lower promised utility earns a lower consumption and is asked to exercise higher effort. Thus, the country with the lower promised utility is expected to recover faster from the recession.

As the recession ends, the promised utility increases (cf. part III of Proposition 9) and effort goes to zero. Consumption may either remain constant or increase depending on whether the PC binds. Interestingly, the set of states such that the PC binds expands. Namely, there are realizations of ϕ such that consumption rises only if the recession ends. In contrast, for sufficiently large ϕ's, the agent's PC is binding irrespective of whether the recession continues or ends. In this case, consumption remains constant. In other words, because of limited commitment, the agent is offered partial insurance against the continuation of the recession.

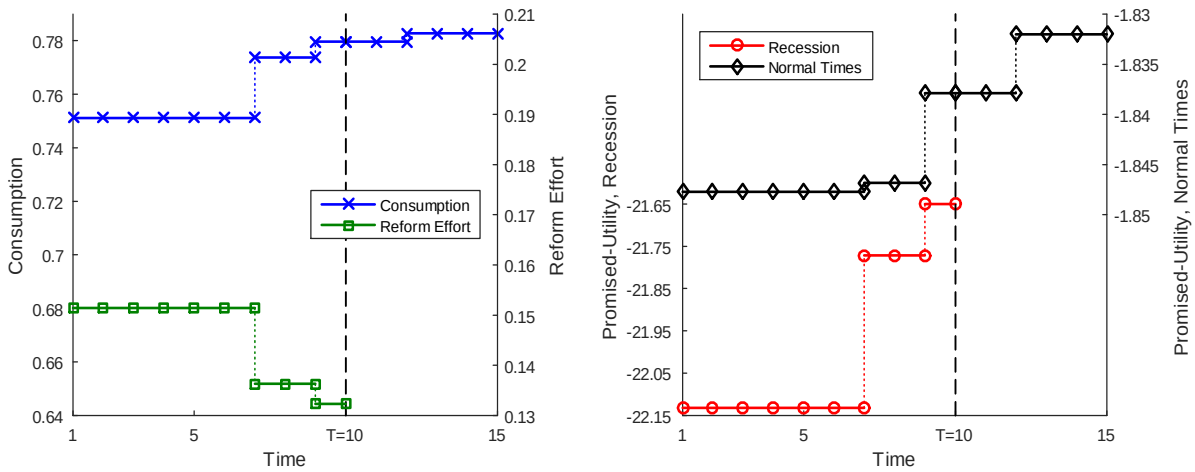

Figure 4: Simulation of consumption, effort, and promised utilities for a particular sequence of ϕ's in the constrained optimum. In this simulation the recession ends at time $T = 10$.

Figure 4 illustrates an example of the time path of consumption and effort (left panel) and of the corresponding promised utilities (right panel) in the constrained efficient allocation when the economy starts in a recession.

4.2 Self-enforcing reform effort

Thus far we have assumed that the planner can dictate the reform effort during recession as long as the agent stays within the contract.[13] In this section, we consider an alternative environment where the planner can verify reform effort only at the end of each period. More formally, the allocation is identical to the solution to the planning problem (22)–(24) subject to the additional incentive constraint (IC) stipulating that, for all $\phi \in \aleph$,

$$-X\left(p_\phi\right) + \beta\left(\left(1 - p_\phi\right)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right) \geq \beta \underline{\nu} - \zeta, \tag{29}$$

where $\zeta \geq 0$ is an exogenous direct punishment that the planner can inflict on the agent when she does not exert the prescribed reform effort (note that in this case the country pays no default cost).

If the IC (29) is slack, the allocation of Proposition 9 is not susceptible to profitable deviations, and the solution is as in Proposition 9. Note that, if the IC constraint is not binding at t, it will never bind in future, since the allocation of Proposition 9 entails a non-decreasing promised utility and a non-increasing effort path. The following Lemma establishes properties of the constrained allocation when the IC is binding.

[13] Assuming that reforms are observable seems natural to us. It is possible, for instance, to verify whether a country introduces labor market reforms, removes product market regulation, cuts employment in the public sector, or passes legislative measures to curb tax evasion (e.g., by intensifying tax audits and enforcing penalties). Nevertheless, it may be difficult to prevent deviations such as delays, lack of implementation, or weak enforcement of agreed-upon reforms. In other words, the agent might accept the consumption decided by the planner but not exert the agreed effort.

Lemma 5 *When the IC is binding, effort and promised utilities are constant at the levels* $(p^*, \underline{\omega}^*, \bar{\omega}^*)$*, where the triplet is uniquely determined by the equations*

$$\underline{\nu} - \tilde{\zeta} = -X(p^*) + \beta\left((1-p^*)\underline{\omega}^* + p^*\bar{\omega}^*\right), \tag{30}$$

$$\bar{P}'(\bar{\omega}^*) = \underline{P}'(\underline{\omega}^*), \tag{31}$$

$$X'(p^*) = \beta(\bar{\omega}^* - \underline{\omega}^*) - \frac{\beta}{\underline{P}'(\underline{\omega}^*)}\left(\bar{P}(\bar{\omega}^*) - \underline{P}(\underline{\omega}^*)\right), \tag{32}$$

where $\tilde{\zeta} \equiv \zeta - (1-\beta)\underline{\nu}$*, and the profit functions* $\bar{P}$ *and* $\underline{P}$ *are defined as in Section 4.*

Equation (30) yields the IC when it holds with equality. Equations (31) and (32) then follow from the FOCs (see equations (66)-(68) in the appendix). These two conditions hold true regardless of whether or not the IC constraint is binding.[14] Although the profit functions of the problem with an IC constraint generally differ from those of the problem without IC constraint, we prove that the profit functions coincide when evaluated at the promised utilities $\underline{\omega}^*$ and $\bar{\omega}^*$.

The following proposition characterizes the equilibrium dynamics.

Proposition 10 *Suppose that the economy starts in a recession, and is endowed with the initial promised utility* ν*.*

1. *If* $\nu \geq \underline{\omega}^*$*, then the IC is never binding, and the constrained optimal allocation,* $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$*, is identical to that in Proposition 9.*

2. *If* $\nu < \underline{\omega}^*$*, then there exist two thresholds,* ϕ^* *and* $\tilde{\phi}(\nu)$*, where* $\phi^* = \tilde{\phi}(\underline{\omega}^*)$ *(expressions in the proof in the appendix) such that:*

   (a) *If* $\phi < \phi^*$*, the PC is binding while the IC is not binding. The solution is not history-dependent and is determined as in Proposition 9 (in particular,* $\underline{\omega}_\phi > \underline{\omega}^*$ *and* $p_\phi < p^*$*).*

   (b) *If* $\phi \in [\phi^*, \tilde{\phi}(\nu)]$*, both the PC and the IC are binding. Effort and promised utilities are equal to* $(p^*, \underline{\omega}^*, \bar{\omega}^*)$ *as given by Lemma 5. Consumption is determined by equations (24) and (29) which yield:*

   $$c_\phi^* = u^{-1}\left(\tilde{\zeta} - \phi\right). \tag{33}$$

   Consumption and effort are lower than in the allocation of Proposition 9.

   (c) *If* $\phi > \tilde{\phi}(\nu)$*, the IC is binding, while the PC is not binding. Effort and promised utilities are equal to* $(p^*, \underline{\omega}^*, \bar{\omega}^*)$*. The consumption level is determined by the promise-keeping constraint (23). In particular, consumption is constant across* ϕ *and given by:*

   $$c_{\tilde{\phi}(\nu)}^* = u^{-1}\left(\tilde{\zeta} - \tilde{\phi}(\nu)\right). \tag{34}$$

   For given ν *and* ϕ*, consumption and effort are lower than in the allocation of Proposition 9.*

[14] To see why the solution to (29)–(32) is unique, note that the concavity and monotonicity of $\underline{P}$ and $\bar{P}$ imply that Equation (31) determines a positive relationship between $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$. Thus, Equation (32) yields an implicit decreasing relationship between p_ϕ and $\underline{\omega}_\phi$, while (29) yields an implicit increasing relationship between p_ϕ and $\underline{\omega}_\phi$.

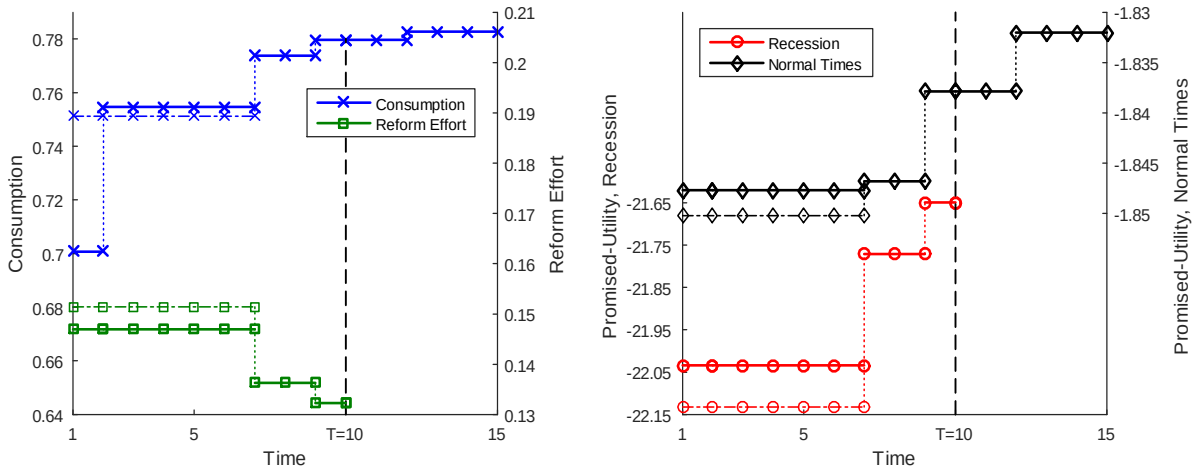

Figure 5: Simulation of consumption, effort, and promised utilities for a particular sequence of ϕ's where the IC is initially binding. Solid lines refer to an economy with an IC constraint. Dashed lines refer to the economy without an IC constraint. In this particular siumulation the recession ends at time $T = 10$.

Consider an economy where, initially, $\nu < \underline{\omega}^*$. If the first realization of ϕ is sufficiently low (case 2.a of Proposition 10), the binding PC induces an effort level so low that the IC is not binding. Therefore, the allocation is not history-dependent, and the characterization of Proposition 9 applies. If the first realization of ϕ is larger than ϕ^*, the IC is binding, and Lemma 5 implies that effort and promised utility are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$, i.e. the maximum effort and the minimum promised future utilities consistent with the IC. If $\phi \in [\phi^*, \tilde{\phi}(\nu)]$ (case 2.b), consumption is pinned down jointly by the PC and the promise-keeping constraint. In this case, consumption is decreasing in ϕ. Finally, if $\phi > \tilde{\phi}(\nu)$ (case 2.c) the PC imposes no constraint, and the initial consumption is determined only by the promise-keeping constraints. When the IC is binding (cases 2.b and 2.c), both consumption and effort are lower than in the second-best solution of Proposition 9. Intuitively, the planner cannot set effort at the efficient level due to the IC, and adjusts optimally to the constraint by reducing current consumption and effort, and increasing promised utilities relative to the case of no IC constraint. Thus, the contract provides less consumption insurance but more effort smoothing. After one period the equilibrium is characterized as in the constrained optimum of Proposition 9.

Figure 5 is the analogue of Figure 4 in an economy in which the IC is binding in the initial period, i.e., $\nu < \underline{\omega}^*$. It shows simulated paths of consumption, effort and promised utility in the constrained optimal allocation for two otherwise identical economies where one economy (solid lines) is subject to the IC constraint, while the other economy (dashed lines) has no such constraint. The initial promised utility ν (not displayed) is lower than $\underline{\omega}^*$ implying that the IC is binding. In the first period, consumption and effort are lower in the economy with an IC constraint. In contrast, promised utility is higher. In other words, the planner provides less insurance by making consumption and effort initially lower, but growing at a higher speed. As of the second period, the dynamics of both economies are the same as in Figure 4.

4.3 Comparison between constrained optimum and laissez-faire equilibrium

In this section, we compare the constrained optimum with the laissez-faire equilibrium.[15] To this aim, we assume that the borrower's outside options corresponds to outright default followed by a permanent reversion to the Markov equilibrium allocation characterized in Section 3. More formally, we let $\bar{\nu} = W(0, \bar{w})$ and $\underline{\nu} = W(0, \underline{w})$. We view this as a natural assumption in our environment. The purpose of our analysis is to identify market inefficiencies that could be ameliorated by various institutional arrangements that we discuss below. Our assumption implies that debtors can always quit any institutional arrangement and revert to the market equilibrium.[16]

The assumption above allows us to derive an important equivalence result: in normal times, the constrained efficient allocation of Proposition 8 is identical to the laissez-faire equilibrium. To establish this result we return, first, to the laissez-faire equilibrium. Let

$$\bar{\Pi}(b) = \left(1 - F\left(\bar{\Phi}(b)\right)\right) b + \int_0^{\bar{\Phi}(b)} \hat{b}(\phi, \bar{w})\, dF(\phi) \tag{35}$$

denote the expected value for the creditors of an outstanding debt b before the current-period uncertainty is resolved. Note that $\bar{\Pi}(b)$ yields the expected debt repayment, which is lower than the face value of debt, since in some states of nature debt is renegotiated. To prove the equivalence, we postulate that $\bar{\Pi}(b) = \bar{P}(\nu)$, and show that in this case $\nu = EV(b, \bar{w})$.[17] If the equilibrium were not constrained efficient, the planner could do better, and we would find that $\nu > EV(b, \bar{w})$.

Proposition 11 *Assume that the economy is in normal times and that $\bar{\nu} = W(0, \bar{w})$. The laissez-faire equilibrium is constrained Pareto efficient, namely, $\bar{\Pi}(b) = \bar{P}(\nu) \Leftrightarrow \nu = EV(b, \bar{w})$.*

Intuitively, renegotiation provides the market economy with sufficiently many state contingencies to attain second-best efficiency. This result hinges on two features of the renegotiation protocol. First, renegotiation averts any real loss associated with unordered default. Second, creditors have all the bargaining power in the renegotiation game.[18]

The efficiency result of Proposition 11 does not carry over to recessions. In the laissez-faire equilibrium, consumption is falling (and debt accumulates) when the country honors its debt. In contrast, the planner would insure the agent's consumption by keeping it constant whenever the PC is not binding. Therefore, the market underprovides insurance. The dynamics of the reform effort also are sharply different. In the constrained efficient allocation, effort is a monotone decreasing function of promised utility which is in turn step-wise increasing over time (cf. Figure 4). In contrast, in the laissez-faire equilibrium the reform effort is hump-shaped in debt. Since debt increases over time (unless it is renegotiated), effort is also hump-shaped over time conditional on no renegotiation.

[15] We ignore the incentive constraint discussed in section 4.2, since, as we show, this affects at most the allocation in the initial period.

[16] In principle, one could consider more sophisticated punishment strategies. These would affect the initial value of the outside option without changing the qualitative properties of the constrained optimum allocation.

[17] Recall that $EV(b, \bar{w}) = \int_{\aleph} V(b, \phi, \bar{w})\, dF(\phi)$ denotes the discounted utility accruing to a country with the debt level b in the competitive equilibrium.

[18] We view this as a useful benchmark. In reality, renegotiations may entail costs associated with legal proceeds and lawsuits, trade retaliation, temporary market exclusion, etc. Also, creditors may not have the full ex-post bargaining power at the renegotiation stage as in Yue (2010). This would reduce the amount of loans creditors can recover. In all these cases, the competitive equilibrium would fail to implement the second best.

5 Decentralization

In this section, we discuss policies and institutions that decentralize the constrained efficient allocation.

5.1 Laissez-faire (Markov) equilibrium with state-contingent debt

The analysis of the laissez-faire equilibrium in Section 3 was carried out under the assumption that the government can issue only a non-contingent asset. In this section, we extend the analysis to allow state-contingent debt. We show that a laissez-faire equilibrium with state-contingent debt would attain constrained efficiency if and only if there were no moral hazard. However, when the reform effort is endogenous, the combination of moral hazard and limited commitment curtails the insurance that markets can provide. Consequently, the Markov equilibrium with state-contingent debt is not constrained efficient. In the quantitative analysis of Section 7 below, we show that markets for state-contingent debt yield only small quantitative welfare gains relative to the benchmark laissez-faire economy.

Let $b_{\underline{w}}$ and $b_{\bar{w}}$ denote two securities paying one unit of output if the economy is in a recession or in normal times, respectively. We label these securities *recession-contingent debt* and *recovery-contingent debt*, respectively, and denote by $Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ and $Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ their corresponding prices. The budget constraint in a recession is given by:

$$Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} = b_{\underline{w}} + c - \underline{w}.$$

Under limited commitment, the price of each security depends on the two outstanding debt levels, as both affect the reform effort and the probability of renegotiation.[19] The value function of the benevolent government can be written as:

$$\begin{aligned} V\left(b_{\underline{w}}, \phi, w\right) \quad = \quad & \max_{\left\{b'_{\underline{w}}, b'_{\bar{w}}\right\}} \left\{u\left[Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \mathbb{B}\left(b, \phi, \underline{w}\right)\right]\right. \\ & \left. -X\left(\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) + \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right] EV\left(b'_{\underline{w}}, \underline{w}\right) + \beta\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) EV\left(b'_{\bar{w}}, \bar{w}\right)\right\}. \end{aligned} \tag{36}$$

Mirroring the analysis in the case of non-state-contingent debt, we proceed in two steps. First, we characterize the optimal reform effort. This is determined by the difference between the discounted utility conditional on the recession ending and continuing, respectively (cf. equation (14)):

$$X'\left(\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) = \beta\left[\int_0^\infty V\left(b'_{\bar{w}}, \phi', \bar{w}\right) dF\left(\phi\right) - \int_0^\infty V\left(b'_{\underline{w}}, \phi', \underline{w}\right) dF\left(\phi\right)\right].$$

Note that the incentive to reform would vanish under full insurance.

Next, we characterize consumption and debt issuance. To this aim, consider first the equilibrium asset prices. The prices of the recession- and recovery-contingent debt are given by, respectively:

$$Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \frac{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) + \frac{1}{b'_{\underline{w}}}\int_0^{\underline{\Phi}\left(b'_{\underline{w}}\right)}\left(\underline{\Phi}^{-1}\left(\phi\right) dF\left(\phi\right)\right)\right), \tag{37}$$

$$Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \frac{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) + \frac{1}{b'_{\bar{w}}}\int_0^{\bar{\Phi}\left(b'_{\bar{w}}\right)}\left(\bar{\Phi}^{-1}\left(\phi\right) dF\left(\phi\right)\right)\right). \tag{38}$$

[19] Note that these assets are not Arrow-Debreu assets since their payoffs are not conditional on the realization of ϕ. An alternative approach would have been to follow Alvarez and Jermann (2000) and issue an Arrow-Debreu asset for each state (w, ϕ) and let the default-driven participation constraint serve as an endogenous borrowing constraint.

The next proposition characterizes the CEE with state-contingent debt.

Proposition 12 *Assume that there exist markets for two securities delivering one unit of output if the economy is in recession and in normal times, respectively, and subject to the risk of renegotiation. Suppose that the economy is initially in recession. The following CEEs are satisfied in the laissez-faire equilibrium:*
(I) If the recession continues,

$$\underbrace{\frac{u'\left(c'|_{H,\underline{w}}\right)}{u'\left(c\right)}}_{MRS\ if\ rec.\ continues} = 1 + \underbrace{\frac{\partial}{\partial b'_{\underline{w}}}\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}_{>0} \times \underbrace{\frac{R \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)\left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)}}_{>0}. \tag{39}$$

(II) If the recession ends,

$$\underbrace{\frac{u'\left(c'|_{H,\bar{w}}\right)}{u'\left(c\right)}}_{MRS\ if\ rec.\ ends} = 1 + \underbrace{\frac{\partial}{\partial b'_{\bar{w}}}\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}_{<0} \times \underbrace{\frac{R \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}}_{>0}, \tag{40}$$

where

$$\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \equiv \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} - \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}}}{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} \geq 0. \tag{41}$$

Moreover,

$$\begin{aligned}
c &= Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \mathbb{B}\left(b, \phi, \underline{w}\right), \\
c'|_{H,\underline{w}} &= Q_{\underline{w}}\left(B_{\bar{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right) \times B_{\underline{w}}\left(b'_{\underline{w}}\right) + Q_{\bar{w}}\left(B_{\underline{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right) \times B_{\bar{w}}\left(b'_{\underline{w}}\right) + \underline{w} - b'_{\underline{w}}, \\
c'|_{H,\bar{w}} &= Q\left(B\left(b'_{\underline{w}}, \bar{w}\right), \bar{w}\right) \times B\left(b'_{\underline{w}}, \bar{w}\right) + \bar{w} - b'_{\bar{w}},
\end{aligned}$$

where $B_{\underline{w}}\left(b_{\underline{w}}\right)$ and $B_{\bar{w}}\left(b_{\underline{w}}\right)$ denote the optimal level of newly-issued recession- and recovery-contingent debt when the recession continues, and debt is honored.

If the probability that the recession ends were exogenous, consumption would be independent of the realization of the aggregate state. In this case, the CEEs imply constant consumption $c'|_{H,\underline{w}} = c'|_{H,\bar{w}} = c$ where, recall, $c'|_{H,w}$ is consumption conditional on debt being honored in the next period. The solution has the same properties as the constrained Pareto optimum without moral hazard: consumption is constant when debt is honored, and increases discretely when it is renegotiated. The next proposition establishes formally that the two allocations are equivalent. To this aim, define $\underline{\Pi}\left(b_{\underline{w}}\right)$ to be the valuation of debt conditional on staying in recession but before the realization of ϕ.

Proposition 13 *If the probability that the recession ends is independent of the reform effort (i.e., $\Psi = p$), then the laissez-faire equilibrium with state-contingent debt is constrained Pareto efficient, namely, $\underline{\Pi}\left(b_{\underline{w}}\right) = \underline{P}\left(\nu\right) \Leftrightarrow \nu = EV\left(b_{\underline{w}}, \underline{w}\right)$.*

An immediate implication of Proposition 13 is that if effort could be contracted before issuing the new debt, then the laissez-faire equilibrium with state-contingent debt would implement the constrained optimal allocation. This equivalence breaks down if there is moral hazard. In this case, the consumption and effort dynamics of the laissez-faire equilibrium are qualitatively different from

those of the constrained optimum. In the laissez-faire equilibrium, consumption falls (and recession-contingent debt increases) whenever the economy remains in recession and debt is honored, as shown by equation (39). On the contrary, consumption increases whenever the recession ends, as shown by equation (40). Therefore, different from the efficient allocation (which, recall, features constant consumption when the outside option is not binding), the laissez-faire equilibrium features imperfect insurance, even conditional on honoring the debt.

The intuition is as follows. By issuing more recession-contingent debt, the country strengthens its incentive to make reforms, since $\partial\Psi/\partial b'_{\underline{w}} > 0$. This induces the government to issue more recession-contingent debt than in the absence of moral hazard. This effect is stronger the larger is $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ which can be interpreted as the net expected gain accruing to the lenders from a marginal increase in the probability that the recession ends. On the contrary, issuing more recovery-contingent debt weakens the incentives to do reform. As a result, consumption increases if the recession ends and falls if the recession continues (and debt is honored). This result highlights the trade-off between insurance and incentives: the country must give up insurance in order to gain credibility about its willingness to do reforms.

The behavior of effort is also different between the laissez-faire equilibrium with state-contingent debt and the constrained efficient allocation. In the planning allocation, effort is constant whenever the outside option is not binding. In contrast, in the equilibrium changes in debt influence the reform effort: this is increasing in the newly-issued recession-contingent debt and decreasing in the newly-issued recovery-contingent debt. In summary, the Markov equilibrium with state-contingent debt is inefficient and provides less smoothing of consumption and reform effort than does the planner.

5.2 Reputational mechanisms

We have focused so far on Markov equilibria abstracting from reputational mechanisms. We find this restriction reasonable in our environment, as the decentralized nature of the international credit market makes it difficult for creditors to coordinate expectations on appropriate punishment schemes. Nevertheless it is interesting to investigate under which conditions non-Markov equilibria can implement the constrained efficient allocation. In this section, we show that a simple reputational equilibrium decentralizes the constrained optimum if markets for state-contingent debt exist and the discount factor is sufficiently high. A thorough investigation of the non-Markov equilibria is beyond the scope of our analysis. Instead, we focus on an implementation where the threat point is the infinite reversion to the Markov equilibrium, in the spirit of Barro and Gordon (1983).

The candidate constrained *efficient* equilibrium has the following features. The agent exerts the second best effort level given by the planner solution above, and the market prices (state-contingent) debt accordingly. The agent receives full consumption insurance if it honors its debt. Debt may be renegotiated along the equilibrium path. However, if the agent does not exert the required effort level, the reputational equilibrium reverts to the Markov equilibrium studied above. We prove in the online appendix that in order for such an equilibrium to be sustained the following condition must hold, for all $b_{\underline{w}}$ and ϕ :

$$X\left(p^C\right) - X\left(\Psi(b'_{\bar{w}}, b'_{\underline{w}})\right) < \beta \left[\begin{array}{c} \left(p^C - \Psi(b'_{\bar{w}}, b'_{\underline{w}})\right) \times \left(EV(b'_{\bar{w}}, \phi', \bar{w}) - EV(b'_{\underline{w}}, \phi', \underline{w})\right) \\ +(1-p^C) \times \left(EV^C(b'_{\underline{w}}, \phi', \underline{w}) - EV(b'_{\underline{w}}, \phi', \underline{w})\right) \end{array} \right], \tag{42}$$

where p^C, $b'_{\bar{w}}$, and $b'_{\underline{w}}$ are the optimal effort and debt issuances given an initial debt $b_{\underline{w}}$ and a realization ϕ, and EV^C denotes the expected utility along the constrained efficient candidate equilibrium. The left-hand side is the value of the deviation in terms of the reduction in the effort cost in the current

period. The right-hand side is the cost (punishment) for abandoning the efficient allocation. This can be decomposed into two parts. The first is the reduction in the probability that the recession ends associated with the deviation times the expected gain of leaving the recession in the Markov equilibrium. The second is welfare loss of switching to the Markov equilibrium in recession. This condition is more likely to be satisfied when β is large. On the contrary the condition necessarily fails when β is sufficiently small.

5.3 Interpreting the optimal plan as an austerity program

In this section, we discuss an institutional interpretation of the constrained optimum allocation. Consider a stand-by program run by an international institution (e.g., the IMF) which, like the planner, can monitor the reform, but cannot get around the limited commitment problem, i.e., the indebted country can walk away unilaterally. We show that the planner allocation can be interpreted as a combination of transfers (or loans), repayment schedules, reform program and renegotiation strategy. This program has two key features. First, the country cannot run an independent fiscal policy, i.e., it is not allowed to issue additional debt in the market. Second, the program is subject to renegotiation. More precisely, whenever the country credibly threatens to abandon the program, the international institution should sweeten the deal by increasing the transfers, reducing the required effort, and reducing the debt the country owes when the recession ends. When no credible threat of default is on the table, consumption and reform effort should be held constant as long as the recession lasts. When the recession ends, the international institution receives a payment from the country, financed by issuing debt in the market.

Let ν denote the present discounted utility guaranteed to the country when the program is first agreed upon. Let $c^*(\nu)$ and $p^*(\nu)$ be the consumption and reform effort associated with the promised utility in the planning problem. Upon entering the program, the country receives a transfer equal to $T(\nu) + b_0$, where $T(\nu) = c^*(\nu) - \underline{w}$ (note that $T(\nu)$ could be negative). In the subsequent periods, the country is guaranteed the transfer flow $T(\nu)$ so long as the recession lasts and there is no credible request of renegotiating the terms of the austerity program. In other words, the international institution first bails out the country from its obligations to creditors, and then becomes the sole residual claimant of the country's sovereign debt. The country is also asked to exert a reform effort $p^*(\nu)$. If the country faces a low realization of ϕ and threatens to leave the program, the institution improves the terms of the program so as to match the country's outside option. Thereafter, consumption and effort are held constant at new higher and lower levels, respectively, as in the planner's allocation. And so on, for as long as the recession continues.

As soon as the recession ends, the country owes a debt b_N to the international institution, determined by the equation

$$Q(b_N, \bar{w}) \times b_N = c^*(\nu_N) - \bar{w} + b_N.$$

Here ν_N is the expected utility granted to the country after the most recent round of renegotiation. After receiving this payment, the international institution terminates the program and lets the country finance its debt in the market.

This program resembles an *austerity* program, in the sense that the country is prevented from running an independent fiscal policy and reform program. In particular, the country would like to issue extra debt after entering the stand-by agreement, so austerity is a binding constraint. In addition, the country would like to shirk on the reform effort prescribed by the agreement. Thus, the government would like to (temporarily) deviate from the optimal plan, and promises about future transfers is an essential feature of the program.

A distinctive feature of the assistance program is that the international institution sets "harsh" entry conditions in anticipation of future renegotiations. How harsh these conditions are depends on ν. In turn, ν may reflect a political decision about how many (if any) own resources the international institution wishes to commit to rescuing the indebted country. A natural benchmark is to set ν such that the international institution makes zero profits (and zero losses) in expectation. Whether, ex-post, the international institution makes net gains or losses hinges on the duration of the recession and on the realized sequence of ϕ's.

Another result that has important policy implications is that there would be no welfare gain if the international institution committed never to accept any renegotiation. On the contrary, such a policy would lead to welfare losses because, on the one hand, there would be inefficient default in equilibrium; on the other hand, the country could not expect future improvements, and therefore would not accept a very low initial consumption, or a very high reform effort. If the international institution's expected profit were zero in both programs, the country would receive a lower expected utility from the alternative (no renegotiation) program.

In summary, our theory prescribes a pragmatic approach to debt renegotiation. Credible threats of default should be appeased by reducing the debt and softening the austerity program. Such approach is often criticized for creating bad incentives. In our model, such appeasement is precisely the optimal policy under the reasonable assumption that penalties on sovereign countries for breaking an agreement are limited.

6 Extension: learning

In our theory, renegotiation is unambiguously good for the borrower. On the one hand, consumption always increases upon renegotiation, in line with the empirical evidence documented by Reinhart and Trebesch (2016). On the other, renegotiations do not affect the terms at which the country can borrow in future. In particular, conditional on the debt level, the risk premium is independent of the country's credit history. In this section, we sketch an extension where bond prices depend on the frequency of previous renegotiations. We assume that there is imperfect information about the distribution from which countries draw their realizations of ϕ. In particular, there are two types of countries, creditworthy (CW) and not creditworthy (NC), that draw from different distributions.[20] In particular, $F_{NC}(\phi) \geq F_{CW}(\phi)$, with strict inequality holding for some ϕ, implying that the NC country is more likely to have lower realizations of the default cost. We assume that priors are common knowledge, and denote by π the belief that the borrower is CW. Beliefs are updated according to Bayes' rule:

$$\pi' = \frac{f_{CW}(\phi)}{f_{CW}(\phi) \times \pi + f_{NC}(\phi) \times (1-\pi)} \pi \equiv \Gamma(\phi, \pi).$$

Moreover, we define

$$F(\phi|\pi) \equiv \pi F_{CW}(\phi) + (1-\pi) F_{NC}(\phi),$$

and restrict attention to laissez-faire equilibria during normal times ($w = \bar{w}$). For the ease of exposition, normal times variables will be indicated with a bar on top for the rest of this section.

[20] One could assume that the distributions have a common support in order to rule out perfectly revealing realizations. However, this is not essential.

In the new environment, the price of debt depends on the prior about the country's type, i.e., $\bar{Q}(b',\pi)$. No arbitrage implies the following bond price:

$$\bar{Q}(b',\pi) = \frac{1}{R}\left(\begin{array}{c} (1-F(\Phi^*(b',\pi)|\Gamma(\Phi^*(b',\pi),\pi))) + \\ \frac{\pi}{b'}\int_0^{\Phi^*(b',\pi)} \hat{b}(\phi,\Gamma(\phi,\pi))\ dF_{CW}(\phi) + \frac{1-\pi}{b'}\int_0^{\Phi^*(b',\pi)} \hat{b}(\phi,\Gamma(\phi,\pi))\, dF_{NC}(\phi) \end{array}\right)$$

where $\Phi^*(b',\pi)$ denotes the threshold ϕ' such that debt will be honored next period if and only if $\phi' \geq \Phi^*(b',\pi)$. More formally, Φ^* is the unique fixed point of the following equation

$$\Phi^* = \bar{\Phi}\left(b',\Gamma(\Phi^*,\pi)\right).$$

The function Φ^* takes into account that the realization of ϕ' will itself alter next-period beliefs, which in turn affect the country's incentive to renegotiate.[21] The bond price is falling in b' and increasing in π.

Consider, next, the consumption-savings decision. The CEE yields:[22]

$$\begin{aligned} 1-F\left(\Phi^*\left(\bar{B}(b,\pi),\pi\right)|\Gamma(\Phi^*,\pi)\right) &= \pi\int_{\Phi^*(b,\pi)}^{\infty} \frac{u'\left[\bar{C}\left(\Gamma\left(\phi',\pi\right),\bar{B}(b,\pi)\right)\right]}{u'\left[\bar{C}(b,\pi)\right]} dF_{CW}\left(\phi'\right) \\ &\quad + (1-\pi)\int_{\Phi^*(b,\pi)}^{\infty} \frac{u'\left[\bar{C}\left(\Gamma\left(\phi',\pi\right),\bar{B}(b,\pi)\right)\right]}{u'\left[\bar{C}(b,\pi)\right]} dF_{NC}\left(\phi'\right). \end{aligned}$$

If next-period consumption conditional on honoring the debt did not depend on ϕ', then the CEE would boil down to equation (13). However, in this extension, the realized consumption growth depends on ϕ' because creditors learn over time about the borrowers' types. For example, take two realization of ϕ', say ϕ'_h and ϕ'_l, such that $\phi'_h > \phi'_l$, neither inducing renegotiation. Here, consumption will be larger under ϕ'_h because the larger realization has a stronger positive effect on the belief that the country is CW. This improves the terms of borrowing, and hence consumption. Note that renegotiation might be associated with a fall in consumption – for example if the realized ϕ is just below $\Phi^*(b',\pi)$ the effect of a very small renegotiation is more than offset by that of Bayesian updating. Conversely, a country experiencing a sequence of large ϕ's which induces it to honor debt for a long time will enjoy an increasing consumption.

In summary, this simple extension shows that our theory can incorporate learning effects through which countries prone to renegotiation are punished by the market with high interest rates.

7 Quantitative Analysis

In this section, we study the quantitative properties of the model. To this end, we calibrate the model economy to match salient facts on default premia, investor losses, and debt-to-GDP ratios.[23] Our main purpose here is to evaluate the welfare gain of going from the laissez-faire equilibrium to the constrained optimal allocation. We also assess the quantitative importance of limited commitment, non-contractible effort, and incomplete financial markets.

[21] Note that some functions must be redefined to take into account their dependence on public beliefs. Apart from $\bar{Q}(b',\pi)$, defined in the text, $\hat{b}(\phi,\pi)$ is the renegotiated debt given ϕ and π. Moreover, $\bar{\Phi}(b',\pi)$ denotes the threshold that makes the country indifferent between honoring the debt level b' and defaulting, conditional on the realized belief π.

[22] We defer the derivation to the online appendix.

[23] The model is solved by discretizing the state space. The benchmark calibration uses 5,000 points for debt and 600 points for ϕ. The numerical approximation is remarkably accurate in the sense that the Euler equation errors are negligible. See the online appendix for further details.

7.1 Calibration

A model period corresponds to one year. We normalize the GDP during normal times to $\bar{w} = 1$ and assume that the recession causes a drop in income of 38%, i.e., $\underline{w} = 0.62 \times \bar{w}$. This corresponds to the fall of GDP per capita for Greece between 2007 and 2013, relative to trend.[24] Since we focus on the return on government debt, the annual real gross interest rate is set to $R = 1.02$, implying $\beta = 1/R = 0.98$. The utility function is assumed to be CRRA with a relative risk aversion of 2.

We assume an isoelastic effort cost function; $X(p) = \frac{\xi}{1+1/\varphi}(p)^{1+1/\varphi}$, where ξ regulates the average level of effort and φ regulates the elasticity of reform effort to changes in the return to reforms. We set the two parameters, φ and ξ, so as to match two points on the equilibrium effort function $\Psi(b)$. In particular, we assume that the effort at the debt limit is $\Psi\left(\bar{b}\right) = 10\%$, so that a country at the debt limit would choose an effort inducing an expected duration of the recession of one decade (we have Greece in mind). Moreover, we assume that the maximum effort is $\max_b \Psi(b) = 20\%$, inducing an expected recession duration of five years (we have Iceland and Ireland in mind). This implies setting $\varphi = 20.68$ and $\xi = 23.70$.

Finally, we determine the distribution of the default cost ϕ, assumed to have bounded support $\left[0, \bar{\phi}\right]$. We calibrate the distribution $f(\phi)$ so that the model matches key moments of quantities and prices of sovereign debt. One common problem in the quantitative literature on sovereign debt is that those models fail to match observed values of debt-to-GDP ratios under realistic parametrization (Arellano 2008; Yue 2010). This is not a problem in our model. In fact, the maximum default cost realization $\bar{\phi}$ is set so that the debt limit during normal times is $\bar{b}/\bar{w} = 180\%$. Moreover, the distribution f is chosen so that the model matches an average default premium of 4% for a country which has a debt-output ratio of 100% during recession.[25] This was the average debt and average default premium for Greece, Ireland, Italy, Portugal, and Spain (GIIPS) during 2008-2012. To match this target we assume that ϕ is distributed according to a Beta distribution with c.d.f. given by $F(\phi; \bar{\phi}, \eta) = \Upsilon(\phi/\bar{\phi}, \eta)/\Upsilon(1, \eta)$, where $\Upsilon(x, \eta) = \int_0^x (1-t)^{\eta-1} dt$. By setting $\eta = 0.139$ the target is met.

7.2 Welfare Comparison

We use the calibrated economy to evaluate the welfare gains of different policy arrangements. The welfare gains are measured as the equivalent variation in terms of the market value of an initial debt to output reduction in the market economy, namely, the market value of a reduction in initial debt required to make the borrower indifferent between staying in the market arrangement (with the reduction in debt) and moving to an alternative allocation.

We assume that the economy starts in a recession and has an initial debt-output ratio of $b_0/y_0 = 100\%$. Table 1 reports the welfare gains of the policy arrangements relative to the market allocation. The results for the benchmark calibration with a relative risk aversion of two, are listed in the middle column of the table. Given the initial b_0, the welfare gain of going from the market allocation to

[24] GDP per capita of Greece fell from 18,924 to 14,551 Euro between 2007 and 2013 (Eurostat). The annualized growth rate between 1997 and 2007 was 3.8%. The fall in output between 2007 and 2013 relative to trend is therefore 38%.

[25] Our calibration yields simulation results that are also in line with the findings of Reinhart and Trebesch (2016) who document an average debt relief of 40% of external government debt across both the 1930s and the 1980s/1990s. Moreover, the average debt relief is reported to be 21% of GDP for advanced economies in the 1930s, and 16% of GDP for emerging market economies in the 1980s/1990s. Even though we do not target these moments of the data in the benchmark calibration, our simulations yield an average face value debt relief of 38% corresponding to a reduction worth 29% of recession GDP at market value, which is within the ballpark of their estimates.

	RRA		
	1	2	3
Market Equilibrium	0%	0%	0%
First-best	76.7%	143.2%	198.1%
Second-best	21.2%	44.6%	64.1%
State-contingent	1.2%	5.0%	5.4%
Grexit	6.3%	-5.7%	-19.5%

Table 1: Welfare equivalent transfer (market value, % of GDP)

the first best is equivalent to a one-time transfer of 143% of GDP. Similarly, the gain of going to the constrained optimum is equivalent to a one-time transfer of 45% of GDP. The large welfare difference between the first- and the second-best allocation shows the large losses associated with the limited enforcement.

Allowing for state-contingent debt, on the other hand, yields a gain equivalent to a mere 5% one-time transfer, substantially less than the second-best gains. As discussed above, this illustrates that the trade-off between moral hazard and insurance renders access to full insurance not very useful. Moreover, it shows that the large gains of the constrained optimum must be originating from the planner's ability to mitigate the moral hazard problem.

To understand the role of risk aversion, we vary the relative risk aversion from 1 to 3 (see Table 1).[26] As expected, the welfare gains relative to the market allocation are increasing in the degree of risk aversion.

7.3 Consumption volatility and recession duration

In this section we report some important moments of our simulation results. Recall that for each considered policy arrangement, we start the simulation at the initial debt level (or a level of promised utility) corresponding to a zero profit intervention in the market equilibrium at a debt-output ratio of 100%.

	RRA		
	1	2	3
Market Equilibrium	0.0209	0.0190	0.0162
First-best	0	0	0
Second-best	0.0080	0.0066	0.0055
State-contingent	0.0187	0.0152	0.0132
Grexit	0.0312	0.0338	0.0351

Table 2: Consumption volatility over the first 10 years

Table 2 shows the variance of log consumption over the first ten simulation periods. The constrained optimum (second best) yields substantially smoother consumption than the market allocation. The consumption volatility is not much affected by the degree of relative risk aversion.

[26] When changing the relative risk aversion, the internally calibrated parameters, φ, ξ, $\bar{\phi}$, and η, are recalibrated so the model meets the stated empirical moments.

Table 3 reports the expected duration (the cross-sectional mean duration resulting from a large number of stochastic simulations) of the recession which is inversely related to the average level of reform effort over the recession period. For example, in the benchmark calibration the first-best allocation implies an expected recession of 16.1 years, which corresponds to a constant recovery probability (reform effort level) of 6.2%.

	RRA		
	1	2	3
Market Equilibrium	6.2	6.7	7.2
First-best	15.1	16.1	16.9
Second-best	7.0	6.7	6.3
State-contingent	7.4	8.2	8.5
Grexit	8.0	6.6	5.7

Table 3: Expected duration of the recession (in years)

Interestingly, the expected duration of the recession is roughly equal for the market equilibrium and the constrained optimum (where the ranking depends on the relative risk aversion). In contrast, the allocation with state-contingent debt yields significantly lower reform effort. This illustrates a main point of our paper: access to better insurance magnifies the moral hazard problem.

In the last row of Table 1 we consider a simple version of an "austerity program" with the following features: (1) the external institution guarantees the debt of the borrower (so the market price is always $1/R$), (2) the borrower cannot issue additional debt in the private market (i.e., "fiscal austerity"), (3) the institution commits to terminate the arrangement and induce an outright default ("Grexit") if the borrower attempts to renegotiate the debt obligations (in which case the institution steps in an honors the issued guarantees) or increase the debt level, (4) once the recession is over, no new guarantees will be issued, and (5) the initial debt is increased to a level such that the institution has zero expected profits (i.e., the austerity program breaks even in expectation). However, there are no conditions imposed on the reform effort.

One the one hand, this program could potentially be good because it mitigates the moral hazard program. To see this, note that with a debt guarantee the bond price does not change when the recession ends, so more of the gains from reform effort will accrue to the borrower. On the other hand, the program will induce inefficient default. For the benchmark calibration, we find that such an austerity program yields a loss equivalent to a one-time tax of 5.7%, so the borrower would prefer the market to this austerity program. However, when the risk aversion is sufficiently low, the austerity program is actually better than the market allocation. With a lower risk aversion, the insurance value is smaller and the improvements in the moral hazard problem outweighs the inefficiencies.

8 Conclusions

This paper presents a theory of sovereign debt dynamics under limited commitment. A sovereign country issues debt to smooth consumption during a recession whose duration is uncertain and endogenous. The expected duration of the recession depends on the intensity of (costly) structural reforms. Both elements – the risk of repudiation and the need for structural reforms – are salient features of the recent European debt crisis.

The laissez-faire equilibrium features repeated debt renegotiations. Renegotiations are more likely to occur during recessions and when the country has accumulated a high level of debt. As a recession drags on, the country has an incentive to go deeper into debt. A higher level of debt in turn may deter rather than stimulate economic reforms.

The theory bears normative predictions that are relevant for the management of the European crisis. The market equilibrium is inefficient for two reasons. On the one hand, the government of the sovereign country underinvests in structural reforms. The intuitive reason is that the short-run cost of reforms is borne entirely by the country, while future benefits of reforms accrue in part to the creditors in the form of an *ex-post* increased price of debt, due to a reduction in the probability of renegotiation. On the other hand, the limited commitment to honor debt induces high risk premia and excess consumption volatility. A well-designed intervention by an international institution can improve welfare, as long as the institution can monitor the reform process. While we assume, for tractability, that the international institution can monitor reforms perfectly, our results carry over to a more realistic scenario where reforms are only imperfectly monitored. The optimal policy also entails an assistance program whereby an international organization provides the country with a constant transfer flow, deferring the repayment of debt to the time when the recession ends. The optimal contract takes into account that this payment is itself subject to renegotiation risk.

A second implication is that, when the government of the indebted country credibly threatens to renege on an existing agreement, concessions should be made to avoid an outright repudiation. Contrary to a common perception among policy makers, a rigid commitment to enforce the terms of the original agreement is not optimal. Rather, the optimal policy entails the possibility of multiple renegotiations, which are reflected in the terms of the initial agreement.

To retain tractability, we make important assumptions that we plan to relax in future research. First, in our theory the default cost follows an exogenous stochastic process. In a richer model, this would be part of the equilibrium dynamics. Strategic delegation is a potentially important extension. In the case of Greece, voters may have an incentive to elect a radical government with the aim of delegating the negotiation power to an agent that has or is perceived to have a lower default cost than voters do (cf. Rogoff 1985). In our current model, however, the stochastic process governing the creditor's outside option is exogenous, and is outside of the control of the government and creditors.

Second, again for simplicity, we assume that renegotiation is costless, that creditors can perfectly coordinate and that they have full bargaining power in the renegotiation game. Each of these assumptions could be relaxed. For instance, in reality the process of negotiation may entail costs. Moreover, as in the recent contention between Argentina and the so-called vulture funds, some creditors may hold out and refuse to accept a restructuring plan signed by a syndicate of lenders. Finally, the country may retain some bargaining power in the renegotiation. All these extensions would introduce interesting additional dimensions, and invalidate some of the strong efficiency results (for instance, the result that the laissez-faire economy attains the constrained optimum in the absence of income fluctuations). However, we are confident that the gist of the results is robust to these extensions.

Finally, by focusing on a representative agent, we abstract from conflicts of interest between different groups of agents within the country. Studying the political economy of sovereign debt would be an interesting extension. We leave the exploration of these and other avenues to future work.

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A Appendix: Proofs of lemmas, propositions, and corollaries

Proof of Proposition 1. The case of normal times is straightforward. Consider an economy starting in a recession. Perfect insurance implies constant consumption and effort. The efficient solution maximizes the discounted utility, $W = u(c)/(1-\beta) - X(p)/(1-\beta(1-p))$ with respect to c and p, subject to the following intertemporal budget constraint:

$$\frac{1}{1-\beta(1-p)}(\underline{w}-c) + \frac{\beta}{1-\beta}\frac{p}{1-\beta(1-p)}(\bar{w}-c) = b. \tag{43}$$

Note that since insurance is provided in actuarially fair markets the budget constraint holds in expected terms. Writing the Lagrangean, and differentiating it with respect to c yields $u'(c) = \lambda$. Differentiating the Lagrangean with respect to p and simplifying terms yields

$$X'(p)\left(\frac{1-\beta}{\beta} + p\right) - X(p) = (\bar{w} - \underline{w}) \times u'(c), \tag{44}$$

which is identical to equation (1) in the text. Equation (43) defines a positively sloped locus in the plane (p, c), while equation (44) defines a negatively sloped locus in the same plane. Unless the solution for effort is a corner, the two equations pin down a unique interior solution for p and c. Consider the comparative statics with respect to b (note that b only features in equation (43)). An increase in b yields a decrease in c and an increase in p. This concludes the proof. ■

Proof of Lemma 1. The first part follows from the definitions of Φ and $\hat{b}$. For the second part, note that $\Phi(b, w) = W(0, w) - W(b, w)$ such that if $W(b, w)$ is strictly decreasing in b, then $\Phi(b, w)$ must be strictly increasing in b. Thus, the inverse function of $\Phi(b, w)$ exits and is given by the expressions stated in Lemma 1. ■

Proof of Proposition 2. We prove the existence of the Markov equilibrium by showing that the value function W is a fixed-point of a monotone mapping following Stokey and Lucas (1989, Theorem 17.7).

Let $\Gamma(\underline{w})$ be the space of bounded, continuous, and decreasing functions defined over $[\underline{b}, \tilde{b}]$. Moreover, let d_∞ denote the supremum norm such that (Γ, d_∞) is a complete metric space. Let $z \in \Gamma(\underline{w})$ and γ be a real constant representing the outside option under outright default. $T(z;\gamma)$ is similar to the Bellman equation of the Markov equilibrium in the text, but differs in that the value of outright default in the recession state is *exogenously* given by $\gamma - \phi$. We first establish the existence of an equilibrium for an exogenous γ, and then extend the argument to an endogenous outside option as in the Markov equilibrium.

Define the following mapping:

$$T(z;\gamma,w)(b) = \max_{b' \in [\underline{b}, \tilde{b}]} u(Q(b';z,\gamma)b' - b + w) + Z(b';z,\gamma,w), \tag{45}$$

where $\Phi(x;z,\gamma) = \gamma - z(x)$, and $z\left(\hat{b}(\phi;z,\gamma)\right) = \gamma - \phi$. In addition, let $T^0(z;\gamma) = z$, $T^n(z;\gamma) = T(T^{n-1}(z;\gamma);\gamma)$, $n = 1, 2, \ldots$, and $z^n \equiv T^n(z;\gamma)$. Moreover, when $w = \bar{w}$,

$$\begin{aligned} Z(x;z,\gamma,\bar{w}) &= \beta E\left[\max\left\{z(x), \gamma - \phi'\right\}\right], \text{ and} \\ Q(x;z,\gamma,\bar{w})x &= R^{-1}E\left[\min\left\{x, \hat{b}(\phi;z,\gamma)\right\}\right]. \end{aligned}$$

Correspondingly, when $w = \underline{w}$

$$\begin{aligned} Z(x;z,\gamma,\underline{w}) &= -X(\Psi(x;z,\gamma)) + \beta \begin{bmatrix} \Psi(x;z,\gamma)E\left[\max\left\{W(x,\bar{w}), W(0,\bar{w}) - \phi'\right\}\right] \\ +(1-\Psi(x;z,\gamma))E\left[\max\left\{z(x), \gamma - \phi'\right\}\right] \end{bmatrix}, \\ X'(\Psi(x;z,\gamma)) &= \beta \begin{bmatrix} \left[1 - F(\Phi(x,\bar{w}))\right] W(x,\bar{w}) \\ -\left[1 - F(\Phi(x;z,\gamma))\right] z(x) \end{bmatrix}, \\ Q(x;z,\gamma,\underline{w})x &= R^{-1}E \begin{bmatrix} \Psi(x;z,\gamma)\min\left\{x, \hat{b}(\phi,\bar{w})\right\} \\ +(1-\Psi(x;z,\gamma))\min\left\{x, \hat{b}(\phi;z,\gamma)\right\} \end{bmatrix}. \end{aligned}$$

Note that the mapping during recession, $T(z;\gamma,\underline{w})$, takes as given the existence of the equilibrium functions $W(b,\bar{w})$, $\Phi(b,\bar{w})$ and $\hat{b}(\phi,\bar{w})$. This is legitimate because one can prove existence recursively, first for normal times and then in recession.

We define upper and lower bounds for the value functions. More formally, $W_{MIN} \equiv u(\underline{w})/(1-\beta) - \phi_{\max}$, and $W_{MAX} \equiv u(\bar{w})/(1-\beta)$. It is straightforward to see that W_{MIN} and W_{MAX} are, respectively, lower and upper bounds to the present utility the country can attain in equilibrium.

We establish first that the operator $T(z;\gamma,w)(b)$ is a uniformly continuous, bounded and decreasing (in b) mapping of the function space Γ into itself. Continuity follows by the Theorem of the Maximum. Boundedness follows from the fact that utility is bounded because consumption, reform effort, the support of the default cost, and the elements of Γ are bounded. Finally, to establish that the mapping T is decreasing in b note that, for any $\Delta > 0$, $T(z;\gamma,w)(b+\Delta) < T(z;\gamma,w)(b)$ since

$$T(z;\gamma,w)(b+\Delta) = \max_{b' \in [\underline{b}, \tilde{b}]} u\left(Q(b';z,\gamma,w)b' + w - (b+\Delta)\right) + Z\left(b';z,\gamma,w\right)$$

$$\begin{aligned} &= u\left(Q(B(b+\Delta;z,\gamma,w);z,\gamma,w)\cdot B(b+\Delta;z,\gamma,w) + w - (b+\Delta)\right) + Z\left(B(b+\Delta;z,\gamma,w);z,\gamma,w\right) \\ &< u\left(Q(B(b+\Delta;z,\gamma,w);z,\gamma,w)\cdot B(b+\Delta;z,\gamma,w) + w - b\right) + Z\left(B(b+\Delta;z,\gamma,w);z,\gamma,w\right) \\ &\leq u\left(Q(B(b;z,\gamma,w);z,\gamma,w)\cdot B(b;z,\gamma,w) + w - b\right) + Z\left(B(b;z,\gamma,w);z,\gamma,w\right) = T(z;\gamma,w)(b), \end{aligned}$$

where $B(b;z,\gamma,w) = \arg\max_{b'\in[\underline{b},\tilde{b}]} u(Q(b';z,\gamma,w)b' - b + w) + Z(b';z,\gamma,w)$.

Next, we establish that the mapping $T(z;\gamma,w)$ is monotone in z, that its fixed-point $z^*(z;\gamma,w)(b) = \lim_{n\to+\infty} T^n(z;\gamma,w)(b)$ exists, and that $z^* \in \Gamma$. To this aim, consider $z, z^+ \in \Gamma$ with $z(b) < z^+(b)$, $\forall b \in [\underline{b},\tilde{b}]$. Since z and z^+ are decreasing in b, implying that $z^+\left(\hat{b}(\phi;z^+,\gamma)\right) = \gamma - \phi > z\left(\hat{b}(\phi;z^+,\gamma)\right)$, it follows immediately that $\hat{b}(\phi;z,\gamma) < \hat{b}(\phi;z^+,\gamma)$. Consequently, $\Phi(b;z,\gamma) \geq \Phi(b;z^+,\gamma)$ for all $b \in [\underline{b},\tilde{b}]$, which in turn implies that $Q(b;z,\gamma,w) \leq Q(b;z^+,\gamma,w)$. Consider first the case of $w = \bar{w}$. We establish that $z^+ > z \Leftrightarrow T(z^+;\gamma,\bar{w})(b) > T(z;\gamma,\bar{w})(b)$, since

$$\begin{aligned}
T\left(z^+;\gamma,\bar{w}\right)(b) &= \max_{b'\in[\underline{b},\tilde{b}]} u\left(Q(b';z^+,\gamma,\bar{w})b' - b + \bar{w}\right) + Z\left(b';z^+,\gamma,\bar{w}\right) \\
&= u\left(Q(B\left(b;z^+,\gamma,\bar{w}\right);z^+,\gamma,\bar{w}) \cdot B\left(b;z^+,\gamma,\bar{w}\right) - b + \bar{w}\right) + Z\left(B\left(b;z^+,\gamma,\bar{w}\right);z^+,\gamma,\bar{w}\right) \\
&\geq u\left(Q(B\left(b;z,\gamma,\bar{w}\right);z^+,\gamma,\bar{w}) \cdot B\left(b;z,\gamma,\bar{w}\right) - b + \bar{w}\right) + Z\left(B\left(b;z,\gamma,\bar{w}\right);z^+,\gamma,\bar{w}\right) \\
&> u\left(Q(B\left(b;z,\gamma,\bar{w}\right);z,\gamma,\bar{w}) \cdot B\left(b;z,\gamma,\bar{w}\right) - b + \bar{w}\right) + Z\left(B\left(b;z,\gamma,\bar{w}\right);z,\gamma,\bar{w}\right) \\
&= \max_{b'\in[\underline{b},\tilde{b}]} u(Q\left(b';z,\gamma,\bar{w}\right)b' + \bar{w} - b) + Z\left(b';z,\gamma,\bar{w}\right) = T\left(z;\gamma,\bar{w}\right)(b).
\end{aligned}$$

where the first inequality follows from the fact that $B(b;z,\gamma,\bar{w})$ yields a lower utility relative to the optimal $B(b;z^+,\gamma,\bar{w})$. The second inequality follows from the fact that $Q(b;z,\gamma,\bar{w}) \leq Q(b;z^+,\gamma,\bar{w})$ for all $b \in [\underline{b},\tilde{b}]$.

Consider, next, the case of $w = \bar{w}$. Let $E\bar{V}(b) \equiv E[V(b,\phi,\bar{w})]$. We establish that $z^+ > z \Leftrightarrow T(z^+;\gamma,\underline{w})(b) > T(z;\gamma,\underline{w})(b)$, since

$$\begin{aligned}
T\left(z^+;\gamma,\underline{w}\right)(b) &= \max_{b'\in[\underline{b},\tilde{b}]} u\left(Q(b';z^+,\gamma)b' - b + \underline{w}\right) + Z\left(b';z^+,\gamma\right) \\
&= u\left(Q(B\left(b;z^+,\gamma\right);z^+,\gamma) \cdot B\left(b;z^+,\gamma\right) - b + \underline{w}\right) - X\left(\Psi\left(B\left(b;z^+,\gamma\right);z^+,\gamma\right)\right) \\
&\quad + \left[\begin{array}{c} \Psi\left(B\left(b;z^+,\gamma\right);z^+,\gamma\right)\beta E\bar{V}\left(B\left(b;z^+,\gamma\right)\right) + \\ \left(1-\Psi\left(B\left(b;z^+,\gamma\right);z^+,\gamma\right)\right)\beta E\left[\max\left\{\gamma-\phi, z^+\left(B\left(b;z^+,\gamma\right)\right)\right\}\right] \end{array}\right] \\
&\geq u\left(Q(B\left(b;z,\gamma\right);z^+,\gamma) \cdot B\left(b;z,\gamma\right) - b + \underline{w}\right) - X\left(\Psi\left(B\left(b;z,\gamma\right);z,\gamma\right)\right) \\
&\quad + \left[\begin{array}{c} \Psi\left(B\left(b;z,\gamma\right);z,\gamma\right)\beta E\bar{V}\left(B\left(b;z,\gamma\right)\right) + \\ \left(1-\Psi\left(B\left(b;z,\gamma\right);z,\gamma\right)\right)\beta E\left[\max\left\{\gamma-\phi, z^+\left(B\left(b;z,\gamma\right)\right)\right\}\right] \end{array}\right] \\
&> u\left(Q\left(B\left(b;z,\gamma\right);z,\gamma\right) \cdot B\left(b;z,\gamma\right) - b + \underline{w}\right) - X\left(\Psi\left(B\left(b;z,\gamma\right);z,\gamma\right)\right) \\
&\quad + \left[\begin{array}{c} \Psi\left(B\left(b;z,\gamma\right);z,\gamma\right)\beta E\bar{V}\left(B\left(b;z,\gamma\right)\right) + \\ \left(1-\Psi\left(B\left(b;z,\gamma\right);z,\gamma\right)\right)\beta E\left[\max\left\{\gamma-\phi, z\left(B\left(b;z,\gamma\right)\right)\right\}\right] \end{array}\right] \\
&= \max_{b'\in[\underline{b},\tilde{b}]} u(Q(b';z,\gamma)b' - b + \underline{w}) + Z(b';z,\gamma) = T(z;\gamma)(b,\underline{w}),
\end{aligned}$$

where the same logic as above applies.

We have established that $T(z;\gamma,w)$ is a monotone mapping with the sup norm. This mapping is an equicontinuous family (each function in Γ is uniformly continuous and the continuity is uniform for all functions in Γ). Then, Stokey and Lucas (1989, Theorem 17.7) ensures that the fixed point of $T(z;\gamma,w)$ exists, is an element of Γ and is given by $z^*(z;\gamma,w) = \lim_{n\to+\infty} T^n(z;\gamma,w)$.

Thus far, we have proven the existence of at least one fixed point of the mapping T for any exogenous outside option, $\gamma \in [W_{MIN}, W_{MAX}]$. We now use a different fixed point argument to show that, conditional on an initial z, there exists a unique fixed point that the outside options in normal times and recession are, respectively, $\gamma_z^*(\bar{w})$ and $\gamma_z^*(\underline{w})$ with the following properties: $z^*(z;\gamma_z^*(\bar{w}),\bar{w})(0) = \gamma_z^*(\bar{w}) = W_{MAX}$ and $z^*(z;\gamma_z^*(\underline{w}),\underline{w})(0) = \gamma_z^*(\underline{w}) \in (W_{MIN}, W_{MAX})$. To

see why, note that, by the Theorem of the Maximum, $z^*(z;\gamma,w)(b) = \lim_{n\to+\infty} T^n(z;\gamma,w)(b)$ is continuous in γ. Moreover, z^* is bounded since $z^*(z;\gamma,w) \in [W_{MIN}, W_{MAX}]$. Thus, the Brouwer fixed-point theorem ensures that there exists a $\gamma_z \in [W_{MIN}, W_{MAX}]$ such that $z^*(z;\gamma_z,w)(0) = \gamma_z$. Since $z^*(z;\gamma_z,w)(0) = \gamma_z - \Phi(0;z^*,\gamma_z)$, this is equivalent to say that, at each fixed point, $\Phi(0;z^*,\gamma_z^*(w)) = 0$. To prove uniqueness, we note then that $\Phi\left(0;z_\gamma^*,\gamma\right)$ is monotone increasing for all $\gamma \in [W_{MIN}, W_{MAX}]$, as the set of (potential) states of nature in which the outside option is attractive expands when γ increases. Therefore, there exists a unique fixed point $\gamma_z^*(w)$ such that $\Phi(0;z^*,\gamma_z^*(w)) = 0$. In particular in normal times $\gamma_z^*(\bar{w}) = W(0,\bar{w}) = W_{MAX}$. In recession, $\gamma_z^*(\underline{w}) = W(0,\underline{w}) \in (W_{MIN}, W_{MAX})$.

The results proven thus far allow us to claim the existence of an equilibrium value function W such that $W(b,w) = T(W;W(0,w),w)(b)$. The definition of the remaining equilibrium functions follow from Definition 1. This establishes the existence of a Markov equilibrium. The continuity of the value function $W(b,w)$ in b follows from the Theorem of the Maximum, and implies that also the equilibrium functions Φ, Q and Ψ are continuous in b. It is also straightforward to show that W is strictly decreasing in b and, hence, that $\Phi(b,w) = W(0,w) - W(b,w)$ is strictly increasing in b. Finally, we claim that the bond revenue, $Q(b',w)b'$, is (weakly) monotone increasing in b'. This follows from equations (6)–(8), the fact that Φ increasing in b, and from Lemma 1.

Next, we prove that B is monotone decreasing in b by applying Topkis' Theorem. To this aim, define $B(b,w) = \arg\max_{b'\in[\underline{b},\tilde{b}]} O(b',b,w)$ where

$$O(b',b,w) = u(Q(b',w)b' - b + w) + \beta Z(b',w). \tag{46}$$

We first establish that the objective function $O(b',b,w)$ is supermodular in (b',b), i.e., if $b'_H > b'_L$ and $b_H > b_L$, then, $O(b'_H,b_H,w) - O(b'_L,b_H,w) \geq O(b'_H,b_L,w) - O(b'_L,b_L,w)$. To this aim, note that

$$\begin{aligned} & u(Q(b'_H,w)b'_H - b_H + w) - u(Q(b'_L,w)b'_L - b_H + w) \\ \geq\ & u(Q(b'_H,w)b'_H - b_L + w) - u(Q(b'_L,w)b'_L - b_L + w). \end{aligned} \tag{47}$$

This inequality follows from the concavity of the utility function and the fact that $Q(b,w)b$ is increasing with b, implying that, for $\Delta > 0$,

$$u'(Q(x,w)x - b + w) - u'(Q(x+\Delta,w)(x+\Delta) - b + w) \geq 0.$$

Rearranging terms in (47), and adding and subtracting continuation values on both sides of the inequality yields

$$\begin{aligned} & u(Q(b'_H,w)b'_H - b_H + w) - u(Q(b'_H,w)b'_H - b_L + w) \\ & +\beta Z(b'_H,w) - \beta Z(b'_H,w) \\ \geq\ & u(Q(b'_L,w)b'_L - b_H + w) - u(Q(b'_L,w)b'_L - b_L + w) \\ & +\beta Z(b'_L,w) - \beta Z(b'_L,w) \end{aligned}$$

that is equivalent to $O(b'_H,b_H,w) - O(b'_H,b_L,w) \geq O(b'_L,b_H,w) - O(b'_L,b_L,w)$. This establishes that $O(b',b,w)$ is supermodular in (b',b). Topkis' Theorem implies then that $B(b_H,w) \geq B(b_L,w)$. This concludes the proof of Proposition 2. ■

Corollary 1 *Let the operator T be defined as in equation (45). If, for $w \in \{\underline{w},\bar{w}\}$, $W(b,w) = \lim_{n\to\infty} T^n(W_{MIN};W(b,w),w)(b) = \lim_{n\to\infty} T^n(W_{MAX};W(b,w),w)(b)$, then the Markov equilibrium is unique (i.e., there exists a unique equilibrium value function $W(b,w)$ satisfying Definition 1).*

Proof of Corollary 1. The proof follows immediately from the Corollary to Theorem 17.7 in Stokey and Lucas (1989). ■

Proof of Proposition 3. The proof is an application of the generalized envelope theorem in Clausen and Strub (2013) which allows for discrete choices (i.e., repayment or renegotiation) and non-concave value functions. Consider the program $W(b,w) = \max_{b' \in [\underline{b}, \tilde{b}]} O(b', b, w)$ where O is defined in equation (46). Theorem 1 in Clausen and Strub (2013) ensures that if we can find a *differentiable lower support function* (DLSF) for O, then O is differentiable for all $b' \in \mathcal{B}(w)$.

We start by proving the lemma for the case of $w = \underline{w}$. The strategy of the proof involves finding DLSF for the equilibrium functions $Q(b', \underline{w})b'$ and $Z(b', \underline{w})$. To this aim, we follow the strategy of Benveniste and Scheinkman (1979), and consider the value function of a pseudo-borrower that chooses debt issuance $b' = B(x,w)$ instead of the optimal $b' = B(b,w)$,

$$\widetilde{W}(b,x,w) \equiv u\left(Q\left(B(x,w),w\right)B(x,w) - b + w\right) + Z\left(B(x,w),w\right).$$

Note that $\widetilde{W}$ is differentiable and strictly decreasing in b. Since debt issuance is chosen suboptimally, it must be the case that $\widetilde{W}(b,x,w) \leq W(b,w)$ with equality holding at $x = b$. Furthermore, let the pseudo-borrower set the default threshold at the level $\widetilde{\Phi}(b,x,w) = W(0,w) - \widetilde{W}(b,x,w)$, where $\widetilde{\Phi}(b,x,w) \geq \Phi(b,w)$. Thus, the pseudo-borrower will find it optimal to renegotiate for a range of ϕ larger than $\Phi(b,w)$. Note that $\widetilde{\Phi}(b,x,w)$ is differentiable and strictly increasing in b. Thus, the inverse function exists and is such that $\widetilde{\Phi}^{-1}_{x,w}(\phi) \leq \hat{b}(\phi,w)$ (where we define $\widetilde{\Phi}_{x,w}(b) \equiv \widetilde{\Phi}(b,x,w)$).

Consider first the case in which $w = \underline{w}$. Let

$$\widetilde{O}\left(b', b, x, \underline{w}\right) = u\left(\widetilde{Q}(b', x, \underline{w})b' - b + \underline{w}\right) + \widetilde{Z}(b', x, \underline{w}),$$

where the pseudo bond revenue function is given by

$$\begin{aligned}\widetilde{Q}\left(b', x, \underline{w}\right)b' &= \frac{1}{R}\widetilde{\Psi}(b',x)\left(\left[1 - F(\widetilde{\Phi}\left(b', x, \bar{w}\right))\right]b' + \int_0^{\widetilde{\Phi}(b',\bar{w})} \widetilde{\Phi}^{-1}_{x,\bar{w}}(\phi)dF(\phi)\right) \\ &\quad + \frac{1}{R}(1 - \widetilde{\Psi}(b',x))\left(\left[1 - F(\widetilde{\Phi}\left(b', x, \underline{w}\right)\right]b' + \int_0^{\widetilde{\Phi}(b',\underline{w})} \widetilde{\Phi}^{-1}_{x,\underline{w}}(\phi)dF(\phi)\right),\end{aligned}$$

and the continuation value is given by

$$\widetilde{Z}(b', x, \underline{w}) = -X(\widetilde{\Psi}(b',x)) + \beta\left[\begin{array}{c}\widetilde{\Psi}(b',x)E\max\left\{\widetilde{W}\left(b', x, \bar{w}\right), \widetilde{W}(0, x, \bar{w}) - \phi'\right\} \\ +(1 - \widetilde{\Psi}(b',x))E\max\left\{\widetilde{W}\left(b', x, \underline{w}\right), \widetilde{W}(0, x, \underline{w}) - \phi'\right\}\end{array}\right],$$

having defined $\widetilde{\Psi}$ as

$$\widetilde{\Psi}(b',x) = \left(X'\right)^{-1}\left(\beta\left[\begin{array}{c}E\max\left\{\widetilde{W}(b', x, \bar{w}), \widetilde{W}(0, x, \bar{w}) - \phi'\right\} \\ -E\max\left\{\widetilde{W}(b', x, \underline{w}), \widetilde{W}(0, x, \underline{w}) - \phi'\right\}\end{array}\right]\right).$$

Note that $\widetilde{Q}$, $\widetilde{Z}$ and $\widetilde{\Psi}$ are differentiable in b' since we established above that $\widetilde{W}$ and $\widetilde{\Phi}$ are differentiable.

Then, $\widetilde{O}$ is a DLSF for O such that $\widetilde{O}(b', b, x, \underline{w}) \leq O(b', b, \underline{w})$ with equality (only) at $b' = x$. Thus, Theorem 1 in Clausen and Strub (2013) ensures that the objective function $O(b', b, \underline{w})$ is differentiable

in b' at $b' = B(b, \underline{w})$ and that $\partial O(b', b, \underline{w})/\partial b' = \partial \widetilde{O}(b', b, B(b, \underline{w}), \underline{w})/\partial b' = 0$. In this case, a standard first-order condition yields

$$\frac{\partial u\left(Q(b', \underline{w})b' - b + \underline{w}\right)}{\partial b'} + \frac{\partial Z(b', \underline{w})}{\partial b'} = 0.$$

Moreover, Lemma 3 in Clausen and Strub (2013) ensures that the functions $W(b, \underline{w})$, $Z(b, \underline{w})$, $\Phi(b, \underline{w})$, $Q(b, \underline{w})$, and $\Psi(b)$ are differentiable and that a standard envelope condition applies, namely,

$$\begin{aligned}
\frac{\partial Z(b', \underline{w})}{\partial b'} &= \beta \left[\begin{array}{c} \Psi(b')\left[1 - F(\Phi(b', \bar{w}))\right] \frac{\partial W(b', \bar{w})}{\partial b'} \\ +(1 - \Psi(b'))\left[1 - F(\Phi(b', \underline{w}))\right] \frac{\partial W(b', \underline{w})}{\partial b'} \end{array} \right], \\
\frac{\partial W(b, \underline{w})}{\partial b} &= -u'\left(Q(B(b, \underline{w}), \underline{w})B(b, \underline{w}) - b + \underline{w}\right) < 0.
\end{aligned}$$

The proof for the case of $w = \bar{w}$ follows the same strategy and is therefore omitted. ■

Proof of Lemma 2. In normal times, the bond revenue function can be written as

$$Q(b', \bar{w})b' = R^{-1} E \min\left\{b', \hat{b}(\phi, \bar{w})\right\}.$$

Note that the minimum function in the expectation operator is concave in b'. Since the sum of concave functions is still concave, then also $E\left\{\min b', \hat{b}(\phi, \bar{w})\right\}$ must be concave in b'. This implies that the marginal bond revenue is falling in b'. Differentiating $Q(b', \bar{w})b'$ with respect to $b' \in [\underline{b}, \tilde{b}]$ yields:

$$\begin{aligned}
\frac{\partial\left(Q(b', \bar{w})b'\right)}{\partial b'} &= Q\left(b', \bar{w}\right) + \frac{\partial}{\partial b'}\left(R^{-1}\left(1 - F\left(\Phi\left(b', \bar{w}\right)\right)\right) + (Rb')^{-1}\int_0^{\Phi(b', \bar{w})} \bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right) \times b' \\
&= Q\left(b', \bar{w}\right) - R^{-1} b' f\left(\Phi\left(b', \bar{w}\right)\right) \times \frac{\partial \Phi\left(b', \bar{w}\right)}{\partial b'} \\
&\quad + b'(Rb')^{-1}\bar{\Phi}^{-1}\left(\Phi\left(b', \bar{w}\right)\right) f\left(\Phi\left(b', \bar{w}\right)\right) \\
&\quad \times \frac{\partial \Phi\left(b', \bar{w}\right)}{\partial b'} - \underbrace{\frac{1}{R}\frac{1}{b'}\int_0^{\Phi(b', \bar{w})}\left(\bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right)}_{=Q(b', \bar{w}) - \frac{1}{R}\left(1 - F(\Phi(b', \bar{w}))\right)} \\
&= R^{-1}\left(1 - F\left(\Phi\left(b', \bar{w}\right)\right)\right).
\end{aligned}$$

■

Proof of Proposition 4. The first-order condition of (10) for $w = \bar{w}$ yields:

$$\frac{\partial}{\partial b'}\left\{b' \times Q\left(b', \bar{w}\right)\right\} \times u'(c) + \frac{\partial}{\partial b'}\beta E V\left(b', \bar{w}\right) = 0,$$

The function V has a kink at $b' = \hat{b}(\phi, \bar{w})$. Consider, first, the range of realizations $\phi > \bar{\Phi}(b)$ implying that $b' < \hat{b}(\phi, \bar{w})$. Differentiating V in this range yields:

$$\frac{\partial}{\partial b} V(b, \phi, \bar{w}) = -u'\left[Q\left(B(b, \bar{w}), \bar{w}\right) \times B(b, \bar{w}) + \bar{w} - b\right].$$

Next, consider the region of renegotiation, $\phi < \bar{\Phi}(b)$, implying that $b > \hat{b}(\phi, \bar{w})$. In this case, $\frac{\partial}{\partial b} V(b, \phi, w) = 0$. Using the results above one obtains:

$$\begin{aligned}
\frac{\partial}{\partial b} EV(b, \bar{w}) &= \int_0^{\Phi(b,\bar{w})} \frac{\partial}{\partial b} V(b, \phi, \bar{w})\, dF(\phi) + \int_{\Phi(b,\bar{w})}^{\infty} \frac{\partial}{\partial b} V(b, \phi, \bar{w})\, dF(\phi) \\
&= -\int_{\Phi(b,\bar{w})}^{\infty} u'\left[Q\left(B(b, \bar{w}), \bar{w}\right) \times B(b, \bar{w}) + \bar{w} - b\right] dF(\phi) \\
&= -\left[1 - F\left(\Phi(b, \bar{w})\right)\right] \times u'\left[Q\left(B(b, \bar{w}), \bar{w}\right) \times B(b, \bar{w}) + \bar{w} - b\right] \qquad (48)
\end{aligned}$$

Plugging this expression back into the FOC, and leading the expression by one period, yields

$$0 = \frac{\partial}{\partial b'} \left\{b' \times Q\left(b', \bar{w}\right)\right\} \times u'(c) - \beta \left[1 - F\left(\Phi\left(b', \bar{w}\right)\right)\right] \times u'\left(c'|_{H,\bar{w}}\right).$$

Finally, recall that $\frac{\partial}{\partial b}\left\{b \times Q(b, \bar{w})\right\} = \frac{1}{R}\left(1 - F\left(\Phi(b, \bar{w})\right)\right)$, as established in the proof of Lemma 2. Thus, for $F(\Phi(b, \bar{w})) < 1$, the above equation is equivalent to the CEE (13) in the proposition. Although the first-order condition is also satisfied at $b' = \bar{b}$, it is possible to show that this is never an optimal solution as long as $b < \bar{b}$ (details in the online appendix).

Consider, next, the properties of the equilibrium functions C and W. Using the definition of C in equation (11) and Lemma 2, standard algebra shows that C is continuously differentiable and decreasing, with derivative $\partial C(b, \bar{w})/\partial b = R^{-1}\left[1 - F(\Phi(b, \bar{w}))\right] - 1 < 0$. Since $B(b, \bar{w}) = b$, then B maps the complete domain of b into itself. Proposition 3 implies then that value function is differentiable everywhere, and that the envelope condition $\partial W(b, \bar{w})/\partial b = -u'(C(b, \bar{w}))$ applies. Differentiating this condition with respect to b yields $\partial^2 W(b, \bar{w})/(\partial b)^2 = -u''(C(b, \bar{w}))\partial C(b, \bar{w})/\partial b = u''(C(b, \bar{w}))\left[1 - R^{-1}\left[1 - F(\Phi(b, \bar{w}))\right]\right] < 0$. This establishes that the value function W is twice continuously differentiable and strictly concave. This concludes the proof of the proposition. ■

Proof of Proposition 5. We start by claiming that $c^{FB}(b, w) > C(b, w)$, where $c^{FB}(b, w)$ was defined in Proposition 1 and $C(b, w)$ denotes consumption in the laissez-faire equilibrium when debt is honored. To see why, note that $\partial/\partial b\left\{W^{FB}(b, w)\right\} \geq \partial/\partial b\left\{W(b, w)\right\}$. This follows from observing that the difference between $W^{FB}(b, w)$ and $W^{FB}(b + \Delta, w)$, where $\Delta > 0$, merely reflects the utility loss from a permanent reduction in consumption $c^{FB}(b, w)$ and (in recession) a permanent increase in effort $p^{FB}(b)$. In contrast, in the market equilibrium a larger debt induces a higher volatility of consumption (recall that $\Phi(b, w)$ is monotone increasing in b – see Lemma 1 – so higher debt increases the probability of renegotiation) and (in recession) effort. Since $\partial/\partial b\left\{W^{FB}(b, w)\right\} = -u'\left(c^{FB}(b, w)\right)$ and $\partial/\partial b\left\{W(b, w)\right\} = -u'\left(C(b, w)\right)$, then the claim that $c^{FB}(b, w) > C(b, w)$ follows.

Let $\phi_{\min} \equiv \left[u(\bar{w}) - u\left((1 - \beta)(\bar{w} - \underline{w})\right)\right]/(1 - \beta)$ and $b_{PV} \equiv \underline{w} + \frac{\beta}{1-\beta}\bar{w}$. The assumption in the Proposition implies that $F(\phi_{\min}) = 0$ and $b_{PV} < \bar{w}$. Consider first the range $b \leq b_{PV}$. In this range, $W(b, \bar{w}) = W^{FB}(b, \bar{w})$. To see this, note that $W(b_{PV}, \bar{w}) = u\left(\bar{w} - (1 - \beta) b_{PV}\right)/(1 - \beta) = \phi_{\min}$. Since by assumption $F(\phi_{\min}) = 0$, no renegotiation is possible for $b \leq b_{PV}$, so the claim follows. The two claims above and Proposition 1 imply that $C(b, \bar{w}) = c^{FB}(b, \bar{w}) > c^{FB}(b, \underline{w}) \geq C(b, \underline{w})$ in the range $b \leq b_{PV}$. Consider next the range $b \in (b_{PV}, \bar{b}]$. In this range, $C(b, \underline{w}) \leq 0$ since debt exceeds the maximum present value of future income. In contrast, $C(b, \bar{w}) > 0$ since $b_{PV} < \bar{w}$. We have therefore established that $C(b, \bar{w}) > C(b, \underline{w})$ for all $b \leq \bar{b}$. ■

Proof of Lemma 3. If in the initial period (but not later) the country can contract on effort while issuing new debt, the problem becomes

$$\max_{b', p^*} \left\{u(c) - X(p^*) + \beta p^* \times EV\left(b', \bar{w}\right) + \beta(1 - p^*) \times EV\left(b', \underline{w}\right)\right\}.$$

Note that the next-period value function V is the same as in the benchmark problem with non-contractible effort, since we are considering a one-period deviation. The first-order condition with respect to p yields

$$\begin{aligned} 0 &= \frac{d}{dp^*}\left\{Q\left(b',\underline{w}\right)b'\right\}\times u'(c) - X'(p^*) + \beta\left(EV\left(b',\bar{w}\right) - EV\left(b',\underline{w}\right)\right) \\ &\Rightarrow X'(p^*) = \left[Q\left(b',\bar{w}\right) - \hat{Q}\left(b',\underline{w}\right)\right]b'\times u'(c) \\ &\quad +\beta\left[\int_0^\infty V\left(b',\phi',\bar{w}\right)dF(\phi) - \int_0^\infty V\left(b',\phi',\underline{w}\right)dF(\phi)\right] \\ &> \beta\left[\int_0^\infty V\left(b',\phi',\bar{w}\right)dF(\phi) - \int_0^\infty V\left(b',\phi',\underline{w}\right)dF(\phi)\right], \end{aligned} \tag{49}$$

where the last equation follows from the facts that $Q\left(b',\bar{w}\right) > \hat{Q}\left(b',\underline{w}\right)$, and that

$$\begin{aligned} \frac{d}{dp}\left\{Q\left(b',\underline{w}\right)b'\right\} &= \frac{d}{dp}\left\{\left[pQ\left(b',\bar{w}\right) + (1-p)\hat{Q}\left(b',\underline{w}\right)\right]b'\right\} \\ &= \left[Q\left(b',\bar{w}\right) - \hat{Q}\left(b',\underline{w}\right)\right]b'. \end{aligned}$$

The right-hand side of the inequality in equation (49) is the optimal effort in the benchmark case with non-contractible effort, given in equation (14). This establishes the lemma. ■

Proof of Proposition 6. Consider, first, the range $b \in [0, b_1)$. Differentiate equation (14) with respect to b',

$$\begin{aligned} X''\left(\Psi\left(b'\right)\right)\Psi'\left(b'\right) &= \beta\left[\int_0^\infty \frac{\partial}{\partial b'}V\left(b',\phi',\bar{w}\right)dF(\phi) - \int_0^\infty \frac{\partial}{\partial b'}V\left(b',\phi',\underline{w}\right)dF(\phi)\right] \\ &= -\beta\left[1 - F\left(\bar{\Phi}\left(b'\right)\right)\right]\times u'\left[Q\left(B\left(b',\bar{w}\right),\bar{w}\right)\times B\left(b',\bar{w}\right) + \bar{w} - b'\right] \\ &\quad +\beta\left[1 - F\left(\underline{\Phi}\left(b'\right)\right)\right]\times\left[Q\left(B\left(b',\underline{w}\right)\right)\times B\left(b',\underline{w}\right) + \underline{w} - b'\right] \end{aligned} \tag{50}$$

Taking the limit of equation (50) as $b' \to 0$ yields

$$\begin{aligned} X''\left(\Psi(0)\right)\Psi'(0) &= \beta\left[1 - F\left(\underline{\Phi}(0)\right)\right]\times\left[Q\left(B\left(0,\underline{w}\right)\right)\times B\left(0,\underline{w}\right) + \underline{w} - 0\right] \\ &\quad -\beta\left[1 - F\left(\bar{\Phi}(0)\right)\right]\times u'\left[Q\left(B\left(0,\bar{w}\right),\bar{w}\right)\times B\left(0,\bar{w}\right) + \bar{w} - 0\right] \\ &= \beta\left[u'\left(Q\left(B\left(0,\underline{w}\right)\right)\times B\left(0,\underline{w}\right) + \underline{w} - 0\right) - u'\left(\bar{w}\right)\right] > 0, \end{aligned} \tag{51}$$

where the last equation uses the facts that $\bar{\Phi}(0) = \underline{\Phi}(0) = F(0) = 0$ and that during normal times $c = \bar{w}$ if $b = 0$. Note that during recession, the annualized present value of income is strictly smaller than $\bar{w}$. Therefore, it can never be optimal to choose consumption during recession larger than or equal to $\bar{w}$ when $b = 0$. Since the marginal utility of consumption is larger in a recession than during normal times, the right-hand side of equation (51) is strictly positive. Since $X'' > 0$, then $\lim_{b\to 0}\Psi'(b) = \Psi'(0) > 0$. By continuity, it follows then that $\Psi'(b)$ will be positive for a range of b close to $b = 0$, so there must exist a $b_1 > 0$ such that $\Psi'(b) > 0$ for all $b \in [0, b_1)$.
Consider, next, the range $b \in \left[b^-, \bar{b}\right)$, in which case $F\left(\underline{\Phi}(b)\right) = 1$ and $F\left(\bar{\Phi}(b)\right) < 1$. This implies that equation (50) can be written as

$$X''\left(\Psi(b)\right)\Psi'(b) = -\beta\left[1 - F\left(\bar{\Phi}(b)\right)\right]\times u'\left[Q\left(B\left(b,\bar{w}\right),\bar{w}\right)\times B\left(b,\bar{w}\right) + \bar{w} - b\right] < 0,$$

which establishes that $\Psi'(b) < 0$ for all $b \in \left[b^-, \bar{b}\right)$ and with strict inequality also for $b = b^-$. By continuity, it follows then that there exists a $b_2 < b^-$ such that $\Psi'(b) < 0$ for all $b \in \left(b_2, \bar{b}\right)$. Finally, in the range where $b \geq \bar{b}$, $F(\underline{\Phi}(b)) = F\left(\bar{\Phi}(b)\right) = 1$ so the right-hand side of equation (50) becomes zero, implying that $\Psi'(b) = 0$. ■

Proof of Lemma 4. Differentiating the bond revenue with respect to b yields

$$\begin{aligned}
\frac{d}{db}\left\{Q(b,\underline{w})\, b\right\} &= \frac{d}{db}\left\{pbQ(b,\bar{w}) + (1-p)\, b\hat{Q}(b,\underline{w})\right\} + \Psi'(b) \times \left(Q(b,\bar{w}) - \hat{Q}(b,\underline{w})\right) b \\
&= \Psi(b) \times \frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1-\Psi(b)) \times \frac{1}{R}\left(1 - F\left(\underline{\Phi}(b)\right)\right) \\
&\quad + \Psi'(b) \times \left(Q(b,\bar{w}) - \hat{Q}(b,\underline{w})\right) b,
\end{aligned} \tag{52}$$

where the second equality can be derived as following:

$$\begin{aligned}
\frac{d}{db}\left\{pbQ(b,\bar{w}) + (1-p)\, b\hat{Q}(b,\underline{w})\right\} &= p\frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1-p)\,\hat{Q}(b,\underline{w}) \\
&\quad + (1-p)\left[\begin{array}{c} -\frac{b}{R} f\left(\underline{\Phi}(b)\right) \times \underline{\Phi}'(b) - \\ \frac{1}{R}\frac{1}{b}\int_0^{\underline{\Phi}(b)}\left(\underline{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right) + \frac{1}{R} b f\left(\underline{\Phi}(b)\right)\underline{\Phi}'(b) \end{array}\right] \\
&= p\frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1-p)\frac{1}{R}\left(1 - F\left(\underline{\Phi}(b)\right)\right)
\end{aligned}$$

Consider, first, the case in which $\Psi(b)$ is constant, $\Psi(b) = p$. In this case, debt revenue is increasing for all $b < \bar{b}$, since, then, $p/R \times \left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1-p)/R \times \left(1 - F\left(\underline{\Phi}(b)\right)\right) > 0$. Moreover, it reaches a maximum at $b = \bar{b}$ (recall that $F\left(\bar{\Phi}(b)\right) < F\left(\underline{\Phi}(b)\right)$ for all $b < \bar{b}$). This establishes that, if Ψ is constant, then $\bar{b}^R = \bar{b}$.

Consider, next, the general case. Proposition 6 implies that, in the range where $b \in [b_2, \bar{b}]$, $\Psi'(b) < 0$. Since $Q(b,\bar{w}) > \hat{Q}(b,\underline{w})$, then, in a left neighborhood of $\bar{b}$, $\Psi'(b) \times \left[Q(b,\bar{w}) - \hat{Q}(b,\underline{w})\right] b < 0$. This means that, starting from $\bar{b}$, one can increase the debt revenue by reducing debt, i.e., $\bar{b}^R < \bar{b}$. ■

Proof of Proposition 7. The procedure is analogous to the derivation of the CEE in normal times. The first-order condition of (10) for $w = \underline{w}$ yields

$$0 = \frac{d}{db'}\left\{Q\left(b',\underline{w}\right) \times b'\right\} \times u'(c) + \beta\left[1 - \Psi\left(b'\right)\right]\frac{d}{db'}EV\left(b',\underline{w}\right) + \beta\Psi\left(b'\right)\frac{d}{db'}EV\left(b',\bar{w}\right),$$

where a term has been cancelled by the envelope theorem. Using the same argument as in the proof of Proposition 4 we can write:

$$\frac{d}{db}EV(b,\underline{w}) = -\left[1 - F\left(\underline{\Phi}(b)\right)\right] \times u'\left[Q\left(B(b,\underline{w}),\underline{w}\right) \times B(b,\underline{w}) + \underline{w} - b\right].$$

Plugging this back into the FOC (after leading the expression by one period) yields the CEE

$$\begin{aligned}
0 &= u'(c) \times \left\{\Psi'\left(b'\right) \times R\left[Q\left(b',\bar{w}\right) - \hat{Q}\left(b',\underline{w}\right)\right] b' + \right. \\
&\quad \left. + \Psi\left(b'\right) \times \left(1 - F\left(\bar{\Phi}\left(b'\right)\right)\right) + \left[1 - \Psi\left(b'\right)\right] \times \left(1 - F\left(\underline{\Phi}\left(b'\right)\right)\right)\right\} \\
&\quad - \beta R\left(\left[1 - \Psi\left(b'\right)\right] \times \left[1 - F\left(\underline{\Phi}\left(b'\right)\right)\right] u'\left(c'|_{H,\underline{w}}\right) + \Psi\left(b'\right) \times \left[1 - F\left(\bar{\Phi}\left(b'\right)\right)\right] u'\left(c'|_{H,\bar{w}}\right)\right),
\end{aligned}$$

where the equality follows from Lemma 4. Rearranging terms yields equation (16). ■

Proof of Proposition 8. We write the Lagrangian,

$$\begin{aligned}\bar{\Lambda} &= \int_{\aleph}\left[\bar{w}-c_{\phi}+\beta\bar{P}\left(\bar{\omega}_{\phi}\right)\right]dF\left(\phi\right)+\bar{\mu}\left(\int_{\aleph}\left[u\left(c_{\phi}\right)+\beta\bar{\omega}_{\phi}\right]dF\left(\phi\right)-\nu\right)\\ &\quad+\int_{\aleph}\bar{\lambda}_{\phi}\left[u\left(c_{\phi}\right)+\beta\bar{\omega}_{\phi}-\bar{\nu}+\phi\right]dF\left(\phi\right),\end{aligned}$$

with the associated multipliers $\bar{\mu}$ and $\bar{\lambda}_{\phi}$. The first-order conditions yield

$$f\left(\phi\right) = u'\left(c_{\phi}\right)\left(\bar{\mu}f\left(\phi\right)+\bar{\lambda}_{\phi}\right), \tag{53}$$

$$\bar{\lambda}_{\phi}+\bar{\mu}f\left(\phi\right) = -\bar{P}'\left(\bar{\omega}_{\phi}\right)f\left(\phi\right). \tag{54}$$

The envelope condition yields

$$-\bar{P}'\left(\nu\right)=\bar{\mu} \tag{55}$$

The two first-order conditions and the envelope condition jointly imply that

$$u'\left(c_{\phi}\right) = -\frac{1}{\bar{P}'\left(\bar{\omega}_{\phi}\right)} \tag{56}$$

$$\bar{P}'\left(\bar{\omega}_{\phi}\right) = \bar{P}'\left(\nu\right)-\frac{\bar{\lambda}_{\phi}}{f\left(\phi\right)}. \tag{57}$$

Note that (56) is equivalent to (20) in the text. Consider, next, two cases, namely, when the PC is binding and when it is not binding.

When the PC is binding, $\bar{\lambda}_{\phi}>0$. (57) implies then that $\bar{\omega}_{\underline{\phi}}>\nu$. Then, (56) and (21) determine jointly the solution for $\left(c_{\phi},\bar{\omega}_{\phi}\right)$. When the PC is not binding, $\bar{\lambda}_{\phi}=0$. (57) implies then that $\bar{\omega}_{\phi}=\nu$ and $c_{\phi}=c\left(\nu\right)$.

What remains to be shown is that the first-order conditions are sufficient. The proof follows Thomas and Worrall (1990, Proof of Proposition 1). The details of this proof are in Proposition 14 in the online appendix. ■

Proof of Proposition 9. We write the Lagrangian,

$$\begin{aligned}\underline{\Lambda} &= \int_{\aleph}\left[\underline{w}-c_{\phi}+\beta\left(\left(1-p_{\phi}\right)\underline{P}\left(\underline{\omega}_{\phi}\right)+p_{\phi}\bar{P}\left(\bar{\omega}_{\phi}\right)\right)\right]dF\left(\phi\right)\\ &\quad+\underline{\mu}\left(\int_{\aleph}\left(u\left(c_{\phi}\right)-X\left(p_{\phi}\right)+\beta\left(\left(1-p_{\phi}\right)\underline{\omega}_{\phi}+p_{\phi}\bar{\omega}_{\phi}\right)\right)dF\left(\phi\right)-\nu\right)\\ &\quad+\int_{\aleph}\underline{\lambda}_{\phi}\left(u\left(c_{\phi}\right)-X\left(p_{\phi}\right)+\beta\left(\left(1-p_{\phi}\right)\underline{\omega}_{\phi}+p_{\phi}\bar{\omega}_{\phi}\right)-\underline{\nu}+\phi\right)dF\left(\phi\right),\end{aligned}$$

where $\underline{\mu}$ and $\underline{\lambda}_{\phi}$ denote the multipliers in recession. The first-order conditions yield:

$$f\left(\phi\right) = u'\left(c_{\phi}\right)\left(\underline{\mu}\,f\left(\phi\right)+\underline{\lambda}_{\phi}\right) \tag{58}$$

$$\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right) = -\underline{P}'\left(\underline{\omega}_{\phi}\right)\,f\left(\phi\right) \tag{59}$$

$$\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right) = -\bar{P}'\left(\bar{\omega}_{\phi}\right)\,f\left(\phi\right) \tag{60}$$

$$\beta\left(\bar{P}\left(\bar{\omega}_{\phi}\right)-\underline{P}\left(\underline{\omega}_{\phi}\right)\right)\,f\left(\phi\right) = \left(\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right)\right)\left(X'\left(p_{\phi}\right)-\beta\left(\bar{\omega}_{\phi}-\underline{\omega}_{\phi}\right)\right) \tag{61}$$

The envelope condition yields:

$$-\underline{P}'(\nu) = -\underline{\mu} \tag{62}$$

Combining the first-order conditions and the envelope condition yields:

$$\begin{aligned} u'(c_\phi) &= -\frac{1}{\underline{P}'(\underline{\omega}_\phi)} \\ \underline{P}'(\underline{\omega}_\phi) &= \underline{P}'(\nu) - \frac{\underline{\lambda}_\phi}{f(\phi)} \\ \underline{P}'(\underline{\omega}_\phi) &= \bar{P}'(\bar{\omega}_\phi) \\ X'(p_\phi) &= \beta\left(u'(c_\phi)\left(\bar{P}(\bar{\omega}_\phi) - \underline{P}(\underline{\omega}_\phi)\right) + \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right) \end{aligned} \tag{63}$$

We distinguish two cases, namely, when the PC is binding and when it is not binding.

(I) When the PC is binding and the recession continues, $\underline{\lambda}_\phi > 0$, $\underline{\omega}_\phi > \nu$, and

$$u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right) = \underline{\nu} - \phi$$

Then, (25), (27), (28) and (26) determine jointly the solution for $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$. In this case, there is no history dependence, i.e., ν does not matter.

(II) When the PC is not binding, $\underline{\lambda}_\phi = 0$. Then, $\underline{\omega}_\phi = \nu$ and $c_\phi = \underline{c}(\nu)$. The solution is history dependent. Moreover, (28) implies that

$$\beta \times \left\{u'(\underline{c}(\nu))\left[\bar{P}(\bar{\omega}(\nu)) - \underline{P}(\nu)\right] + \left[\bar{\omega}(\nu) - \nu\right]\right\} = X'(p(\nu)), \tag{64}$$

namely, the planner requires constant effort over the set of states for which the constraint is not binding: $p_\phi = p(\nu)$. Differentiating the left-hand side yields

$$\begin{aligned} & \underbrace{u''(\underline{c}(\nu))\,\underline{c}'(\nu) \times \left[\bar{P}(\bar{\omega}(\nu)) - \underline{P}(\nu)\right]}_{<0} + \left[u'(\underline{c}(\nu))\,\underline{P}'(\nu) + 1\right]\left(\bar{\omega}'(\nu) - 1\right) \\ = \; & u''(\underline{c}(\nu))\,\underline{c}'(\nu) \times \left(\bar{P}(\bar{\omega}(\nu)) - \underline{P}(\nu)\right) < 0 \end{aligned}$$

since, recall, (25) implies that $\bar{P}'(\nu) = -1/u'(\underline{c}(\nu))$. This implies that the right-hand side must also be decreasing in ν. Since X is concave and increasing, this implies in turn that $p(\nu)$ must be increasing in ν.

(III) Finally, we prove that $\underline{\omega}_\phi < \bar{\omega}_\phi$. To this aim, note that $\underline{\mu} > \bar{\mu}$ since it is more expensive to deliver a given promised utility during recession than in normal times. Hence, the respective envelope conditions, (55) and (62), imply that, for any x, $\underline{P}'(x) < \bar{P}'(x)$. Next, note that since both $\underline{P}(x)$ and $\bar{P}(x)$ are decreasing concave functions, then $\underline{P}'(x_1) = \bar{P}'(x_2) \Leftrightarrow x_1 < x_2$. Therefore, the condition $\underline{P}'(\underline{\omega}_\phi) = \bar{P}'(\bar{\omega}_\phi)$ (see equation (63)) implies that $\underline{\omega}_\phi < \bar{\omega}_\phi$. ■

Proof of Lemma 5. The Lagrangian of the planner's problem reads as

$$\begin{aligned} \underline{\Lambda} = & \int_{\aleph}\left[\underline{w} - c_\phi + \beta\left((1-p_\phi)\underline{P}(\underline{\omega}_\phi) + p_\phi\bar{P}(\bar{\omega}_\phi)\right)\right] dF(\phi) \\ & + \underline{\mu}\left(\int_{\aleph}\left(u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right)\right) dF(\phi) - \nu\right) \\ & + \int_{\aleph}\underline{\lambda}_\phi\left(u(c_\phi) - X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right) - \underline{\nu} + \phi\right) dF(\phi) \\ & + \int_{\aleph}\gamma_\phi\left(\underline{\nu} - \tilde{\zeta} - X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right)\right) dF(\phi), \end{aligned}$$

where the Lagrange multipliers of the PC and IC must be non-negative for all ϕ, $\underline{\lambda}_\phi \geq 0, \gamma_\phi \geq 0$. The first-order conditions yield:

$$\begin{aligned} f(\phi) &= u'(c_\phi)\left(\underline{\mu} f(\phi) + \underline{\lambda}_\phi\right), & (65)\\ \underline{\lambda}_\phi + \underline{\mu} f(\phi) + \gamma_\phi &= -\underline{P}'\left(\underline{\omega}_\phi\right) f(\phi), & (66)\\ \underline{\lambda}_\phi + \underline{\mu} f(\phi) + \gamma_\phi &= -\bar{P}'(\bar{\omega}_\phi) f(\phi), & (67)\\ \beta\left(\bar{P}(\bar{\omega}_\phi) - \underline{P}\left(\underline{\omega}_\phi\right)\right) f(\phi) &= \left(\underline{\lambda}_\phi + \underline{\mu} f(\phi) + \gamma_\phi\right)\left(X'(p_\phi) - \beta\left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right), & (68) \end{aligned}$$

while the envelope condition yields $\underline{P}'(\nu) = \underline{\mu}$.

The first-order conditions (66)–(68) imply equations (31)-(32) in the text. Since $\underline{P}$ and $\bar{P}$ are monotonic and concave, equation (31) implies a positive relationship between $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$. Equation (32) yields then a negative relationship between p_ϕ and $\underline{\omega}_\phi$. Consider, next, the IC constraint. When the IC constraint is binding, equations (30), (31), and (32) pin down a unique solution for p_ϕ, $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$, denoted by $(p^*, \underline{\omega}^*, \bar{\omega}^*)$. ■

Proof of Proposition 10. This proof builds on the proof of Lemma 5. Suppose $\nu > \underline{\omega}^*$ (case 1). Then the IC is not binding in the initial period. Moreover, by Proposition 9, the promised utility is non-decreasing over time. Thus, the IC will never bind in the future, and can be ignored altogether. Suppose, next, that $\nu \leq \underline{\omega}^*$ (case 2). We first determine the upper bound on ϕ, denoted by ϕ^*, such that the PC is binding while the IC is not binding. Let $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ denote the solution characterized in Proposition 9 when the IC is not binding and $(c^*_\phi, p^*, \underline{\omega}^*, \bar{\omega}^*)$ the solution characterized in Proposition 10 when the IC is binding. Note that c^*_ϕ is defined in (33). At the threshold realization ϕ^*, the two allocations must be equivalent, i.e.,

$$(c_{\phi^*}, p_{\phi^*}, \underline{\omega}_{\phi^*}, \bar{\omega}_{\phi^*}) = (c^*_{\phi^*}, p^*, \underline{\omega}^*, \bar{\omega}^*).$$

Given a promised utility of $\underline{\omega}^*$, the promise-keeping constraint implies:

$$\begin{aligned} \underline{\omega}^* &= \underline{\omega}_{\phi^*} = \int_0^{\phi^*} \left[u(c_\phi) - X(p_\phi) + \beta\left[(1-p_\phi)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right]\right] dF(\phi)\\ &\quad + \int_{\phi^*}^{\phi_{\max}} \left[u(c_{\phi^*}) - X(p_{\phi^*}) + \beta\left[(1-p_{\phi^*})\underline{\omega}_{\phi^*} + p_{\phi^*}\bar{\omega}_{\phi^*}\right]\right] dF(\phi)\\ &= \int_0^{\phi^*} (\underline{v} - \phi)\, dF(\phi) + (\underline{v} - \phi^*)\left[1 - F(\phi^*)\right] = \underline{v} - \int_0^{\phi^*} \phi dF(\phi) - \phi^*\left[1 - F(\phi^*)\right]. \quad (69) \end{aligned}$$

Since $\underline{\omega}_\phi$ is decreasing in ϕ, then ϕ^* is unique. Moreover, if $\phi < \phi^*$, then $\underline{\omega}_\phi > \underline{\omega}^*$. In this case, the solution is not history-dependent and is determined as in Proposition 9 (case 2.a). If, to the opposite, $\phi \geq \phi^*$, then $\underline{\omega}_\phi = \underline{\omega}^*$. Two subcases must be distinguished here. First, if $\phi \geq \phi^*$ and $\nu = \underline{\omega}^*$, then the multipliers of both the IC and PC must be zero, $\underline{\lambda}_\phi = \gamma_\phi = 0$, because the planner keeps the triplet $(p^*, \underline{\omega}^*, \bar{\omega}^*)$ constant. More formally, the envelope condition together with equation (66) implies that

$$\underline{P}'(\nu) = \underline{P}'\left(\underline{\omega}_\phi\right) + \frac{\underline{\lambda}_\phi + \gamma_\phi}{f(\phi)}.$$

Thus, $\nu = \underline{\omega}_\phi = \underline{\omega}^*$, and both the multiplier of the IC and that of the PC must be zero. In other words, as long as the IC was binding in the previous period, and continues to bind in the current period, the planner keeps consumption, effort and promised utilities constant.

Second, if $\phi \geq \phi^*$ and $\nu < \underline{\omega}^*$, then the planner must adjust promised utility, $\underline{\omega}_\phi = \underline{\omega}^*$, to satisfy the IC. In this case, the multiplier of the IC must be strictly positive, $\gamma_\phi > 0$.

For the determination of consumption, two separate cases must be distinguished. In the first case (2.b), $\phi \leq \tilde{\phi}(\nu)$ (where the expression for $\tilde{\phi}(\nu)$ is given below), so that both the IC and the PC bind. In this case, $\underline{\lambda}_\phi > 0$, and the IC and the PC determine jointly the consumption level, whose level is given by c^*_ϕ as defined in equation (33). In the second case (2.c), $\phi > \tilde{\phi}(\nu)$, so that the PC does not bind. In this region, $\underline{\lambda}_\phi = 0$, and the consumption level provided by the planner is pinned down by the promise-keeping constraint (23), and given by equation (34).

Next, we determine the unique threshold, $\tilde{\phi}(\nu)$, that sets apart case (2.b) from case (2.c) and show that $\tilde{\phi}(\nu) \geq \phi^*$. Because the PC holds with equality at the threshold realization $\tilde{\phi}(\nu)$, the consumption level for all realizations of $\phi > \tilde{\phi}(\nu)$ where the PC is not binding must be given by

$$c^*_{\tilde{\phi}(\nu)} = u^{-1}\left(\tilde{\zeta} - \tilde{\phi}(\nu)\right).$$

This condition, together with the promise-keeping constraint, fully characterizes the threshold $\tilde{\phi}(\nu)$:

$$\begin{aligned}
\nu &= \int_0^{\phi^*} \left[u(c_\phi) - X(p_\phi) + \beta\left[(1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right]\right] dF(\phi) \\
&\quad + \int_{\phi^*}^{\tilde{\phi}(\nu)} \left[u(c^*_\phi) - X(p^*) + \beta\left[(1-p^*)\underline{\omega}^* + p^*\bar{\omega}^*\right]\right] dF(\phi) \\
&\quad + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[u(c^*_{\tilde{\phi}(\nu)}) - X(p_{\phi^*}) + \beta\left[(1-p_{\phi^*})\underline{\omega}_{\phi^*} + p_{\phi^*}\bar{\omega}_{\phi^*}\right]\right] dF(\phi) \\
&= \int_0^{\phi^*} (\underline{v} - \phi)\, dF(\phi) + \int_{\phi^*}^{\tilde{\phi}(\nu)} (\underline{v} - \phi)\, dF(\phi) + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[u(c^*_{\tilde{\phi}(\nu)}) + Z(b_0)\right] dF(\phi) \\
&= F(\tilde{\phi}(\nu))\underline{v} - \int_0^{\tilde{\phi}(\nu)} \phi dF(\phi) + \left(\underline{v} - \tilde{\phi}(\nu)\right)\left[1 - F(\tilde{\phi}(\nu))\right] = \underline{v} - \int_0^{\tilde{\phi}(\nu)} \phi dF(\phi) - \tilde{\phi}(\nu)\left[1 - F(\tilde{\phi}(\nu))\right] \qquad (70)
\end{aligned}$$

If $\nu = \underline{\omega}^*$ then equations (69) and (70) imply that $\phi^* = \tilde{\phi}(\nu)$. The right-hand side of (70) is falling in $\tilde{\phi}$. It follows that if $\nu < \underline{\omega}^*$, then $\tilde{\phi}(\nu) \geq \tilde{\phi}(\underline{\omega}^*) = \phi^*$. ■

Proof of Proposition 11. We prove the proposition by deriving a contradiction. To this aim, suppose that, for $\bar{\Pi}(b) = \bar{P}(\nu)$, the planner can deliver more utility to the agent than can the laissez-faire equilibrium. Namely, $\nu > EV(b, \bar{w})$. Then, since $\bar{P}$ is a decreasing strictly concave function, we must have that $\bar{P}(EV(b, \bar{w})) > \bar{P}(\nu)$ and $\bar{P}'(EV(b, \bar{w})) > \bar{P}'(\nu)$. We show that this inequality, along with the set of optimality conditions, induces a contradiction.

First, recall, that equation (6) implies that $\bar{\Pi}(b) = RQ(b, \bar{w})\, b$. Thus,

$$\bar{P}(EV(b, \bar{w})) > \bar{P}(\nu) = RQ(b, \bar{w})\, b, \qquad (71)$$

where $EV(b, \bar{w})$ is decreasing in b. Differentiating the two sides of the inequality (71) with respect to b yields

$$\bar{P}'(EV(b, \bar{w})) \times \frac{d}{db} EV(b, \bar{w}) > \frac{d}{db}\left[Q(b, \bar{w})\, b\right] \times R = 1 - F\left(\bar{\Phi}(b)\right), \qquad (72)$$

where the right-hand side equality follows from the proof of Lemma 2. Next, equation (48) implies that

$$\frac{d}{db} EV(b, \bar{w}) = -\left[1 - F\left(\bar{\Phi}(b)\right)\right] \times u'\left[C(b, \bar{w})\right],$$

where $C(b,\bar{w}) = Q(B(b,\bar{w}),\bar{w}) \times B(b,\bar{w}) + \bar{w} - b$ is the consumption level in the laissez-faire equilibrium when the debt b is honored. Plugging the expression of $\frac{d}{db}EV(b,\bar{w})$ into (72), and simplifying terms, yields

$$u'(C(b,\bar{w})) > -\frac{1}{\bar{P}'\left(EV\left(\bar{b},\bar{w}\right)\right)}. \tag{73}$$

Next, note that $C(b,\bar{w}) = c(\nu)$. Equation (73) yields $u'(c(\nu)) > -1/\bar{P}'\left(EV\left(\bar{b},\bar{w}\right)\right)$, while (56) yields that $u'(c(\nu)) = -1/\bar{P}'(\nu)$. Thus, the two conditions jointly imply that $\bar{P}'(\nu) > \bar{P}'\left(EV\left(\bar{b},\bar{w}\right)\right)$ which in turn implies that $\nu < EV\left(\bar{b},\bar{w}\right)$, since $\bar{P}$ is decreasing and concave. This contradicts the assumption that $\nu > EV\left(\bar{b},\bar{w}\right)$.

The analysis thus far implies that $\nu \leq EV\left(\bar{b},\bar{w}\right)$. We can also rule out that $\nu < EV(b,\bar{w})$, because it would contradict that the allocation chosen by the planner is constrained efficient. Therefore, $\nu = EV(b,\bar{w})$. ■

Proof of Proposition 12. We proceed in two steps: first, we derive the CEEs (step A), and then we show that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$ (step B).

Step A: The first-order conditions with respect to $b'_{\bar{w}}$ and $b'_{\underline{w}}$ in problem (36) yields

$$0 = u'(c) \times \frac{d}{db'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + \beta\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \frac{d}{db'_{\bar{w}}} EV\left(b'_{\bar{w}}, \bar{w}\right),$$

$$0 = u'(c) \times \frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right] \frac{d}{db'_{\underline{w}}} EV\left(b'_{\underline{w}}, \underline{w}\right),$$

where $REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \equiv b'_{\underline{w}} \times Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + b'_{\bar{w}} \times Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ is the bond revenue and c is defined in the Proposition. Note that both equations have been simplified using an envelope condition. The value function has a kink at $b_{\underline{w}} = \hat{b}(\phi, \underline{w})$. Consider, first, the range of realizations $\phi \in [\underline{\Phi}(b), \infty)$, implying that $b_{\underline{w}} < \hat{b}(\phi, \underline{w})$. Differentiating the value function yields:

$$\begin{aligned}
\frac{d}{db}V(b,\phi,\bar{w}) &= -u'\left[Q(B(b,\bar{w}),\bar{w}) \times B(b,\bar{w}) + \bar{w} - b\right], \\
\frac{d}{db}V(b,\phi,\underline{w}) &= -u'\left[Q_{\underline{w}}\left(B_{\underline{w}}(b), B_{\bar{w}}(b)\right) \times B_{\underline{w}}(b) + Q_{\bar{w}}\left(B_{\underline{w}}(b), B_{\bar{w}}(b)\right) \times B_{\bar{w}}(b) + \underline{w} - b\right],
\end{aligned}$$

where $B_{\underline{w}}$ and $B_{\bar{w}}$ denote the optimal issuance of the two assets, respectively. Next, consider the range of realizations $\phi < \underline{\Phi}(b)$, implying that $b_{\underline{w}} \geq \hat{b}(\phi, \underline{w})$. In this case, $\frac{d}{db}V(b,\phi,\underline{w}) = 0$.

In analogy with equation (48), we obtain:

$$\frac{d}{db_{\underline{w}}} EV\left(b_{\underline{w}}, \underline{w}\right) = -\left[1 - F\left(\underline{\Phi}\left(b_{\underline{w}}\right)\right)\right] \times u'\left(c'|_{H,\underline{w}}\right). \tag{74}$$

Plugging (48) and (74) into the respective first-order conditions, and leading by one period, yields

$$u'(c) \times \frac{d}{db'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \beta\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \left[1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right] \times u'\left(c'|_{H,\bar{w}}\right),$$

$$u'(c) \times \frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right] \times \left[1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right] \times u'\left(c'|_{H,\underline{w}}\right).$$

The marginal revenues from issuing recession-contingent debt is given by:

$$\begin{aligned}
&\frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \\
= \quad & \frac{1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) - \frac{\frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}}}{1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + b'_{\bar{w}} \times \frac{\partial}{\partial b'_{\underline{w}}} Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \\
= \quad & \frac{1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) + \frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}} \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right), \qquad (75)
\end{aligned}$$

where, note, $\frac{\partial}{\partial b'_{\underline{w}}} Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}} \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}$ follows from applying standard differentiation to the definition of $Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ in equation (38). Applying the same methodology to the recovery-contingent debt, we obtain:

$$\frac{d}{db'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \frac{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) + \frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\bar{w}}} \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right).$$

The CEEs conditional on the recession continuing and ending, respectively, are then:

$$\begin{aligned}
\beta \frac{u'\left(c'|_{H,\underline{w}}\right)}{u'(c)} &= \frac{1}{R} + \frac{\partial}{\partial b'_{\underline{w}}} \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \frac{\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)\left[1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right]} \\
\beta \frac{u'\left(c'|_{H,\bar{w}}\right)}{u'(c)} &= \frac{1}{R} + \frac{\partial}{\partial b'_{\bar{w}}} \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \frac{\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}.
\end{aligned}$$

Setting $\beta R = 1$ yields equations (39)-(40).

Step B: Next, we prove that, in equilibrium, $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. To prove the claim, it is useful to define the two functions

$$\begin{aligned}
\theta_{\bar{w}}\left(b'_{\bar{w}}\right) &\equiv \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} = \frac{1}{R}\left(\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) b'_{\bar{w}} + \int_0^{\bar{\Phi}\left(b'_{\bar{w}}\right)} \bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right), \\
\theta_{\underline{w}}\left(b'_{\underline{w}}\right) &\equiv \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}}}{1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} = \frac{1}{R}\left(\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) b'_{\underline{w}} + \int_0^{\underline{\Phi}\left(b'_{\underline{w}}\right)} \underline{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right),
\end{aligned}$$

where, recall, $\underline{\Phi}(x) > \bar{\Phi}(x)$ and $F\left(\underline{\Phi}(x)\right) > F\left(\bar{\Phi}(x)\right)$. Note that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \theta_{\bar{w}}\left(b'_{\bar{w}}\right) - \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, where both $\theta_{\bar{w}}$ and $\theta_{\underline{w}}$ are increasing functions in the relevant range, i.e., $b'_{\bar{w}} \leq \bar{b}$ and $b'_{\underline{w}} \leq \bar{b}$. We proceed in two steps. First, we show that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}}$ (step B1). Next, we show that $b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$ (step B2). Steps B1 and B2 establish jointly a contradiction ruling out that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$ (step B3).

Step B1: Suppose that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$. Then, the CEEs (39)–(40) and the assumption that $u'' < 0$, imply that

$$c'|_{H,\bar{w}} \leq c \leq c'|_{H,\underline{w}}. \qquad (76)$$

Suppose, to derive a contradiction, that $b'_{\underline{w}} \geq b'_{\bar{w}}$. Recall that, if the recession ends and debt is honored, debt remains constant, i.e., $b'' = B\left(b'_{\bar{w}}\right) = b'_{\bar{w}}$. Moreover, $Q\left(b'_{\bar{w}}, \bar{w}\right) = Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) / \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$. Thus,

$c'|_{H,\bar{w}} = Q\left(b'_{\bar{w}}, \bar{w}\right) b'_{\bar{w}} + \bar{w} - b'_{\bar{w}} = \theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \bar{w} - b'_{\bar{w}}.$

$$\begin{aligned} c'|_{H,\bar{w}} &= \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \bar{w} - b'_{\bar{w}} \\ &\geq \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\bar{w}}\left(b'_{\underline{w}}\right) + \bar{w} - b'_{\underline{w}} \\ &\geq \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\underline{w}}\left(b'_{\underline{w}}\right) + \bar{w} - b'_{\underline{w}} \\ &> \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\underline{w}}\left(b'_{\underline{w}}\right) + \underline{w} - b'_{\underline{w}} = c'|_{H,\underline{w}}. \end{aligned}$$

The first inequality follows from the assumption that $b'_{\underline{w}} \geq b'_{\bar{w}}$ and the fact that $(1-p)\,\theta_{\bar{w}}(x) - x < 0$ for any $p \in [0,1]$, which is due to the fact that $\theta_{\bar{w}}(x) \leq x/R < x$ for any x. The second inequality follows from the fact that $\theta_{\bar{w}}\left(b'_{\underline{w}}\right) \geq \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, see equation (77) below. The last inequality follows from the maintained assumption that $\bar{w} > \underline{w}$. We have therefore proven that if $b'_{\underline{w}} \geq b'_{\bar{w}}$ then $c'|_{H,\bar{w}} > c'|_{H,\underline{w}}$, which contradicts (76) and, hence, implies that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \geq 0$. We conclude from Step B1 that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}}$.

Step B2: Suppose that $b'_{\underline{w}} = b'_{\bar{w}} = x$. Then, for any x:

$$\begin{aligned} \Delta(x,x) &= \theta_{\bar{w}}(x) - \theta_{\underline{w}}(x) = \frac{1}{R}\underbrace{\int_{\bar{\Phi}(x)}^{\underline{\Phi}(x)} \left(\left(x - \underline{\Phi}^{-1}(\phi)\right) \times f(\phi)\ d\phi\right)}_{>0} \\ &\quad + \frac{1}{R}\underbrace{\int_{0}^{\bar{\Phi}(x)} \left(\left(\bar{\Phi}^{-1}(\phi) - \underline{\Phi}^{-1}(\phi)\right) \times f(\phi)\ d\phi\right)}_{>0} > 0. \end{aligned} \tag{77}$$

Since $\theta_{\bar{w}}(x)$ is an increasing function for $x \leq \bar{b}$, equation (77) implies that $\theta_{\bar{w}}\left(b'_{\bar{w}}\right) > \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, for all $b'_{\underline{w}} < b'_{\bar{w}} \leq \bar{b}$. We conclude from Step B2 that $b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$.

Step B3: Putting together the conclusions of Step B1 and Step B2, we derive a contradiction: $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. Therefore, we must have that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. ■

Proof of Proposition 13. The strategy of the proof is the same as that of Proposition 11. We prove the proposition by deriving a contradiction. To this aim, suppose that, for $\underline{\Pi}\left(b_{\underline{w}}\right) = \underline{P}(\nu)$, the planner can deliver more utility than the agent gets in the laissez-faire equilibrium. Namely, $\nu > EV\left(b_{\underline{w}}, \underline{w}\right)$. Then, since $\underline{P}$ is a decreasing strictly concave function, we must have that $\underline{P}\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) > \underline{P}(\nu)$ and $\underline{P}'\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) > \underline{P}'(\nu)$. Note that, absent moral hazard, the price of recession-contingent debt is independent of the amount of recovery-contingent debt. It is therefore legitimate to define $\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) \equiv Q_{\underline{w}}\left(b_{\underline{w}}, b_{\bar{w}}\right)$.

First, the same argument invoked in the proof of Proposition 11 implies that $\underline{\Pi}\left(b_{\underline{w}}\right) = \frac{R}{1-p}\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) b_{\underline{w}}$. Hence,

$$\underline{P}\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) > \underline{P}(\nu) = \frac{R}{1-p}\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) b_{\underline{w}}. \tag{78}$$

where $EV\left(b_{\underline{w}}, \underline{w}\right)$ is decreasing in $b_{\underline{w}}$. Differentiating the two sides of the inequality (78) with respect to $b_{\underline{w}}$ yields:

$$\begin{aligned} &\underline{P}'\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) \times \frac{d}{db_{\underline{w}}} EV\left(b_{\underline{w}}, \underline{w}\right) \\ > \ &\frac{R}{1-p}\frac{d}{db_{\underline{w}}}\left(\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) b_{\underline{w}}\right) = -\left[1 - F\left(\underline{\Phi}\left(b_{\underline{w}}\right)\right)\right], \end{aligned} \tag{79}$$

where the right-hand side equality follows from equation (75). Next, equation (48) implies that

$$\frac{d}{db}EV\left(b,\bar{w}\right)=-\left[1-F\left(\bar{\Phi}\left(b\right)\right)\right]\times u'\left(C\left(b,\bar{w}\right)\right),$$

where $C\left(b_{\underline{w}},\underline{w}\right)$ is the consumption level assuming that the recession-contingent debt $b_{\underline{w}}$ is honored. Plugging in the expression of $\frac{d}{db_{\underline{w}}}EV\left(b_{\underline{w}},\bar{w}\right)$ allows us to simplify (79) as follows:

$$u'\left(C\left(b_{\underline{w}},\underline{w}\right)\right)>-\frac{1}{\underline{P}'\left(EV\left(b_{\underline{w}},\underline{w}\right)\right)}. \tag{80}$$

Next, note that $C\left(b_{\underline{w}},\underline{w}\right)=c\left(\nu\right)$. Equation (80) yields $u'\left(c\left(\nu\right)\right)>-\frac{1}{\underline{P}'(EV(b_{\bar{w}},\bar{w}))}$, while (63) yields that $u'\left(c\left(\nu\right)\right)=-\frac{1}{\underline{P}'(\nu)}$. Thus, the two conditions jointly imply that $-\frac{1}{\underline{P}'(\nu)}>-\frac{1}{\underline{P}'(EV(b_{\bar{w}},\underline{w}))}$ which in turn implies that $\nu<EV\left(b_{\bar{w}},\underline{w}\right)$, since $\underline{P}$ is decreasing and concave. This contradicts the assumption that $\nu>EV\left(b_{\bar{w}},\underline{w}\right)$.

The analysis thus far establishes that $\nu\leq EV\left(b_{\bar{w}},\underline{w}\right)$. We can also rule out that $\nu<EV\left(b_{\bar{w}},\underline{w}\right)$ because it would contradict that the allocation chosen by the planner is constrained efficient. Therefore, $\nu=EV\left(b_{\bar{w}},\underline{w}\right)$. ■

Proof of results in Section 6. The price of state-contingent debt along the equilibrium path yields:

$$\begin{aligned}
Q^C_{\underline{w}}\left(b'_{\underline{w}},p\right) &= \frac{1-p}{R}\hat{Q}^C_{\underline{w}}\left(b'_{\underline{w}}\right), \qquad Q^C_{\bar{w}}\left(b'_{\bar{w}},p\right)=\frac{p}{R}\hat{Q}^C_{\bar{w}}\left(b'_{\bar{w}}\right),\\
\hat{Q}^C_{w}\left(b'_{w}\right) &= \frac{1}{R}\left(1-F\left(\bar{\Phi}^C_w\left(b'_w\right)\right)\right)+\frac{1}{b'_w}\int_0^{\bar{\Phi}^C(b'_w)}\left(\bar{\Phi}^{C-1}_w\left(\phi\right)\,dF\left(\phi\right)\right).
\end{aligned}$$

The corresponding value function is:

$$\begin{aligned}
V^C\left(b_{\underline{w}},\phi,\underline{w}\right) &= \max_{\left\{b'_{\underline{w}},b'_{\bar{w}},p\right\}}\left\{u\left[\frac{1-p}{R}\hat{Q}^C_{\underline{w}}\left(b'_{\underline{w}}\right)\times b'_{\underline{w}}+\frac{p}{R}\hat{Q}^C_{\bar{w}}\left(b'_{\bar{w}}\right)\times b'_{\bar{w}}+\underline{w}-\mathbb{B}\left(b,\phi,\underline{w}\right)\right]\right.\\
&\quad\left.-X\left(p\right)+\beta\left[1-p\right]EV\left(b'_{\underline{w}},\underline{w}\right)+\beta pEV\left(b'_{\bar{w}},\bar{w}\right)\right\}.
\end{aligned}$$

The efficient effort is pinned down by the following condition:

$$X'\left(p\right)=u'\left(c\right)\times\left(\hat{Q}^C_{\underline{w}}\left(b'_{\underline{w}}\right)\times b'_{\underline{w}}-\hat{Q}^C_{\bar{w}}\left(b'_{\bar{w}}\right)\times b'_{\bar{w}}\right)+\beta\left[1-p\right]EV^C\left(b'_{\underline{w}},\underline{w}\right)+\beta pEV\left(b'_{\bar{w}},\bar{w}\right).$$

The CEEs yield

$$\frac{u'\left(c'|_{H,\underline{w}}\right)}{u'\left(c\right)}=1, \qquad \frac{u'\left(c'|_{H,\bar{w}}\right)}{u'\left(c\right)}=1.$$

Consider, next a deviation such that the agent exercises the ex-post optimal effort level conditional on the debt issued. In this case, the agent is punished by a permanent reversion to the Markov equilibrium. The value of such deviation is

$$V^{DEV}\left(b_{\underline{w}},\phi,w\right)=u(c)-X\left(\Psi(b'_{\bar{w}},b'_{\underline{w}})\right)+\beta\left[\Psi(b'_{\bar{w}},b'_{\underline{w}})EV(b'_{\bar{w}},\phi',\bar{w})+(1-\Psi(b'_{\bar{w}},b'_{\underline{w}}))EV(b'_{\underline{w}},\phi',\underline{w})\right].$$

The incentive compatibility constraint can then be expressed as

$$\begin{aligned}
&-X\left(p^C\right)+\beta\left[p^C\times EV(b'_{\bar{w}},\phi',\bar{w})+(1-p^C)\times EV^C(b'_{\underline{w}},\phi',\underline{w})\right]\\
>\;&-X\left(\Psi(b'_{\bar{w}},b'_{\underline{w}})\right)+\beta\left[\Psi(b'_{\bar{w}},b'_{\underline{w}})EV(b'_{\bar{w}},\phi',\bar{w})+(1-\Psi(b'_{\bar{w}},b'_{\underline{w}}))EV(b'_{\underline{w}},\phi',\underline{w})\right],
\end{aligned}$$

that simplifies to equation (42) in the text. ■

Online Appendix of
"Sovereign Debt and Structural Reforms"

Andreas Müller, Kjetil Storesletten and Fabrizio Zilibotti

B Additional proofs and details on numerical algorithm

This online appendix contains technical analysis and additional proofs related to Sections 3.4, 4 and 6. It als contains details of the numerical algorithm in Section 7.

B.1 Details of the proof of Proposition 4

We have established in Lemma 3 that an interior solution necessarily satisfies the FOC

$$\begin{aligned} & [1 - F(\Phi(B(b,\bar{w}),\bar{w}))]\, u'(Q(B(b,\bar{w}),\bar{w})B(b,\bar{w}) - b + \bar{w}) \\ = \; & [1 - F(\Phi(B(b,\bar{w}),\bar{w}))]\, u'(Q(B(B(b,\bar{w}),\bar{w}))B(B(b,\bar{w}),\bar{w}) - B(b,\bar{w}) + \bar{w}). \end{aligned}$$

Both, constant debt accumulation, $B(b,\bar{w}) = b$ and maximial debt accumulation $B(b,\bar{w}) = \bar{b}$ are obvious solution candidates. Note however, that $B(b,\bar{w}) = \bar{b}$ can only be a global maximum when the outstanding debt level is at the maximum, $b = \bar{b}$ (where the two solution candidates coincide), because otherwise the objective is strictly falling in the left neigborhood of $\bar{b}$ since

$$u'(Q(\bar{b},\bar{w})\bar{b} - b + \bar{w}) < u'(Q(\bar{b},\bar{w})\bar{b} - \bar{b} + \bar{w}), \quad b < \bar{b}.$$

Thus, we are therefore left to show that $B(b,\bar{w}) = b$ is the unique solution that satisfies the FOC. For the ease of exposition, let us rewrite the FOC as

$$\begin{aligned} & \left[1 - F(\Phi(b',\bar{w}))\right] u'(Q(b',\bar{w})b' - b + \bar{w}) \\ = \; & \left[1 - F(\Phi(b',\bar{w}))\right] u'(Q(b'',\bar{w})b'' - b' + \bar{w}). \end{aligned}$$

Suppose there existed a solution candidate where the current debt accumulation b' was *strictly reduced*, $b - b' > 0$. Because the marginal bond revenue is falling and smaller than R^{-1} this leads to a smaller reduction of today's consumption relative the increase in next period's consumption for a given b''. Therefore, b'' has to be lowered even further to equalize consumption intertemporally, $b' - b'' > b - b' > 0$. This argument can be expanded to further periods such that the equilibrium would feature accelerated *asset* accumulation and ever falling consumption which contradicts the requirement that it is a global maximum. Suppose to the contrary that current debt accumulation b' was *strictly increased*, $b' - b > 0$. Then, by the same argument as before, b'' has to be increased even further to equalize consumption intertemporally, $b'' - b' > b' - b > 0$, and the equilibrium would

feature accelerated *debt* accumulation. This implies that the economy will hit the upper bound on debt accumulation $\bar{b}$ for some outstanding debt level below the maximum, $b < \bar{b}$. However, we have already shown that this cannot be optimal. Thus, $B(b, \bar{w}) = b$ is the unique maximizer of the objective function.

B.2 Formal properties of the constrained efficient allocation (Section 4)

Proposition 14 *There exists unique profit functions $\bar{P}$ and $\underline{P}$ that solve the programs (17) and (22), respectively. Moreover, $\bar{P}$ and $\underline{P}$ are continuously differentiable and strictly concave. Given the promised utility ν and the realization ϕ, (i) if $w = \bar{w}$ there exists a unique optimal pair of promised utility and consumption $\{\bar{\omega}_\phi(\nu), c_\phi(\nu)\}$; (ii) if $w = \underline{w}$ there exists a unique optimal 4-tuple of promised utilities, consumption and effort, $\left\{\underline{\omega}_\phi(\nu), \bar{\omega}_\phi(\nu), p_\phi(\nu), c_\phi(\nu)\right\}$. The first-order conditions in Propositions 8 and 9 are necessary and sufficient when the solution is interior.*

The proof strategy follows Thomas and Worrall (1990, Proof of Proposition 1), i.e., we show that the problem is a contraction mapping to establish the uniqueness and strict concavity of $\bar{P}$ and $\underline{P}$. The differentiabilty of $\bar{P}$ and $\underline{P}$ follows from an application of Lemma 1 in Benveniste and Scheinkman (1979). Finally, we prove that $\bar{P}$ and $\underline{P}$ pin down uniquely promised utilities, effort and consumption.

The arguments used to prove Proposition 14 in normal times and recession are mirror image of each other, except that the recession case is complicated by the presence of an effort choice. For this reason, we prove the results when $w = \underline{w}$ (assuming the properties of $\bar{P}$ follow the proposition), omitting the simpler proof for the case in which $w = \bar{w}$ [more precisely, the arguments are extended by setting $X(p_\phi) = 0$ and $p_\phi = 1$].

We prove the results in the form of three lemmas and one corollary.

Define, first, the mapping $\underline{T}(x)(\nu)$ as the right-hand side of the planner's functional equation

$$\underline{T}(x)(\nu) = \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi\in\aleph})\in\underline{\Lambda}(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi)x(\underline{\omega}_\phi) \end{array} \right]\right] dF(\phi)$$

where maximization is constrained by the set $\underline{\Lambda}(\nu)$ defined by

$$\begin{aligned} \int_\aleph \left[u(c_\phi) - X(p_\phi) + \beta\left[p_\phi\bar{\omega}_\phi + (1-p_\phi)\underline{\omega}_\phi\right]\right] dF(\phi) &\geq \nu \\ u(c_\phi) - X(p_\phi) + \beta\left[p_\phi\bar{\omega}_\phi + (1-p_\phi)\underline{\omega}_\phi\right] &\geq \underline{\nu} - \phi, \quad \forall\phi\in\aleph, \\ c_\phi \in [0, \bar{w}],\ p_\phi \in [\underline{p}, \bar{p}],\ \nu, \underline{\omega}_\phi \in [\underline{\nu} - E[\phi], \underline{\nu}],\ \bar{\omega}_\phi &\in [\bar{\nu} - E[\phi], \bar{\nu}]. \end{aligned}$$

Recall that $\underline{\nu}$ and $\bar{\nu}$ are the values of the outside option during recession and normal times, respectively. We take as given the uniqueness, strict concavity, and differentiability of the profit function in normal times, $\bar{P}$. Moreover, let the profit in normal times be bounded between $\bar{P}_{MIN} = 0$ and $P_{MAX} = \bar{w}/(1-\beta)$.

Lemma 6 *$\underline{T}(x)$ maps concave functions into strictly concave functions.*

Proof. Let $\nu' \neq \nu'' \in [\underline{\nu} - E[\phi], \underline{\nu}]$, $\delta \in (0,1)$, $\nu^o = \delta\nu' + (1-\delta)\nu''$, $\underline{P}_k(\nu) = \underline{T}(\underline{P}_{k-1})(\nu)$, and $\underline{P}_{k-1}$ be concave. Then,

$$\underline{P}_{k-1}(\delta\nu' + (1-\delta)\nu'') \geq \delta\underline{P}_{k-1}(\nu') + (1-\delta)\underline{P}_{k-1}(\nu'').$$

We follow the strategy of Thomas and Worrall (1990, Proof of Proposition 1), i.e., we construct a feasible but (weakly) suboptimal contract, $\left\{c^o_\phi, p^o_\phi, \bar{\omega}^o_\phi, \underline{\omega}^o_\phi\right\}_{\phi\in\aleph}$, such that even the profit generated by the suboptimal contract $\underline{P}^o_k(\delta\nu' + (1-\delta)\nu'') \leq \underline{P}_k(\delta\nu' + (1-\delta)\nu'')$ dominates the linear combination of maximal profits $\delta\underline{P}_k(\nu') + (1-\delta)\underline{P}_k(\nu'')$. Define the weights $\underline{\delta}, \bar{\delta} \in (0,1)$ and the 4-tuple $(c^o_\phi, p^o_\phi, \underline{\omega}^o_\phi, \bar{\omega}^o_\phi)$ such that

$$\begin{aligned}
\underline{\delta} &\equiv \frac{\delta\left[1 - p_\phi(\nu')\right]}{\delta(1 - p_\phi(\nu')) + (1-\delta)(1 - p_\phi(\nu''))} \equiv \delta\frac{1 - p_\phi(\nu')}{1 - p^o_\phi(\nu^o)} \\
\bar{\delta} &\equiv \frac{\delta p_\phi(\nu')}{\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')} \equiv \delta\frac{p_\phi(\nu')}{p^o_\phi(\nu^o)} \\
\underline{\omega}^o_\phi(\nu^o) &= \underline{\delta}\underline{\omega}_\phi(\nu') + (1-\underline{\delta})\underline{\omega}_\phi(\nu'') \\
\bar{\omega}^o_\phi(\nu^o) &= \bar{\delta}\bar{\omega}_\phi(\nu') + (1-\bar{\delta})\bar{\omega}_\phi(\nu'') \\
c^o_\phi(\nu^o) &= u^{-1}\left[\delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu''))\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
(1 - p^o_\phi(\nu^o))\underline{\omega}^o_\phi(\nu^o) &= \delta\left(1 - p_\phi(\nu')\right)\underline{\omega}_\phi(\nu') + (1-\delta)(1 - p_\phi(\nu''))\underline{\omega}_\phi(\nu'') \\
p^o_\phi(\nu^o)\bar{\omega}^o_\phi(\nu^o) &= \delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu'')\bar{\omega}_\phi(\nu'')
\end{aligned}$$

By construction the suboptimal allocation satisfies

$$c^o_\phi \in [0, \bar{w}],\ p^0_\phi \in [\underline{p}, \bar{p}],\ \underline{\omega}^o_\phi \in [\underline{\nu} - E[\phi], \underline{\nu}],\ \bar{\omega}^o_\phi \in [\bar{\nu} - E[\phi], \bar{\nu}],$$

and, given the promised-utility ν^o, is also consistent with the promise-keeping constraint

$$\begin{aligned}
&\int_\aleph \left[u\left(c^o_\phi(\nu^o)\right) - X(p^o_\phi(\nu^o)) + \beta\left[(1 - p^o_\phi(\nu^o))\underline{\omega}^o_\phi(\nu^o) + p^o_\phi(\nu^o)\bar{\omega}^o_\phi(\nu^o)\right]\right] dF(\phi) \\
= &\int_\aleph \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right) + (1-\delta)\left(c_\phi(\nu'')\right) - X\left(\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')\right) \\ +\beta\left[\delta(1 - p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\delta)(1 - p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu'')\right)\bar{\omega}_\phi(\nu'')\right] \end{array}\right] dF(\phi) \\
> &\int_\aleph \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right) + (1-\delta)\left(c_\phi(\nu'')\right) - \left[\delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))\right] \\ +\beta\left[\delta(1 - p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\delta)(1 - p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu'')\right)\bar{\omega}_\phi(\nu'')\right] \end{array}\right] dF(\phi) \\
= &\ \delta\nu' + (1-\delta)\nu'' = \nu^o.
\end{aligned}$$

The fact that $X\left(\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')\right) < \delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))$ follows from the convexity

of X. Moreover, the participation constraint for any ϕ yields

$$
\begin{aligned}
& u\left(c_\phi^o(\nu^o)\right) - X(p_\phi^o(\nu^o)) + \beta\left[(1-p_\phi^o(\nu^o))\underline{\omega}_\phi^o(\nu^o) + p_\phi^o(\nu^o)\bar{\omega}_\phi^o(\nu^o)\right] \\
= & \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right) + (1-\delta)\left(c_\phi(\nu'')\right) - X(\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')) \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right] \\
> & \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right) + (1-\delta)\left(c_\phi(\nu'')\right) - [\delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))] \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right] \\
= & \delta\left[u\left(c_\phi(\nu')\right) - X(p_\phi(\nu') + \beta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu') + \beta p_\phi(\nu')\bar{\omega}_\phi(\nu')\right] \\
& +(1-\delta)\left[c_\phi(\nu'') - X(p_\phi(\nu'')) + \beta(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'') + \beta p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \\
\geq & \delta\left(\underline{\nu} - \phi\right) + (1-\delta)\left(\underline{\nu} - \phi\right) = \underline{\nu} - \phi,
\end{aligned}
$$

where we used again the strict convexity of the cost function X. Thus, we have proven that the suboptimal allocation $\left\{c_\phi^o, p_\phi^o, \underline{\omega}_\phi^o, \bar{\omega}_\phi^o\right\}_{\phi\in\aleph}$ is feasible. Namely, it satisfies the participation constraints and delivers at least the promised utility ν^o. The profit function evaluated at the optimal contract $\left\{c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right\}_{\phi\in\aleph}$ then implies the following inequality,

$$
\begin{aligned}
& \delta\underline{P}_k(\nu') + (1-\delta)\underline{P}_k(\nu'') \\
= & \delta\underline{T}(\underline{P}_{k-1})(\nu') + (1-\delta)\underline{T}(\underline{P}_{k-1})(\nu'') \\
= & \int_\aleph \left[\begin{array}{c} \underline{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] + \\ \beta\left[\delta(1-p_\phi(\nu'))\underline{P}_{k-1}(\underline{\omega}_\phi(\nu')) + (1-\delta)(1-p_\phi(\nu''))\underline{P}_{k-1}(\underline{\omega}_\phi(\nu''))\right] \\ \beta\left[\delta p_\phi(\nu')\bar{P}(\bar{\omega}_\phi(\nu')) + (1-\delta)p_\phi(\nu'')\bar{P}(\bar{\omega}_\phi(\nu''))\right] \end{array}\right] dF(\phi) \\
= & \int_\aleph \left[\begin{array}{c} \underline{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] + \\ \beta(1-p_\phi^o(\nu^o))\left[\underline{\delta}\underline{P}_{k-1}(\underline{\omega}_\phi(\nu')) + (1-\underline{\delta})\underline{P}_{k-1}(\underline{\omega}_\phi(\nu''))\right] \\ \beta p_\phi^o(\nu^o)\left[\bar{\delta}\bar{P}(\bar{\omega}_\phi(\nu')) + (1-\bar{\delta})\bar{P}(\bar{\omega}_\phi(\nu''))\right] \end{array}\right] dF(\phi) \\
< & \int_\aleph \left[\begin{array}{c} \underline{w} - u^{-1}\left(\delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu''))\right) + \\ \beta(1-p_\phi^o(\nu^o))\underline{P}_{k-1}(\underline{\delta}\underline{\omega}_\phi(\nu') + (1-\underline{\delta})\underline{\omega}_\phi(\nu'')) \\ \beta p_\phi^o(\nu^o)\bar{P}(\bar{\delta}\bar{\omega}_\phi(\nu') + (1-\bar{\delta})\bar{\omega}_\phi(\nu'')) \end{array}\right] dF(\phi) \\
= & \int_\aleph \left[\underline{w} - c_\phi^o(\nu^o) + \beta\left[p_\phi^o(\nu^o)\bar{P}(\bar{\omega}_\phi^o(\nu^o)) + (1-p_\phi^o(\nu^o))\underline{P}_{k-1}(\underline{\omega}_\phi^o(\nu^o))\right]\right] dF(\phi) \\
\equiv & \underline{P}_k^o(\nu^o) \leq \underline{P}_k(\nu^o) = \underline{P}_k(\delta\nu' + (1-\delta)\nu'').
\end{aligned}
$$

The first inequality follws from the strict concavity of the utility function and the profit function in normal times, along with the assumed concavity of $\underline{P}_{k-1}$. The second inequality, $\underline{P}_k(\nu^o) \geq \underline{P}_k^o(\nu^o)$, follows from the fact that the optimal allocation delivers (weakly) larger profits than the suboptimal one. We conclude that $\underline{P}_k(\delta\nu' + (1-\delta)\nu'') > \delta\underline{P}_k(\nu') + (1-\delta)\underline{P}_k(\nu'')$, i.e., $\underline{P}_k$ is strictly concave. This concludes the proof of the lemma. ■

Let $\underline{\Omega}$ denote the space of continuous functions defined over the interval $[\underline{\nu} - E[\phi], \underline{\nu}]$ and bounded between $\underline{P}_{MIN} = -(\bar{w} - \underline{w})/(1-\beta)$ and $P_{MAX} = \bar{w}/(1-\beta)$. Moreover, let d_∞ denote the supremum norm, such that $(\underline{\Omega}, d_\infty)$ is a complete metric space.

Lemma 7 *The mapping $\underline{T}(x)$ is an operator on the complete metric space $(\underline{\Omega}, d_\infty)$, $\underline{T}(x)$ is a contraction mapping with a unique fixed-point $\underline{P} \in \underline{\Omega}$.*

Proof. By the Theorem of the Maximum $\underline{T}(x)(\nu)$ is continuous in ν. Moreover, $\underline{T}(x)(\nu)$ is bounded between $\underline{P}_{MIN}$ and P_{MAX} since even choosing zero consumption for any realization of ϕ would induce profits not exceeding P_{MAX}

$$\begin{aligned} \underline{w} + \beta \int_{\aleph} \left[p_\phi \bar{P}(\bar{\omega}_\phi) + (1-p_\phi) x(\underline{\omega}_\phi) \right] dF(\phi) &< \bar{w} + \beta/(1-\beta)\bar{w} \\ &= \bar{w}/(1-\beta) = P_{MAX}, \end{aligned}$$

and choosing the maximal consumption of $\bar{w}$ for any ϕ would induce profits no lower than $\underline{P}_{MIN}$,

$$-(\bar{w} - \underline{w}) + \beta \int_{\aleph} x(\omega_\phi) dF(\phi) \geq \underline{P}_{MIN}.$$

Thus, $\underline{T}(x)(\nu)$ is indeed an operator on $(\underline{\Omega}, d_\infty)$.

According to Blackwell's sufficient conditions $\underline{T}$ is a contraction mapping (see Lucas and Stokey (1989, Theorem 3.3) if: (i) $\underline{T}$ is monotone, (ii) $\underline{T}$ discounts.

1. Monotonicity: Let $x, y \in \underline{\Omega}$ with $x(\nu) \geq y(\nu)$, $\forall \nu \in [\underline{\nu} - E[\phi], \underline{\nu}]$. Then

$$\begin{aligned} \underline{T}(x)(\nu) &= \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi \in \aleph}) \in \underline{\Lambda}(\nu)} \int_{\aleph} \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi) x(\underline{\omega}_\phi) \end{array} \right] \right] dF(\phi) \\ &\geq \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi \in \aleph}) \in \underline{\Lambda}(\nu)} \int_{\aleph} \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi) y(\underline{\omega}_\phi) \end{array} \right] \right] dF(\phi) \\ &= \underline{T}(y)(\nu). \end{aligned}$$

2. Discounting: Let $x \in \underline{\Omega}$ and $a \geq$ be a real constant. Then

$$\begin{aligned} \underline{T}(x+a)(\nu) &= \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi \in \aleph}) \in \underline{\Lambda}(\nu)} \int_{\aleph} \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi) \left(x(\underline{\omega}_\phi) + a \right) \end{array} \right] \right] dF(\phi) \\ &\leq \underline{T}(x)(\nu) + \beta a \\ &< \underline{T}(x)(\nu) + a, \end{aligned}$$

   since $\beta < 1$.

Thus, $\underline{T}$ is indeed a contraction mapping and according to Banach's fixed-point theorem (see Lucas and Stokey (1989, Theorem 3.2)) there exists a unique fixed-point $\underline{P} \in \underline{\Omega}$ satisfying the stationary functional equation,

$$\underline{P}(\nu) = \underline{T}(\underline{P})(\nu).$$

■

Corollary 2 *The profit function $\underline{P}(\nu)$ is strictly concave in $\nu \in [\underline{\nu} - E[\phi], \underline{\nu}]$.*

This follows immediately from Lucas and Stokey (1989, Corollary 1). Since the unique fixed-point of $\underline{T}$ is the limit of applying the operator n times $\underline{T}^n(x)(\nu)$ starting from any (and, in particular the concave ones) element x in $\underline{\Omega}$, and the operator $\underline{T}$ maps concave into strictly concave functions the fixed-point $\underline{P}$ must be strictly concave.

Lemma 8 *The profit function $\underline{P}(\nu)$ is continuously differentiable in $\nu \in [\underline{\nu} - E[\phi], \underline{\nu}]$.*

Proof. The proof is an application of Benveniste and Scheinkman (1979, Lemma 1). Recall that $\bar{P}$ and $\underline{P}$ are strictly concave. Consider the pseudo profit function

$$\begin{aligned}\underline{\widetilde{P}}(\tilde{\nu}, \nu) \equiv & \int_0^{\tilde{\phi}(\tilde{\nu})} \left[\underline{w} - \tilde{c}_\phi(\tilde{\nu}) + \beta\left[p_\phi(\nu)\bar{P}(\bar{\omega}_\phi(\nu)) + (1-p_\phi(\nu))\underline{P}(\underline{\omega}_\phi(\nu))\right]\right] dF(\phi) \\ & + \int_{\tilde{\phi}(\tilde{\nu})}^{\infty} \left[\underline{w} - \tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}) + \beta\left[p_\phi(\nu)\bar{P}(\bar{\omega}_\phi(\nu)) + (1-p_\phi(\nu))\underline{P}(\underline{\omega}_\phi(\nu))\right]\right] dF(\phi)\end{aligned} \tag{81}$$

where the triplet $(p_\phi(\nu), \underline{\omega}_\phi(\nu), \bar{\omega}_\phi(\nu))$ is the same as in the optimal contract given an initial promise ν, and the consumption function, $\tilde{c}_\phi(\tilde{\nu})$, is defined implicitly by the condition

$$\begin{aligned}\tilde{\nu} = & \int_0^{\tilde{\phi}(\tilde{\nu})} \left[u\left(\tilde{c}_\phi(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) + p_\phi(\nu)\bar{\omega}_\phi(\nu)\right]\right] dF(\phi) \\ & + \int_{\tilde{\phi}(\tilde{\nu})}^{\infty} \left[u\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) + p_\phi(\nu)\bar{\omega}_\phi(\nu)\right]\right] dF(\phi).\end{aligned} \tag{82}$$

Note that for $\tilde{\nu} = \nu$, equation (82) is equivalent to the promise-keeping constraint. Moreover, for all states $\phi \leq \tilde{\phi}(\tilde{\nu})$ such that the (pseudo-)participation constraint is binding,

$$u\left(\tilde{c}_\phi(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[\begin{array}{c}(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) \\ +p_\phi(\nu)\bar{\omega}_\phi(\nu)\end{array}\right] = \underline{\nu} - \phi. \tag{83}$$

Otherwise, when $\phi > \tilde{\phi}(\tilde{\nu})$ then consumption and promised utility are history dependent, implying that

$$u\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[\begin{array}{c}(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) \\ +p_\phi(\nu)\bar{\omega}_\phi(\nu)\end{array}\right] = \underline{\nu} - \tilde{\phi}(\tilde{\nu}). \tag{84}$$

Substituting in the right hand-side of (83) and (84), respectively, in the first and second line of (82), pins down the threshold $\tilde{\phi}(\tilde{\nu})$ that that separates states in which the participation constraint is binding from states with in which it is not binding:

$$\tilde{\nu} = \underline{\nu} - \int_0^{\tilde{\phi}(\tilde{\nu})} \phi dF(\phi) - \left(1 - F\left(\tilde{\phi}(\tilde{\nu})\right)\right) \times \tilde{\phi}(\tilde{\nu}).$$

Differentiating equation (83) with respect to $\tilde{\nu}$ shows that, for $\phi \leq \tilde{\phi}(\tilde{\nu})$, $u'\left(\tilde{c}_\phi(\tilde{\nu})\right)\tilde{c}'_\phi(\tilde{\nu}) = 0$. Differentiating equation (82) shows that the consumption function, $\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})$, is also continuously differentiable when $\phi > \tilde{\phi}(\nu)$. In particular,

$$1 = (1 - F(\tilde{\phi}(\nu)))u'\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right)\tilde{c}'_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}). \tag{85}$$

Recall that the function $\widetilde{\underline{P}}$ has the properties that $\widetilde{\underline{P}}(\tilde{\nu},\nu) \leq \underline{P}(\nu)$ with $\widetilde{\underline{P}}(\nu,\nu) = \underline{P}(\nu)$. Thus, Lemma 1 in Benveniste and Scheinkman (1979) implies that the profit function $\underline{P}(\nu)$ is continuously differentiable in $\nu \in [\underline{\nu} - E[\phi], \underline{\nu}]$, with derivative

$$\begin{aligned} \underline{P}'(\nu) &= \widetilde{\underline{P}}_{\tilde{\nu}}(\nu,\nu) = -(1 - F(\tilde{\phi}(\nu)))\tilde{c}'_{\tilde{\phi}(\nu)}(\nu) \\ &= -1/u'[(c(\nu))] < 0. \end{aligned}$$

The value of $\widetilde{\underline{P}}_{\tilde{\nu}}$ follows from the differentiation of (81) using standard methods. The last equality follows from (85) and from the fact that $\tilde{c}_{\tilde{\phi}(\nu)}(\nu) = c(\nu)$. This establishes that the profit function $\underline{P}(\nu)$ is continuously differentiable, concluding the proof.

We can now establish that the constrained allocation is unique. ■

Lemma 9 *The constrained-optimal allocation is characterized by a unique 4-tuple of state-contingent promised utilities, consumption and effort levels,* $\left\{\underline{\omega}_\phi(\nu), \bar{\omega}_\phi(\nu), p_\phi(\nu), c_\phi(\nu)\right\}$.

Proof. Lemma 6 implies that there cannot be two optimal contracts with distinct promised-utilities. Suppose not, so that there exists a 4-tuple of promised utilities $\left\{\underline{\omega}'_\phi, \underline{\omega}''_\phi, \bar{\omega}'_\phi, \bar{\omega}''_\phi\right\}$ such that either $\underline{\omega}'_\phi(\nu) = \underline{\omega}''_\phi(\nu)$ or $\bar{\omega}'_\phi(\nu) \neq \bar{\omega}''_\phi(\nu)$ (or both). Then, from the strict concavity of $\underline{P}$ and $\bar{P}$, it would be possible to construct a feasible allocation that dominates the continuation profit implied by the proposed optimal allocations, i.e., either $\underline{P}(\underline{\delta}\underline{\omega}'_\phi + (1-\underline{\delta})\underline{\omega}''_\phi) > \underline{\delta}\underline{P}(\underline{\omega}'_\phi) + (1-\underline{\delta})\underline{P}(\underline{\omega}''_\phi)$, or $\bar{P}(\bar{\delta}\bar{\omega}'_\phi + (1-\bar{\delta})\bar{\omega}''_\phi) > \bar{\delta}\bar{P}(\bar{\omega}'_\phi) + (1-\bar{\delta})\bar{P}(\bar{\omega}''_\phi)$ (or both). This contradicts the assumption that the proposed allocations are optimal, establishing that the optimal contract pins down a unique pair of promised utilities, $\left\{\underline{\omega}_\phi, \bar{\omega}'_\phi\right\}$.

Finally, we show that a unique pair of promised utilities pins down uniquely effort and consumption. More formally, the first order conditions imply that

$$\begin{aligned} X'(p_\phi) &= \beta\left(-\underline{P}'\left(\underline{\omega}_\phi\right)^{-1}\left(\bar{P}(\bar{\omega}_\phi) - \underline{P}\left(\underline{\omega}_\phi\right)\right) + \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right), \\ -\underline{P}'\left(\underline{\omega}_\phi\right)^{-1} &= u'(c_\phi), \end{aligned}$$

implying that, given ν and ϕ, effort and consumption are uniquely determined. ■

B.2.1 Self-enforcing reform effort

The proof of the profit function's strict concavity and differentiability when reform effort is self-enforcing is by-and-large a corollary of the case without the additional incentive constraint. We know from Proposition 10 that for $\nu > \underline{\omega}^*$ the additional incentive constraint is never relevant, thus strict concavity and the differentiability of the profit function follows immediately from the recession case

discussed above. On the other hand, if $\nu \leq \underline{\omega}^*$, than the profit function evaluated at the optimal contract reads as

$$\begin{aligned}
\underline{P}(\nu) &= \int_0^{\phi^*} \left[\underline{w} - c_\phi + \beta\left[(1-p_\phi)\underline{P}(\underline{\omega}_\phi) + p_\phi \bar{P}(\bar{\omega}_\phi)\right]\right] dF(\phi) \\
&\quad + \int_{\phi^*}^{\tilde{\phi}(\nu)} \left[\underline{w} - c^*_\phi + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^* \bar{P}(\bar{\omega}^*)\right]\right] dF(\phi) \\
&\quad + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[\underline{w} - c^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^* \bar{P}(\bar{\omega}^*)\right]\right] dF(\phi).
\end{aligned}$$

Note that the promised-utility ν only enters the last two terms such that the first derivative of the profit function is given by (we will prove differentiability of the profit function below)

$$\begin{aligned}
\underline{P}'(\nu) &= \left[\underline{w} - c^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^* \bar{P}(\bar{\omega}^*)\right]\right] f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu) \\
&\quad - \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{dc^*_{\tilde{\phi}(\nu)}}{d\tilde{\phi}(\nu)} \tilde{\phi}'(\nu) dF(\phi) \\
&\quad - \left[\underline{w} - c^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^* \bar{P}(\bar{\omega}^*)\right]\right] f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu) \\
&= - \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{dc^*_{\tilde{\phi}(\nu)}}{d\nu} dF(\phi) < 0,
\end{aligned}$$

where equation (34) implies that $dc^*_{\tilde{\phi}(\nu)}/d\nu = -u'(\tilde{\zeta} - \tilde{\phi}(\nu))^{-1}\tilde{\phi}'(\nu) > 0$ as the threshold $\tilde{\phi}(\nu)$ is decreasing in the promised utility. The negative second derivative follows immediately

$$\underline{P}''(\nu) = -\left[\int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{d^2 c^*_{\tilde{\phi}(\nu)}}{d\nu^2} dF(\phi) + \frac{dc^*_{\tilde{\phi}(\nu)}}{d\nu} f(\tilde{\phi}(\nu))(-\tilde{\phi}'(\nu))\right] < 0,$$

because

$$\begin{aligned}
d^2 c^*_{\tilde{\phi}(\nu)}/d\nu^2 &= \left[-u''(\tilde{\zeta} - \tilde{\phi}(\nu))\tilde{\phi}'(\nu)^2 + u'(\tilde{\zeta} - \tilde{\phi}(\nu))\tilde{\phi}''(\nu)\right] \\
&\quad \times \left[u'(\tilde{\zeta} - \tilde{\phi}(\nu))^{-1}\tilde{\phi}'(\nu)\right]^{-2} \\
&> 0.
\end{aligned}$$

The positive sign of the second derivative is based on the fact that $\tilde{\phi}''(\nu) > 0$. This can be verified from totally differentiating equation (70) with respect to ν

$$\begin{aligned}
1 &= -\left[\tilde{\phi}(\nu) f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu)\right] - \left[\tilde{\phi}'(\nu)\left[1 - F(\tilde{\phi}(\nu))\right] - \tilde{\phi}(\nu) f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu)\right] \\
&= -\tilde{\phi}'(\nu)\left[1 - F(\tilde{\phi}(\nu))\right] \quad \Rightarrow \tilde{\phi}'(\nu) = -\left[1 - F(\tilde{\phi}(\nu))\right]^{-1} < 0. \\
\tilde{\phi}''(\nu) &= -f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu) / \left[1 - F(\tilde{\phi}(\nu))\right]^2 > 0.
\end{aligned}$$

Finally, as $\tilde{\phi}(\nu)$ is continuously differentiable, so is consumption, $c^*_{\tilde{\phi}(\nu)}$, and the profit function, $\underline{P}(\nu)$. This concludes the argument.

B.3 Formal properties of the equilibrium with learning (Section 6)

Let the value functions of the government be denoted by $\bar{V}(b,\phi,\pi)$ and $\bar{W}(b,\pi)$. Since outright default is never observed in equilibrium, the value function simplifies to

$$\bar{V}(b,\phi,\pi) = \max_{b'[\underline{b},\bar{b}]} \left\{ u\left(\bar{Q}\left(b',\pi\right) \times b' + \bar{w} - \mathbb{B}\left(b,\phi,\bar{w},\pi\right)\right) + \beta \times EV\left(b',\pi\right) \right\}. \tag{86}$$

where $EV(b',\pi) \equiv E\bar{V}\left(b',\phi',\Gamma\left(\phi',\pi\right)\right)$.

The function $\bar{\Phi}$ is such that $\bar{\Phi}(b,\pi) = \bar{W}(0,\pi) - \bar{W}(b,\pi)$. Given a debt issuance of b' and a current prior of π, debt will be honored next period if $\phi' \geq \Phi^*(b',\pi)$ where Φ^* is the unique fixed point of the following equation

$$\Phi^* = \bar{\Phi}\left(b',\Gamma\left(\Phi^*,\pi\right)\right).$$

The probability of renegotiation is

$$\begin{aligned} E\left\{F\left(\bar{\Phi}\left(b',\pi'\right)|\pi'\right),\pi\right\} &= \pi F_{CW}\left(\bar{\Phi}\left(b',\Gamma\left(\Phi^*,\pi\right)\right)\right) + (1-\pi) F_{NC}\left(\bar{\Phi}\left(b',\Gamma\left(\Phi^*,\pi\right)\right)\right) \\ &= \pi F_{CW}\left(\Phi^*\left(b',\pi\right)\right) + (1-\pi) F_{NC}\left(\Phi^*\left(b',\pi\right)\right) \\ &= F\left(\Phi^*\left(b',\pi\right),\Gamma\left(\Phi^*,\pi\right)\right), \end{aligned}$$

and an arbitrage argument then implies the following bond price

$$\bar{Q}\left(b',\pi\right) = \frac{1}{R}\left(\begin{array}{c} 1 - F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right) + \\ \frac{\pi}{b'}\int_0^{\Phi^*(b',\pi)} \hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)\ dF_{CW}\left(\phi\right) + \frac{1-\pi}{b'}\int_0^{\Phi^*(b',\pi)} \hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right) dF_{NC}\left(\phi\right) \end{array}\right).$$

In what follows, we assume that the relevant equilibrium functions are differentiable in b. Then, differentiating $b \times \bar{Q}(b,\pi)$, with respect to b yields

$$\begin{aligned} \frac{d}{db}\left\{b \times \bar{Q}\left(b,\pi\right)\right\} &= \bar{Q}\left(b,\pi\right) + b \times \frac{d}{db}\bar{Q}\left(b',\pi\right) \\ &= \bar{Q}\left(b,\pi\right) - \frac{b}{R}\left(\frac{\partial F\left(\Phi^*\left(b,\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)}{\partial\phi}\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b}\right) + \\ &\quad + \frac{b}{R}\frac{\pi}{b}\hat{b}\left(\Phi^*\left(b,\pi\right),\Gamma\left(\Phi^*\left(b,\pi\right),\pi\right)\right) f_{CW}\left(\Phi^*\left(b,\pi\right)\right)\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b} \\ &\quad - \frac{1}{R}\frac{\pi}{b}\int_0^{\Phi^*(b,\pi)} \hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)\ dF_{CW}\left(\phi\right) \\ &\quad + \frac{b}{R}\frac{1-\pi}{b}\hat{b}\left(\Phi^*\left(b,\pi\right),\Gamma\left(\Phi^*\left(b,\pi\right),\pi\right)\right) f_{NC}\left(\Phi^*\left(b,\pi\right)\right)\frac{\partial\Phi^*\left(b,\pi\right)}{\partial b} \\ &\quad - \frac{1}{R}\frac{1-\pi}{b}\int_0^{\Phi^*(b,\pi)} \hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right) dF_{NC}\left(\phi\right), \end{aligned}$$

such that

$$\begin{aligned}
\frac{d}{db}\left\{b\times\bar{Q}\left(b,\pi\right)\right\} &= \bar{Q}\left(b,\pi\right)-\underbrace{\frac{b}{R}\left(\frac{\partial F\left(\Phi^{*}\left(b,\pi\right)\left|\Gamma\left(\Phi^{*},\pi\right)\right.\right)}{\partial\phi}\frac{\partial\Phi^{*}\left(b,\pi\right)}{\partial b}\right)}_{A+B}+\\
&\quad+\underbrace{\pi\frac{b}{R}f_{CW}\left(\Phi^{*}\left(b,\pi\right)\right)\frac{\partial\Phi^{*}\left(b,\pi\right)}{\partial b}}_{A}+\underbrace{\left(1-\pi\right)\frac{b}{R}f_{NC}\left(\Phi^{*}\left(b,\pi\right)\right)\frac{\partial\Phi^{*}\left(b,\pi\right)}{\partial b}}_{B}\\
&\quad-\bar{Q}\left(b,\pi\right)+\frac{1}{R}\left(1-F\left(\Phi^{*}\left(b,\pi\right)\left|\Gamma\left(\Phi^{*},\pi\right)\right.\right)\right)\\
&= \frac{1}{R}\left(1-F\left(\Phi^{*}\left(b,\pi\right)\left|\Gamma\left(\Phi^{*},\pi\right)\right.\right)\right).
\end{aligned}$$

Next, consider the consumption-savings decision. The first-order condition of (86) reads

$$\frac{d}{db'}\left\{\bar{Q}\left(b',\pi\right)b'\right\}\times u'\left[\bar{Q}\left(b',\pi\right)\times b'+\bar{w}-\mathbb{B}\left(b,\phi,\bar{w},\pi\right)\right]+\frac{d}{db'}\beta EV\left(b',\pi\right)=0.$$

The value function has a kink at $b=\hat{b}\left(\phi,\pi\right)$. Consider, first, the range where $b<\hat{b}\left(\phi,\pi\right)$. Differentiating the value function yields

$$\frac{d}{db}\bar{V}\left(b,\phi,\pi\right)=-u'\left[\bar{Q}\left(\bar{B}\left(b,\pi\right),\pi\right)\times\bar{B}\left(b,\pi\right)+\bar{w}-b\right],$$

where $\bar{B}$ denotes the optimal issuance of new bonds. Next, consider the region of renegotiation, $b>\hat{b}\left(\phi,\pi\right)$. In this case, $\frac{d}{db}\bar{V}\left(b,\phi,\pi\right)=0$.

Using the results above one obtains

$$\begin{aligned}
\frac{d}{db}EV\left(b,\Gamma\left(\phi,\pi\right)\right) &= \pi\int_{\aleph}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{CW}\left(\phi\right)+\left(1-\pi\right)\int_{\aleph}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\\
&= \pi\left(\begin{array}{c}\int_{0}^{\Phi^{*}(b,\pi)}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{CW}\left(\phi\right)\\+\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{CW}\left(\phi\right)\end{array}\right)\\
&\quad+\left(1-\pi\right)\left(\begin{array}{c}\int_{0}^{\Phi^{*}(b,\pi)}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\\+\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\end{array}\right)\\
&= \pi\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{CW}\left(\phi\right)\\
&\quad+\left(1-\pi\right)\int_{\Phi^{*}(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\\
&= \int_{\Phi^{*}(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF\left(\phi|\pi\right)\\
&= -\int_{\Phi^{*}(b,\pi)}^{\infty}u'\left[\bar{Q}\left(\bar{B}\left(b,\Gamma\left(\phi,\pi\right)\right),\Gamma\left(\phi,\pi\right)\right)\times\bar{B}\left(b,\Gamma\left(\phi,\pi\right)\right)+\bar{w}-b\right]dF\left(\phi|\pi\right).
\end{aligned}$$

Plugging this expression back into the FOC, and leading the expression by one period, yields

$$\begin{aligned} 0 &= \frac{1}{R}\left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)\times u'\left[\bar{Q}\left(b',\pi\right)\times b'+\bar{w}-\mathbb{B}\left(b,\phi,\bar{w},\pi\right)\right] \\ &\quad -\beta\int_{\Phi^*(b,\pi)}^{\infty} u'\left[\bar{Q}\left(\bar{B}\left(b',\Gamma\left(\phi,\pi\right)\right),\Gamma\left(\phi,\pi\right)\right)\times\bar{B}\left(b',\Gamma\left(\phi,\pi\right)\right)+\bar{w}-b\right]dF\left(\phi|\pi\right) \end{aligned}$$

thus

$$\beta R=\left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)\left(\int_{\Phi^*(b,\pi)}^{\infty}\frac{u'\left[\bar{C}\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[\bar{C}\left(\pi,b\right)\right]}dF\left(\phi|\pi\right)\right)^{-1},$$

where the first step uses the fact that $\frac{d}{db'}\left\{\bar{Q}\left(b',\pi\right)b'\right\}=\frac{1}{R}\left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)$, as shown above. Since $\beta R=1$, then

$$\begin{aligned} & 1-F\left(\Phi^*\left(\bar{B}\left(b,\pi\right),\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right) \\ = & \left(\int_{\Phi^*(b,\pi)}^{\infty}\frac{u'\left[\bar{C}\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[\bar{C}\left(\pi,b\right)\right]}dF\left(\phi|\pi\right)\right) \\ = & \pi\int_{\Phi^*(b,\pi)}^{\infty}\frac{u'\left[\bar{C}\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[\bar{C}\left(\pi,b\right)\right]}dF_{CW}\left(\phi\right)+\left(1-\pi\right)\int_{\Phi^*(b,\pi)}^{\infty}\frac{u'\left[\bar{C}\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[\bar{C}\left(\pi,b\right)\right]}dF_{NC}\left(\phi\right). \end{aligned}$$

B.4 Numerical Algorithm

B.4.1 Laissez-faire Equilibrium

We solve for the laissez-faire equilibrium described in Section 3 with an augmented value function iteration algorithm. Let $b = (b_1, b_2, ..., b_N)$ denote the equally spaced and inreasingly ordered grid for sovereign debt. Let $\phi = (\phi_1, \phi_2, ...\phi_S)$ be the increasingly ordered grid for the default cost, where the location of any grid point, ϕ_s, is chosen such that the cumulative weighted sum, $\widetilde{F}(\phi_s) \equiv \sum_{k=1}^{s} 1/S$, approximates the CDF of the default cost shock $F(\phi_s)$. We choose $N = 5'000$ and $S = 600$ to get a solution with high accuracy.[27]

1. Guess the default threshold, $\Phi_0(b, w) \in \phi$, for both aggregate states w and guess the reform effort, $\Psi_0(b)$, over the debt grid b. Compute the associated bond revenue,

$$\begin{aligned} Q_0(b, \bar{w})b &= \hat{Q}_0(b, \bar{w})b \\ Q_0(b, \underline{w})b &= \Psi_0(b)\hat{Q}_0(b, \bar{w})b + (1 - \Psi_0(b))\hat{Q}_0(b, \underline{w})b, \end{aligned}$$

   where the discounted recovery values are given by

$$\hat{Q}_0(b, w)b = R^{-1}\left((1 - \widetilde{F}(\Phi_0(b, w)))b + \sum_{\phi_s \in \phi_1, ..., \Phi_0(b,w)} \hat{b}_0(\phi_s, w)/S\right)$$

   and $\hat{b}_0(\phi, w) \in b$ is the inverse function of $\Phi_0(b, w)$.

2. Guess the value functions conditional on honoring the debt, $W_{0,0}(b, w)$. For any given debt level, $b_n \leq \hat{b}_0(\phi_S, \bar{w})$, on the debt grid update the value function in normal times according to

$$\begin{aligned} W_{i+1,0}(b_n, \bar{w}) &= \max_{b' \in (b_1, ..., \hat{b}_0(\phi_S, \bar{w}))} u\left(Q_0(b', \bar{w})b' + \bar{w} - b_n\right) \\ &\quad + \beta\left[(1 - \widetilde{F}(\Phi_0(b', \bar{w})))W_{i,0}(b', \bar{w}) + \sum_{\phi_s \in \phi_1, ..., \Phi_0(b', \bar{w})} W_{i,0}\left(\hat{b}_0(\phi_s, \bar{w}), \bar{w}\right)/S\right], \end{aligned}$$

   until convergence. For the remaining grid points, $b_n \geq \hat{b}_0(\phi_S, \bar{w})$, set $W_{i+1,0}(b_n, \bar{w}) = W_{i+1,0}(\hat{b}_0(\phi_S, \bar{w}), \bar{w})$. In recession, for any given debt level, $b_n \leq \hat{b}_0(\phi_S, \underline{w})$, on the debt grid, update the value function

[27] Since we are using value function iteration to solve for the decentralized Markov equilibrium, we can use the conditional Euler equation to evaluate the accuracy of the solution in terms of consumption. The mean Euler equation error in normal times (recession) is 1.16×10^{-16} (2.31×10^{-4}). Thus, in recession, there is a \$2.31 mistake for each \$10′000. Note that we approximate and evaluate the accuracy of the solution globally over the full debt grid. Moreover, the conditional Euler equation in recession involves the derivative of an equilibrium function (reform effort) and non-smooth debt accumulation. Thus, we consider the accuracy of the solution to be good. When increasing the number of grid points to $N = 10'000$ and $S = 1'000$, the Euler equation errors can be further reduced to 5.91×10^{20} in normal times and 9.54×10^{-5} in recession at the usual cost of computational time.

according to

$$
\begin{aligned}
W_{i+1,0}(b_n, \underline{w}) &= \max_{b' \in (b_1, \ldots, \hat{b}_0(\phi_S, \bar{w}))} u\left(Q_0(b', \underline{w})b' + \underline{w} - b_n\right) \\
&\quad + \beta\left(1 - \Psi_0(b')\right) \left[\begin{array}{c} (1 - \widetilde{F}(\Phi_0(b', \underline{w})))W_{i,0}(b', \underline{w}) \\ + \sum_{\phi_s \in \phi_1, \ldots, \Phi_0(b', \underline{w})} W_{i,0}\left(\hat{b}_0(\phi_s, \underline{w}), \underline{w}\right)/S \end{array} \right] \\
&\quad + \beta\Psi_0(b') \left[\begin{array}{c} (1 - \widetilde{F}(\Phi_0(b', \bar{w})))W_{\infty,0}(b', \bar{w}) \\ + \sum_{\phi_s \in \phi_1, \ldots, \Phi_0(b', \bar{w})} W_{\infty,0}\left(\hat{b}_0(\phi_s, \bar{w}), \bar{w}\right)/S \end{array} \right],
\end{aligned}
$$

until convergence. For the remaining grid points, $b_n \geq \hat{b}_0(\phi_S, \underline{w})$, set $W_{i+1,0}(b_n, \underline{w}) = W_{i+1,0}(\hat{b}_0(\phi_S, \underline{w}), \underline{w})$. $W_{\infty,0}(b', w)$ denotes the converged value function conditional on the guess for the threshold and the reform effort.

3. Update the default threshold and the reform effort according to

$$\Phi_{j+1}(b, w) = W_{\infty,j}(0, w) - W_{\infty,j}(b, w),$$

and Equation (14). Go back to step 1 and iterate until convergence.

B.4.2 Constrained Pareto Optimum

We solve for the second-best allocation with an augmented function iteration algorithm. Consider the same grid $\phi = (\phi_1, \phi_2, \ldots \phi_S)$ for the default cost that we used above. Let $\nu_w = (\nu_w(\phi_1), \ldots, \nu_w(\phi_S))$ denote the grid for promised utility, where

$$
\begin{aligned}
\nu_{\bar{w}}(\phi_s) &= \bar{\nu} - \sum_{k=1}^{s} \phi_k/S - \phi_s\left(1 - \widetilde{F}(\phi_s)\right) \\
\nu_{\underline{w}}(\phi_s) &= \underline{\nu} - \sum_{k=1}^{s} \phi_k/S - \phi_s\left(1 - \widetilde{F}(\phi_s)\right).
\end{aligned}
$$

Note that given a promised utility, $\nu_w(\phi_s)$, the default cost realization $\phi_s = \tilde{\phi}(\nu_w(\phi_s))$ corresponds to the state s where the participation constraint of the debtor starts binding. It turns out to be convenient to set the promised utility for a continued recession, $\bar{\omega}_{\bar{w}} = \nu_{\bar{w}}$, and the promised utility for a continued recession, $\underline{\omega}_{\underline{w}} = \nu_{\underline{w}}$.

1. Guess the reform effort, $p_{\underline{w},0}(\nu_{\underline{w}})$, over the grid $\nu_{\underline{w}}$.

2. Guess the future consumption, $c'_{w,0,0}(\nu_w)$, and the promised utilitiy, $\bar{\omega}_{\underline{w},0,0}(\nu_{\underline{w}})$, over the grids $\nu_{\bar{w}}$ and $\nu_{\underline{w}}$.

3. Compute current consumption from the Euler equations (which holds for all states s where the participation constraint is not strictly binding)

$$c_{w,0,0}(\nu_w) = \left(u'\right)^{-1}\left[u'\left(c'_{w,0,0}(\nu_w)\right)\beta R\right],$$

and the initial promised utility, $\tilde{\nu}_{w,0}(\nu_w)$, implicitly defined by

$$\begin{aligned}
\bar{\nu} - \tilde{\phi}(\tilde{\nu}_{\bar{w},0,0}(\nu_{\bar{w}})) &= u(c_{\bar{w},0,0}(\nu_{\bar{w}})) + \beta\nu_{\bar{w}} \\
\underline{\nu} - \tilde{\phi}(\tilde{\nu}_{\underline{w},0,0}(\nu_{\underline{w}})) &= u(c_{\underline{w},0,0}(\nu_{\underline{w}})) - X(p_{\underline{w},0}(\nu_{\underline{w}})) + \beta\left[p_{\underline{w},0}(\nu_{\underline{w}})\bar{\omega}_{\underline{w},0,0}(\nu_{\underline{w}}) + (1-p_{\underline{w},0}(\nu_{\underline{w}}))\nu_{\underline{w}}\right],
\end{aligned}$$

4. Update the guess for the future consumption function by interpolating ν_w on the pairs $(\tilde{\nu}_{w,0,0}, c_{w,0,0})$ to yield $c'_{w,1,0}(\nu_w)$ Update the guess for promised utility by interpolating $c'_{\underline{w},1,0}(\nu_{\underline{w}})$ on the pairs $\left(c_{\underline{w},0,0}, \tilde{\nu}_{\underline{w},0,0}\right)$ to yield $\bar{\omega}_{\underline{w},1,0}(\nu_{\underline{w}})$. Go back to step 3 and iterate until convergence. Let $c_{w,\infty,0}(\nu_w)$ and $\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}})$ denote the converged functions given the guess on the reform effort.

5. Guess the profit functions, $\bar{P}_0(\nu_{\bar{w}})$ and $\underline{P}_0(\nu_{\underline{w}})$. Update the profit function in normal times according to

$$\begin{aligned}
\bar{P}_{i+1,0}(\tilde{\nu}_{\bar{w},\infty,0}(\nu_{\bar{w}})) &= \left(1 - \widetilde{F}(\tilde{\phi}(\tilde{\nu}_{\bar{w},\infty,0}(\nu_{\bar{w}})))\right)\left[\bar{w} - c_{\bar{w},\infty,0}(\nu_{\bar{w}}) + R^{-1}\bar{P}_{i,0}(\nu_{\bar{w}})\right] \\
&\quad + \sum_{\phi_s \in \phi_1,\ldots,\tilde{\phi}(\tilde{\nu}_{\bar{w},\infty,0}(\nu_{\bar{w}}))} \left[\bar{w} - c_{\bar{w},\infty,0}(\nu_{\bar{w}}(\phi_s)) + R^{-1}\bar{P}_{i,0}(\nu_{\bar{w}}(\phi_s))\right]/S,
\end{aligned}$$

until convergence, $\bar{P}_{\infty,0}(\nu_{\bar{w}})$. In recession, update according to

$$\begin{aligned}
\underline{P}_{i+1,0}(\tilde{\nu}_{\underline{w},\infty,0}(\nu_{\underline{w}})) &= \left(1 - \widetilde{F}(\tilde{\phi}(\tilde{\nu}_{\underline{w},\infty,0}(\nu_{\underline{w}})))\right)\left[\begin{array}{c} \underline{w} - c_{\underline{w},\infty,0}(\nu_{\underline{w}}) \\ +R^{-1}\left[\begin{array}{c} p_{\underline{w},0}(\nu_{\underline{w}})\bar{P}_{i,0}(\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}})) \\ +(1-p_{\underline{w},0}(\nu_{\underline{w}}))\underline{P}_{i,0}(\nu_{\underline{w}}) \end{array}\right] \end{array}\right] \\
&\quad + \sum_{\phi_s \in \phi_1,\ldots,\tilde{\phi}(\tilde{\nu}_{\underline{w},\infty,0}(\nu_{\underline{w}}))} \left[\begin{array}{c} \underline{w} - c_{\underline{w},\infty,0}(\nu_{\underline{w}}(\phi_s)) \\ +R^{-1}\left[\begin{array}{c} p_{\underline{w},0}(\nu_{\underline{w}}(\phi_s)\bar{P}_{i,0}(\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}}(\phi_s))) \\ +(1-p_{\underline{w},0}(\nu_{\underline{w}}(\phi_s)))\underline{P}_{i,0}(\nu_{\underline{w}}(\phi_s)) \end{array}\right] \end{array}\right]/S.
\end{aligned}$$

6. Update the reform effort function according to

$$p_{\underline{w},j+1}(\nu_{\underline{w}}) = \left(X'\right)^{-1}\left[\begin{array}{c} u'(c_{\underline{w},\infty,j}(\nu_{\underline{w}}))R^{-1}\left(\bar{P}_{\infty,j}(\bar{\omega}_{\underline{w},\infty,j}(\nu_{\underline{w}})) - \underline{P}_{\infty,j}(\nu_{\underline{w}})\right) \\ +\beta\left(\bar{\omega}_{\underline{w},\infty,j}(\nu_{\underline{w}}) - \nu_{\underline{w}}\right) \end{array}\right].$$

Go back to step 3 and iterate until convergence.