Efficient Job Upgrading, Search on the Job and Output Dispersion*

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This version: August 2016

Abstract

A worker's job can be improved through on-the-job search (OJS) and job upgrading. Incorporating these external and internal job dynamics into a directed search model, this paper analytically characterizes the socially efficient allocation and quantitatively evaluates the model. The analysis shows that efficient OJS is front-loaded in a worker's career and stops after a finite number of job switches, but efficient job upgrading continues throughout the career and may be backloaded. OJS and job upgrading generate a job ladder in output and productivity among identical workers, i.e., frictional dispersion. The endogenous ladder also implies a positive return to tenure and a cost of job loss. When the model is calibrated, frictional dispersion, the return to tenure and the cost of job loss are large. Moreover, the analysis reveals the importance of the calibrated feature that the marginal cost of a vacancy increases in the job type. If the marginal cost of a vacancy is non-increasing in the job type, the social planner will choose the starting job to be high and leave very little room for OJS or job upgrading.

JEL classification: E24; J60.

Keywords: Job upgrading; On-the-job search; Frictional dispersion; Return to tenure; Cost of job loss.

* Address: Department of Economics, Pennsylvania State University, 502 Kern Building, University Park, PA 16802, USA. An earlier version of this paper was presented at the NBER summer institute (July 2016), the Society for Economic Dynamics (Toulouse, June 2016), the Shanghai Macro Workshop (SUFE, June 2016), and Carnegie-Mellon University (May 2016). I thank Philipp Kircher and Kyle Herkenhoff for comments. Financial support from the Pennsylvania State University is gratefully acknowledged.

1. Introduction

A worker's job can improve over time in two ways. One is on-the-job search (OJS) that results in a worker moving from one employer to another. The other is job upgrading done by a worker's employer.[1] These external and internal dynamics of jobs can have far-reaching implications for the labor market. Intuitively, job dynamics affect dynamics of wages and productivity, thereby affecting the return to tenure and the cost of job loss. Also, job mobility affects workers' search and firms' job creation. In particular, if unemployed workers expect jobs to improve in the future, they are willing to accept relatively low jobs to start employment, in which case firms are willing to create more vacancies to reduce unemployment. Thus, job dynamics can be viewed as mechanisms to cope with labor market frictions. It is then surprising that the literature has largely focused on the transition of unemployed workers into employment by assuming job dynamics afterward to be exogenous, e.g., Diamond (1982), Mortensen (1982) and Pissarides (2000).[2] In this paper, I characterize the constrained social optimum formally with OJS and job upgrading. Then I calibrate the model to examine the quantitative implications on frictional output dispersion, the return to tenure and the cost of job loss.

A formal model of the efficient allocation with OJS and job upgrading can shed light on a number of questions. What is the socially efficient tradeoff between creating low jobs to reduce unemployment and improving jobs to increase productivity? When jobs can be upgraded, why is it still socially efficient to move workers between jobs through OJS? More generally, how does job mobility in the market affects job upgrading within a firm? Should jobs be upgraded more intensively in the early or the late stage of a worker's career?

The model is also a useful lens to view the quantitative importance of OJS and job upgrading for a number of empirical facts. One is frictional dispersion, i.e., dispersion in output or wages among homogeneous workers. Hornstein et al. (2011) find that most

[1] For the evidence on internal labor mobility, see Baker et al. (1994). Lazear and Oyer (2004) provide empirical evidence on both internal and external labor market mobility. Papageorgiou (2016) provides additional evidence.

[2] A brief review of the literature will appear later in the introduction.

search models fail to produce significant frictional dispersion measured by the mean-min ratio of wages. For example, the canonical search model yields the mean-min ratio lower than 1.05, while the empirical value is between 1.7 and 2. Since job upgrading and OJS stretch the two ends of the distribution in output, it is hopeful that they may increase frictional dispersion significantly. The second fact is the return to tenure (e.g., Mincer and Jovanovic, 1981, and Topel, 1991), and the third fact is the large and persistent cost of job loss (e.g., Jacobson et al., 1993, and Davis and von Wachter, 2011). OJS and job upgrading generate a ladder in output and productivity among homogeneous workers, which induces the return to tenure and the cost of job loss. On all three facts, the empirical literature has struggled with the difficulty in controlling for unobserved heterogeneity and endogenous selection of workers. By assuming workers to be homogeneous, the model is free from this difficulty and gives a controlled environment to view the facts.

The model economy has identical workers and heterogeneous jobs. The job type is modeled as the amount of capital that a firm rents for the job. Net output, defined as output minus the capital cost, increases in the job type initially but eventually reaches the maximum at a final job type. A firm can create any job type by incurring the vacancy cost, which is increasing and convex in the job type. Search is directed toward particular job types.[3] After being matched, a firm can rent more capital to upgrade the job. As an investment, job upgrading has a convex adjustment cost. When a worker separates from a job, the firm can return capital to the owners. However, the firm cannot move capital from one job to another without incurring the adjustment cost or fill a vacant job without incurring the vacancy cost. In this sense, the job type has a match-specific component.

I characterize the socially efficient allocation under the same search frictions as in the market. The focus on the efficient allocation simplifies the analysis by eliminating the complexity of various contractual details. It also puts a strong requirement on the model for generating frictional dispersion in net output. For example, Burdett and Mortensen (1998), Burdett and Coles (2003) and Shi (2009) generate dispersion in wages but not in

[3]For examples of directed search models, see Peters (1991), Moen (1997), Acemoglu and Shimer (1999), Burdett et al. (2001) and Shi (2001, 2009).

net output. In this model, the social planner chooses the target job type for workers to search and the matching rate at the target, both of which can depend on the workers' employment status and the current job type. To deliver the specified matching rate, the planner incurs the vacancy cost to create the right number of vacancies at each search target. For a worker who stays at the current job, the planner chooses the intensity in which the job is upgraded and incurs the adjustment cost to deliver this intensity.

The efficient allocation exhibits the following features. Low type jobs are created to match with unemployed workers. Once employed, a worker's job type starts to be upgraded. At the same time, the worker is allocated to search for a higher job type. If the worker finds a match at the search target, the worker moves to the new job. Thus, both OJS and job upgrading induce a ladder in net output and productivity among homogeneous workers. However, the two types of job dynamics differ in an important aspect: job upgrading is continuous but OJS increases the job type by a discrete amount. The job switching rate declines as the worker's job type increases, and reaches zero after a finite number of job-to-job transitions. In contrast, job upgrading continues until the job type reaches the final level at which net output ceases to increase.

The starting job created for unemployed workers is low because the marginal cost of vacancies is low for low-type jobs and the value of being employed is high. Rather than continuing to search, an unemployed worker can take a low job in order to benefit from job upgrading and OJS. In the early part of a worker's employment, job switching occurs because a job switch increases the job type more quickly than job upgrading does. As the job type increases, the job switching rate declines because the marginal cost of a vacancy increases, but the marginal gain from a higher job type diminishes, in the job type. After a finite number of job switches, the marginal cost of a vacancy exceeds the expected marginal benefit of a vacancy, at which time OJS stops. However, job upgrading continues because the amount of job upgrading can be made sufficiently small.

Job upgrading and OJS are substitutes. Since both increase productivity permanently, it may seem socially efficient to front-load them, i.e., make them happen early in a worker's

career. Indeed, OJS is front-loaded. OJS declines over time and eventually stops because the higher is a worker's current job, the higher is the marginal cost of creating a better job for the worker to move to. However, job upgrading is not front-loaded in general. Because job switching destroys the investment made in upgrading jobs, it is socially efficient to delay job upgrading until the job switching rate becomes small. This delay can be significant if the job switching rate is sufficiently high initially. In this case, job upgrading is hump-shaped over the job type. At low job types, job upgrading is increasing because the job switching rate is declining. At high job types, job upgrading is falling because the marginal benefit of upgrading diminishes.

Job upgrading and OJS induce dispersion in net output among identical workers, a positive return to tenure, and a cost of job loss. They create dispersion in net output by not only allowing the job type to increase in a worker's career, but also reducing the option value of not taking up a job and inducing the starting job type to fall. The return to tenure is positive in the sense that net output increases in tenure, because a longer tenure is associated with a longer duration of job upgrading. When a worker becomes unemployed, the worker will start all over from the bottom of the job ladder, and so net output is lower at reemployment than at the worker's previous job. The higher is the worker's previous job type, the larger is this loss in net output caused by displacement. In this model, output dispersion, the return to tenure and the cost of job loss are tightly related to search frictions and part of the constrained social optimum. In particular, because workers are identical in the model, all dispersion in net output is frictional. If vacancies were costless or matching were frictionless, the social planner would create only the final job type at which net output is maximized, which would eliminate net output dispersion.

When the model is calibrated, it generates the mean-min ratio in net output as 1.81. This ratio can be interpreted as frictional wage dispersion by assuming that wages are proportional to net output. The dispersion is close to the value in the U.S. data and much larger than in Hornstein et al. (2011). It is notable that the calibration meets the same targets as used by Hornstein et al. (2011) on an unemployed worker's job-finding rate and home production. These two targets are the strong disciplines that make most models

unable to generate significant frictional dispersion (see section 5.3 for the explanation). The success of the current model shows that search frictions can be quantitatively important for generating frictional dispersion if they interact with OJS and job upgrading.

The calibrated model also produces significant returns to tenure and persistent losses caused by job displacement. An OLS regression of the model simulated data shows that 10 years of tenure increase net output by 0.5 log points. A closer examination reveals that most of the returns occur in the first five years of tenure. Thus, the model produces larger and faster returns to tenure than in the empirical literature cited earlier. On the cost of job loss, the model shows that when an unemployed worker is reemployed, net output is on average 46% lower than the control group who is continuously employed. The loss can still be 18% five years after reemployment and 12% ten years after reemployment.

Counterfactual experiments reveal that OJS and job upgrading are both potent to generate frictional dispersion. When one is eliminated but the other is kept, the recalibrated model still yields sizable dispersion in net output. However, both types of job dynamics rely on one critical feature of the model – the convexity of the vacancy cost in the job type. In order to match the high job-finding rate of unemployed workers in the data, the model must generate sufficiently many vacancies of a low job type for unemployed workers to search. It is socially efficient to create so many low type jobs relatively to high type jobs only if the marginal cost of a vacancy is increasing steeply in the job type. If the vacancy cost is linear in the job type, then the social planner will make the starting job type so high that leaves little room for job upgrading or OJS. In this case, almost all frictional dispersion in net output, the return to tenure and the cost of job loss will vanish.

The convex vacancy cost can be justified by the implied job dynamics. Using personnel data, Baker et al. (1994) and Lazear and Oyer (2004) find that a typical worker spends a long time in a firm to climb up the job ladder. Part of this slow climb is caused by learning, which has been examined in the literature and is abstracted from in this paper.[4]

[4]Jovanovic (1979) is a classic on the interaction between search and learning about a worker's ability. Papageorgiou (2016) documents the effect of firm size on within-firm labor mobility and uses a learning model to explain this effect. Gervais et al. (2016) incorporates learning into a life-cycle model with search.

Another important cause is simply that high level jobs are scarce because they are much more costly to create. Specifically, the job structure in the firm studied by Baker et al. (1994) was remarkably stable in 20 years even though the firm tripled in size. To sustain such job dynamics, the job ladder must be sufficiently long. The convexity of the vacancy cost in the job type is important to induce such a long ladder by making it optimal to keep the starting job low.[5]

Because job upgrading in this paper has a match-specific component, the theory is related to the literature on match-specific training, e.g., Wasmer (2006) and Lentz and Roys (2015). However, the job type in this model is the result of investment in a job, not in a worker's skill. Moreover, the job type can be created for a vacancy before a match forms, whereas training can happen only after a match is formed. Whether it is socially efficient to create a high job type at the beginning or upgrade a job to a high type later depends on search frictions and especially the vacancy cost. As said above, this feature of the job type enables me to attribute all net output dispersion in the model to search frictions. In addition, the literature on match-specific training, as well as the one on general training (e.g., Acemoglu and Pischke, 1999, Fu, 2011), focuses on whether the market can internalize the pecuniary externality of training in the presence of search frictions. Instead, the current paper focuses on the constrained social optimum instead of an equilibrium. This focus increases the challenge to address the issues. For example, one cannot use market inefficiency to answer the question why job switching may be socially efficient even when job upgrading is available.

This paper is also related to the literature on OJS, which has examined both undirected search (e.g., Burdett and Mortensen, 1998) and directed search (e.g., Shi, 2009, Menzio and Shi, 2011). Some of the papers have incorporated wage-tenure contracts, e.g., Burdett and Coles (2003) and Shi (2009). Since this literature has not allowed job upgrading, it is not able to examine the interaction between job switching and job upgrading. Tsuyuhara (2014) and Lamadon (2014) incorporate moral hazard into an OJS model with wage-tenure

[5]Kaas and Kircher (2015) make a similar argument in a different context. They show that the vacancy cost must be convex in the firm size in order to sustain dynamics in firm size. Here I focus on the job type by assuming the vacancy cost to be linear in the number of employees in a firm.

contracts. Tsuyuhara's model generates a significant mean-min ratio in wages which is similar to the value obtained here with OJS alone. Lise and Robin (2014) introduce ex ante and ex post heterogeneity on both sides of a market to study dynamic sorting, which can also increase dispersion.

Finally, on the cost of job loss, Jarosch (2015) analyzes the persistence of this cost by estimating a model where jobs differ in productivity and the exogenous separation rate. Huckfeldt (2015) analyzes the cyclical pattern of the cost of job loss by constructing a model where skill accumulation differs between two types of jobs. Wee (2015) examines the cost of entering the job market during a recession with a search model where workers learn about the fit with a job and accumulate skills.

A significant part of this paper characterizes the efficient allocation formally. This is necessary because even the basic properties, such as existence and continuity of the social value function, cannot be presumed in the presence of job upgrading and OJS. In addition, the value function is not necessarily concave, which adds complexity to the analysis of the policy functions. By overcoming these difficulties, the formal characterization adds value beyond the particular subject of this paper.

2. The Model

2.1. Model Environment

Time is continuous. Workers and firms have the same time discounting rate $r \in (0, \infty)$. All workers are identical. Jobs are heterogeneous in the type denoted $k \geq 0$. A type k job requires a worker and k units of capital to operate. After a job is filled, the required capital is rented from the rest of the world at the rental rate r. A worker employed at a type k job produces a flow of output $f_a(k)$, where $f_a(0) = 0$, $f_a'(k) > 0$ and $f_a''(k) < 0$ for all k. Denote $f(k) = f_a(k) - rk$ as output net of the capital cost of a type k job. Assume that there is a unique *final job type* $k^* \in (0, \infty)$ such that $f'(k^*) = 0$ and $f(k^*) > 0$. Thus, net output is maximized at k^*. For an unemployed worker, home production is expressed as $f(k_u)$ so that k_u is interpreted as an unemployed worker's equivalent "job type". Assume

that $k_u \in (0, k^*)$ is constant over time.

A firm can create a vacancy of any job type. The flow cost of a type k vacancy is $\psi(k)$, where $\psi'(k) > 0$ and $\psi''(k) \geq 0$ for all $k > 0$, and $\psi(0) = 0$. The vacancy cost must be incurred in advance, in contrast to the cost of capital for a job that can be incurred after a vacancy is filled. Since any job type can be chosen as the starting job, search frictions are the element that may makes it socially inefficient to create only the highest job type, thereby rendering job upgrading necessary.

An employed worker can search on the job with probability λ and an unemployed worker can search with probability one. Search is directed into submarkets described by (ϕ, p), where ϕ is the job type of the search target and p the matching rate for a worker in the submarket. To deliver the matching rate p, the number of vacancies per searching worker must be $\theta(p)$. The matching rate for a vacancy in the submarket is $q(p) = \frac{p}{\theta(p)}$. The function $\theta(p)$ summarizes matching frictions. If the matching rate function for a worker is $P(\theta)$, then $\theta(p)$ is the inverse function of P. Assume:[6]

$$\theta'(p) > \frac{\theta(p)}{p} > 0 \text{ and } \theta''(p) > 0 \text{ for all } p > 0,$$
$$\theta(0) = 0,\ \theta'(0) \in (0, \infty),\ \text{and } \lim_{p\to\infty} \frac{\theta(p)}{p} = \infty.$$

Note that $\theta'(0) > 0$, which requires the number of vacancies to be strictly positive even if the matching rate needed to deliver for a worker is arbitrarily small. The assumption is equivalent to $P'(0) < \infty$. The following example gives two well-known matching functions that satisfy the assumptions on $\theta(p)$:

Example 2.1. *The urn-ball matching function is* $P(\theta) = \frac{\theta}{p_0}\left(1 - e^{-1/\theta}\right)$ *and the generalized telephone matching function is* $P(\theta) = \left[(\theta/p_0)^{-\rho} + 1\right]^{-1/\rho}$, *where* $p_0, \rho \in (0, \infty)$ *are constants. In both cases, the assumptions on* $\theta(.)$ *are satisfied. In particular,* $P'(0) = 1/p_0 < \infty$, *and so* $\theta'(0) = p_0 > 0$.

Once a worker is hired, the capital stock is put into a fixture or specific equipment that can be upgraded only gradually. The flow cost of upgrading a job with an intensity i is

[6] Since $P(\theta(p)) = p$ for all p, the assumptions on $\theta(p)$ are equivalent to the following assumptions on $P(\theta)$: $0 < P'(\theta) < \frac{P(\theta)}{\theta}$ and $P''(\theta) < 0$ for all $\theta > 0$, $P(0) = 0$, $P'(0) \in (0, \infty)$, and $\lim_{\theta\to\infty} \frac{P(\theta)}{\theta} = 0$.

$c(i)$. Let t denote a worker's tenure in the current firm. Then,

$$\frac{\mathrm{d}k(t)}{\mathrm{d}t} = i. \tag{2.1}$$

The upgrading cost satisfies $c(\infty) = \infty$, $c(i) = c'(i) = 0$ for all $i \leq 0$, $c'(i) > 0$ for all $i > 0$, and $c''(i) > 0$ for all $i \geq 0$. These assumptions are common for adjustment costs except the conditions $c(i) = c'(i) = 0$ for all $i < 0$, which imply that downgrading is costless.[7] After incurring the adjustment cost $c(i)$, the increment in the capital stock can be funded immediately at the rental rate r.

An employed worker separates from a job exogenously at the rate $\delta \in (0, \infty)$, in addition to endogenous separation caused by OJS. After a job is left vacant because of separation, the capital stock is returned to the owner.[8]

Some clarifications on jobs and the two costs are useful. First, the job type is not a worker's skill. To be able to hire a worker for a type k job, a firm must incur the vacancy cost $\psi(k)$ independently of the worker's current job type. Second, as most models in the search literature, this paper treats different jobs in the same firm independently and, hence, abstracts from the effect of firm size on internal labor mobility documented by Papageorgiou (2016). Specifically, if a firm wants to move capital from one job to another, the firm must incur the vacancy cost. Third, the difference between the vacancy cost and the upgrading cost captures the difference between a new and an existing job. By incurring the vacancy cost to create a new job, a firm is able to increase the job type in a discrete amount. In contrast, an existing job can only be upgraded smoothly, not because additional capital is not immediately available but because it is costly to put new capital into an existing job structure.

[7] The assumption on the downgrading cost will not be used on the efficient path of k because this path is increasing over time. An example of c is $c(i) = (i+a)^{\gamma} - a^{\gamma} - \gamma a^{\gamma-1} i$ if $i \geq 0$ and $c(i) = 0$ if $i < 0$, where $\gamma > 1$ and $a > 0$. When $\gamma = 2$, a can be 0.

[8] For simplicity, I abstract from the loss in capital caused by a worker's separation from a job. If only a fraction σ of capital survives after separation, the capital owner will incorporate such a loss by increasing the rental rate of capital to $r + [\delta + \lambda p(h)](1 - \sigma)$.

2.2. Planner's Problem

The social planner chooses search and job upgrading to maximize the sum of social values of jobs and unemployed workers. The planner is constrained by search frictions that the matching rates are finite and that the matching process cannot depend on a worker's identity. In particular, the planner's allocation must be the same for all workers employed in the same job type. OJS and job upgrading induce a non-degenerate distribution of workers over job types, which will be analyzed in section 4. This distribution and the measure of unemployed workers are the aggregate state of the economy. However, the efficient choices are independent of this aggregate state, although the evolution of the aggregate state depends on the efficient choices. That is, the allocation is block recursive, as defined by Shi (2009) and Menzio and Shi (2010). Block recursivity arises because the planner can direct the workers at different types to search for separate targets and can create as many vacancies as desired at each target.

For a worker employed at a job type k, the planner chooses the policy functions of job search, $(\phi(k), p(k))$, and the job upgrading intensity, $i(k)$. The function $\phi(k)$ is the target job type to be searched for and $p(k)$ the matching rate for the worker. Let $V(k)$ be the social value of a type k job and V_u the social value of an unemployed worker. I focus on the steady state where $V(k)$ depends on time only through k. That is, if $i = 0$ over time, then $V(k)$ is constant over time. The function $V(k)$ obeys the Bellman equation:

$$rV(k) = f(k) - \delta[V(k) - V_u] + \lambda s(k) + \upsilon(k).$$

This equation equates the flow value of a type k job to the return. In addition to net output and the expected loss due to exogenous separation, the return on the job includes the *social return on job search* $s(k)$ and the *social return on job upgrading* $\upsilon(k)$, which will be computed later. The term, $\mathrm{d}V/\mathrm{d}t$, is present in the equation through $\upsilon(k)$. Rewrite the above equation as[9]

$$(r+\delta)V(k) = f(k) + \delta V_u + \lambda s(k) + \upsilon(k). \tag{2.2}$$

[9]The value of capital is deducted from the social welfare function because capital is rented from the rest of the world. The social welfare function that includes the value of capital is $W(k) = V(k) + k$. The Bellman equation for W is similar to (2.2), with V being replaced by W and f by f_a.

For a worker employed at a type k job, the planner chooses search policy functions $(\phi(k), p(k))$ to maximize the social return on job search:

$$s(k) \equiv \max_{(\phi,p)} \{p[V(\phi) - V(k)] - \psi(\phi)\theta(p)\}. \tag{2.3}$$

The new job at ϕ increases the value by $[V(\phi) - V(k)]$ and the worker succeeds in getting the new job at the rate p. To deliver the matching rate, the number of vacancies needed is $\theta(p)$, and so the vacancy cost is $\psi(\phi)\theta(p)$.

For a type k job, the planner chooses the policy function of the upgrading intensity, $i(k)$, to maximize the return on upgrading:

$$\upsilon(k) \equiv \max_{i} \left[\frac{\mathrm{d}V(k)}{\mathrm{d}t} - c(i)\right] \text{ s.t. (2.1).} \tag{2.4}$$

If $V'(k)$ exists, the problem of job upgrading becomes

$$\upsilon(k) \equiv \max_{i} [V'(k)i - c(i)]. \tag{2.5}$$

For an unemployed worker, the value of unemployment V_u is constant over time because home production is so. Precisely, V_u and s_u satisfy:

$$rV_u = f(k_u) + s_u \tag{2.6}$$

$$s_u = \max_{(p,\phi)} \{p[V(\phi) - V_u] - \psi(\phi)\theta(p)\}. \tag{2.7}$$

Denote efficient search choices for an unemployed worker as (ϕ_u, p_u).

3. Efficient Allocation

Recall that the final job type k^* is defined by $f'(k^*) = 0$. Because net output is decreasing in k for $k > k^*$, it is never socially efficient to have $k > k^*$. Without loss of generality, I will restrict the domain of k to $[0, k^*]$ throughout the analysis.

3.1. Efficient Job Switching and Job Upgrading

The functional equation (2.2) is the Bellman equation of the social value function V. To characterize the efficient allocation, it is necessary to show that the value function exists. Appendix A accomplishes this task and establishes the following lemma:

Lemma 3.1. *There exist a unique function $V(k)$ and a unique number V_u that satisfy (2.2) and (2.6). The derivative $V'(k)$ exists and is given by*

$$V'(k) = c'(i) \text{ for both } i > 0 \text{ and } i = 0, \tag{3.1}$$

where $i = i(k)$. Thus, $V(k)$ is increasing, and strictly so if $i(k) > 0$. Moreover, whenever $\frac{di}{i\,dt}$ exists, the derivative $V''(k)$ exists and is given by

$$V''(k) = c''(i)\frac{di(k(t))}{i\;dt}. \tag{3.2}$$

Thus, $V(k)$ is strictly concave near k if and only if $\frac{di}{i\,dt} < 0$.

When job upgrading is strictly positive, the capital stock is below the final type k^*. At such a capital stock, net output $f(k)$ is strictly increasing in k. As a result, the value function must be strictly increasing in the job type.

Condition (3.1) is the first-order condition of i, which equates the marginal benefit and cost of job upgrading. The value function is differentiable because job upgrading can take place in continuous time and the marginal cost of upgrading is continuous. Any small increase in the job type can be achieved by a finite i in a small interval of time. If the value function were not differentiable at any particular job type, small upgrading near this particular job type would yield a discrete change in the social value of a match, which would contradict continuity of the social value function.[10] Note that (3.1) holds as equality even at the corner $i = 0$, as well as for any interior $i > 0$, because $V'(k) \geq 0 = c'(0)$ while the complementary slackness at $i = 0$ requires $V'(k) \leq c'(0) = 0$.

[10]This argument for differentiability of the value function uses only continuity of the value function, instead of an envelope condition as in a typical analysis (e.g., Stokey et al., 1989).

The value function may not necessarily be concave. The social value function is strictly concave if and only if job upgrading is decreasing over tenure, as shown by (3.2). This result is intuitive. If job upgrading has diminishing marginal gains in the social value, it is socially efficient for job upgrading to taper off over time. If job upgrading has increasing marginal gains in the social value, it is socially efficient for job upgrading to increase over time to explore such increasingly larger marginal gains.

To characterize the efficient allocation in more detail, I focus on the natural case where $\frac{\mathrm{d}i}{\mathrm{d}t}$ exists. The following proposition is proven in Appendix B:

Proposition 3.2. *Assume that $\frac{di}{dt}$ exists. The following results hold:*
(i) The derivatives $V''(k)\,i$ and $v'(k)$ exist, with $v'(k) = V''(k)\,i(k)$.
(ii) The derivative $s'(k)$ exists and is given as $s'(k) = -p(k)\,V'(k)$.
(iii) The envelope condition on (2.2) holds as

$$[r + \delta + \lambda p(k)]\,V'(k) = f'(k) + V''(k)\,i(k)\,. \tag{3.3}$$

Efficient upgrading evolves according to

$$\frac{di}{dt} = \frac{1}{c''(i)}\left\{[r + \delta + \lambda p(k)]\,c'(i) - f'(k)\right\}. \tag{3.4}$$

Result (i) is the envelope condition of the efficient upgrading problem. If the social value function is concave, the marginal return on job upgrading diminishes as the job type increases. Result (ii) is the envelope condition of the efficient job search problem, which shows that the marginal return on job search diminishes as the job type increases.[11]

Since $c'(i) = V'(k)$, (3.4) reveals that job upgrading increases with tenure if and only if $[r + \delta + \lambda p(k)]\,V'(k) > f'(k)$. To explain this result, it is useful to treat a marginally higher job type as an asset. This asset is discounted by the job separation rate $[\delta + \lambda p(k)]$

[11] The proof of result (i) uses the fact that efficient upgrading at each k is uniquely given by (3.1) and, hence, the job upgrading policy function $i(k)$ is continuous. In contrast, the proof of (ii) does not rely on continuity of the search policy functions $(\phi(k), p(k))$, because these functions are yet to be shown to be unique for each k. Instead, the proof of (ii) first uses differentiability of $V(k)$ to establish the existence of $s'(k)$ and then uses $s'(k)$ to prove the envelope condition. With (i), (ii) and the existence of $V'(k)$, every term in (2.2) is differentiable, and (3.3) is the envelope condition on (2.2). The dynamic equation for i, (3.4), arises from substituting $V'(k) = c'(i)$ and $V''(k)\,i = c''(i)\,\frac{\mathrm{d}i}{\mathrm{d}t}$ in (3.3).

as well as r, because job separation destroys the value of the job. Thus, the "permanent income" of a marginally higher k is $[r+\delta+\lambda p(k)]\,V'(k)$. The cash flow of this asset is $f'(k)$, i.e., the value added by a marginally higher job type. If the permanent income of this asset exceeds the cash flow of the asset, the difference must be caused by a capital gain in the asset. In turn, for a job type to have a capital gain, the amount of upgrading must be increasing over tenure. Similarly, if the permanent income of a higher job type is less than the cash flow, there is a capital loss in the asset, in which case job upgrading declines over tenure.

Whether job upgrading increases or decrease over tenure depends on job search. The endogenous job-to-job transition rate appears in (3.4) as part of the effective discounting rate on the current job. To characterize efficient job search, let (i^*, k^*, p^*) denote the *final state* of the efficient allocation, which satisfies $\frac{\mathrm{d}i}{\mathrm{d}t} = 0 = \frac{\mathrm{d}k}{\mathrm{d}t}$. Then,

$$i^* = p^* = 0, \text{ and } f'(k^*) = 0.$$

Note that k^* is consistent with the earlier definition. The following lemma characterizes the policy functions of efficient job search (see Appendix C for a proof):

Lemma 3.3. *Consider any $k \in (0, k^*)$. (i) If $p(k) > 0$, then $\phi(k)$ satisfies $\phi(k) \in (k, k^*)$ and the first-order condition*

$$V'(\phi(k)) = \psi'(\phi(k))\,\frac{\theta(p(k))}{p(k)}. \tag{3.5}$$

Also, ϕ_u satisfies this equation given $p = p_u$.
(ii) Given the efficient choice ϕ, the efficient choice $p(k)$ satisfies:

$$V(\phi(k)) - V(k) - \psi(\phi(k))\,\theta'(p(k)) \leq 0 \text{ and } p(k) \geq 0, \tag{3.6}$$

where the two inequalities hold with complementary slackness. Similarly, given ϕ_u, p_u satisfies this condition with $V(k) = V_u$.
(iii) If $p(k) > 0$, then $V''(\phi) < \psi''(\phi)\,\frac{\theta(p(k))}{p(k)}$ at $\phi = \phi(k)$. If ψ is linear, then $V(\phi)$ is strictly concave at $\phi = \phi(k)$ whenever $p(k) > 0$.

If the efficient job switching rate is strictly positive for a worker at k, then the efficient search target must be interior. The reason is that the marginal benefit of a higher job type is zero at the final job type k^*. The marginal cost of search would exceed the marginal benefit if the search target were set as k^*, which would not be socially efficient. When the efficient search target is interior, it satisfies the first-order condition, because the value function is differentiable. In addition, the net marginal benefit of increasing the job switching rate must be non-positive, and it must be zero if the job switching rate is strictly positive. This complementary slackness condition is (3.6). If $p(k) > 0$, the return on search must be strictly concave in the search target ϕ; otherwise, a small reduction in ϕ accompanied by an increase in p can increase the return on search. This result implies that if the vacancy cost is linear in the job type, then the social value function is strictly concave at the search target whenever $p(k) > 0$.

To characterize efficient job search and upgrading, define k_c and k_T as follows:

$$V'(k_c) = \psi'(k_c)\,\theta'(0) \tag{3.7}$$

$$V(k_T) = V(k_c) - \psi(k_c)\,\theta'(0)\,. \tag{3.8}$$

Recall that $\theta'(0) > 0$. The following proposition holds (see Appendix D for a proof):

Proposition 3.4. *Assuming that $p(k)$ exists and is unique for every k, then (i) and (ii) below hold:*[12]

(i) For all $k \in (0, k^*)$ such that $p(k) > 0$, $\phi(k) \in (k, k^*)$ is unique, $\phi'(k) > 0$, and $p'(k) < 0$.

(ii) $p(k) > 0$ if and only if $k < k_T$. Moreover, $k_T < k_c = \phi(k_T) < k^*$.

In addition, assuming that $\frac{di}{dt}$ exists, then (iii) and (iv) below hold:

(iii) $i(k(t)) > 0$, $\frac{di(k(t))}{dt} < 0$, and $V(k)$ is strictly concave for all $k(t) \in [k_T, k^*)$.

(iv) If ψ is linear, the features in (iii) also hold for all $k(t) \in [\phi_u, k_T]$. If ψ is sufficiently convex, $\frac{di(k(t))}{dt} > 0$ may occur for some $k(t) \in [\phi_u, k_T]$.

[12] Appendix D also gives a sufficient condition for $p(k)$ to be unique for every k in the case where V is concave or linear.

The efficient search target is determined by (3.5), given the efficient job switching rate. The search target is an increasing function of a worker's current job type. That is, the higher a worker's current job type, the higher the target of search for the next job. The efficient job switching rate decreases in a worker's current job type. This is also intuitive. If a worker already has a high job type, further gains from increasing the job type are small. For such job switching, it is socially efficient to create only a small number of vacancies which result in a low job switching rate.

Job switching stops after a finite number of switches. The highest job type at which a worker switches jobs is arbitrarily close to and below k_T. Such a worker searches for the job type k_c, which is strictly below k^*. At and above k_T, there is no job switching and, instead, the job is upgraded continuously toward k^*. OJS stops after a finite number of job switches because of the assumption $\theta'(0) > 0$. Equivalent to $P'(0) < \infty$, this assumption requires that the marginal increase in a worker's matching rate should be bounded even when the number of vacancies in a submarket increases marginally from the initial level 0. Under this assumption, the gain from a job switch must be bounded below by a strictly positive number in order to justify the creation of vacancies for the switch, no matter how small the number of vacancies is. This lower bound on $[V(\phi) - V(k)]$ implies that job switches must stop in a finite number of steps at a job type strictly below the final level. If $\theta'(0) = 0$, contrary to the assumption, it may be possible that both the gain from and the cost of a job switch approach 0 simultaneously as the job type increases toward k^*, in which case job switches can continue indefinitely.[13]

The convergence of the job type to k^* is asymptotic. Specifically, job upgrading is strictly positive and declines over tenure for all $k \in [k_T, k^*)$. To see why these features arise, suppose that job upgrading becomes constant in some interval around tenure t_0 where $k(t_0 + \varepsilon) \in [k_T, k^*)$. Because OJS has stopped after k_T, then $(r + \delta) c'(i) = f'(k) > 0$ at t_0. This implies $i > 0$ at t_0, and so the job type will increase at t_0. Because the marginal productivity of the job type is diminishing, then $(r + \delta) c'(i) > f'(k)$ at tenure slightly

[13] This assumption $\theta'(0) > 0$ is not vacuous: Although it is satisfied by the examples in Example 2.1, it is violated by the Cobb-Douglas matching function.

higher than t_0, say $t_0 + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. This result implies that job upgrading is increasing in tenure at $t_0 + \varepsilon$. Since the marginal cost of upgrading is strictly increasing and the marginal productivity is strictly decreasing, forward induction yields $\mathrm{d}i/\mathrm{d}t > 0$ for all tenure lengths higher than t_0. As a result, job upgrading is bounded above $i(k(t_0))$, which is strictly positive. The job type will surpass k^* in a finite length of time and will keep increasing. Because $f'(k) < 0$ for all $k > k^*$, this path of the job type cannot be socially efficient.

If the vacancy cost is linear in the job type, then job upgrading is positive and declines over tenure also for $k \in [\phi_u, k_T]$. The reason is that the social value function is strictly concave in the job type when the vacancy cost is linear in the job type (see Lemma 3.3). As the marginal value of a higher job type diminishes, the return on upgrading diminishes, and so the efficient amount of upgrading falls.

If the vacancy cost is strictly convex, it is possible that job upgrading can increase in tenure when the job type lies in $[\phi_u, k_T)$. This possibility arises if and only if the social value function is convex and, by (3.3), if and only if

$$[r + \delta + \lambda p(k)] V'(k) > f'(k). \tag{3.9}$$

This condition summarizes the conflict between two opposite forces on efficient job upgrading. One is the gain from front-loading job upgrading. The earlier is a job upgraded, the higher is the joint gain in the present value of the higher productivity. This force exists at all job types and is captured by $f'(k)$ in (3.9). The opposing force is created by job switching. When a worker leaves one job for another, all upgrading occurred in the former match is lost. Thus, job switching increases the effective discount rate on the value of a match, as shown by the presence of $p(k)$ in (3.9). To reduce this loss, the planner can delay upgrading until the job switching rate falls sufficiently. The curvature of the vacancy cost function is an important determinant of which of the two forces dominates because it affects how quickly the job switching rate falls with tenure. The more convex is the vacancy cost function, the lower is the starting job and the higher is the job-switching rate initially. In this case, the social cost of front-loading job upgrading is relatively large,

and so job upgrading can increase over tenure at low job types. On the other hand, if the vacancy cost is linear, then the starting job is high, and so the job-switching rate is low. In this case, the benefit of upgrading the job type early on dominates the opportunity cost, and so job upgrading decreases in tenure at all job types greater than or equal to ϕ_u.

The dynamic pattern of the job type can be summarized as follows. Job upgrading occurs in the entire duration of a worker's employment, but job switches occur only a finite number of times and in the earlier part of employment when the job type is relatively low. When an unemployed worker becomes employed, the job type is the lowest and the job switching rate the highest among all employed workers. If such a worker stays employed, the job type is also upgraded smoothly. As the job type increases because of either job switching or upgrading, the target job type of further search rises and the job switching rate falls. Job upgrading can either increase or decrease in the early stage of employment, depending on the convexity of the vacancy cost function. After a finite number of job switches, job switching stops but job upgrading continues. Job upgrading eventually declines over tenure, and so the job type increases asymptotically toward the level k^*. If the worker ever becomes unemployed, this process starts anew.

3.2. Schedule of Job Types and Effective Tenure

The dynamic pattern of the job type can be illustrated by tracing OJS and job upgrading over time. Consider a hypothetical worker who will never succeed in moving to another job despite OJS. Denote the worker's tenure in the firm as t and use the subscript b to indicate this baseline worker. Immediately after exiting unemployment into a job, the worker's job type is $k(0) = \phi_u$. The path of efficient job upgrading is $\{\hat{\imath}_b(t) : t \geq 0\}$, and the path of efficient job search is $\{(\hat{\phi}_b(t), \hat{p}_b(t)) : t \geq 0\}$, where $\hat{}$ distinguishes the functions of t from the related functions of k.[14] There exists a positive and finite T such that the job switching rate is positive if and only if tenure is less than T; that is, $\hat{p}_b(t) > 0$ if and only if $t < T$. The job switching rate declines over tenure, provided that this rate is positive.

[14] Although the baseline worker will never succeed in moving to another firm, it is still necessary to describe where the worker should search for the next job at every tenure t.

Although the hypothetical worker can have a positive rate of moving to another job, the worker never gets such a lucky draw.

The baseline schedule is valid for the baseline worker not only from the start of employment, but also from any tenure length. If the worker has stayed in the firm for a length of time t_0, then the efficient allocation will be given by the "tail" of the baseline schedule from t_0 onward, which is $\{(\hat{\imath}_b(t), \hat{\phi}_b(t), \hat{p}_b(t)) : t \geq t_0\}$. This result comes from the feature that the efficient allocation is time consistent.

The baseline schedule is also useful for describing the path of efficient upgrading and search for any arbitrary worker. Any job type k possibly reached through a sequence of job switches and job upgrading is equivalent to tenure $\tau(k)$ on the baseline schedule, where $\tau(k)$ solves:

$$k = k_b(0) + \int_0^{\tau(k)} \hat{\imath}_b(t)\,\mathrm{d}t.$$

Let me refer to $\tau(k)$ as the effective tenure of a worker at a job type k. Given k, if the worker will stay in the current job forever until exogenous separation, then the efficient allocation for the worker in the future is:

$$\{(\hat{\imath}(t), \hat{\phi}(t), \hat{p}(t)) : t \geq 0\} = \{(\hat{\imath}_b(t+\tau), \hat{\phi}_b(t+\tau), \hat{p}_b(t+\tau)) : t \geq 0\},$$

where $\tau = \tau(k)$. More generally, when a worker switches from k to $\phi_b(k)$ through OJS, the switch is equivalent to keeping the job in the firm and increasing the effective tenure from $\tau(k)$ to $\tau(\phi_b(k))$. After the switch, the efficient allocation for the worker is the baseline schedule from tenure $\tau(\phi_b(k))$ onward, until the worker moves again. Because the efficient schedule at any arbitrary job type can be described by using the baseline schedule, I will focus on the baseline schedule and omit the subscript b.

Figure 1 depicts the baseline schedule of the job type as the solid curve and a sample path of job switches as the dashed steps. After a worker moves out of unemployment and into employment, the job type increases along the baseline curve as the job is upgraded gradually. At tenure t_1, the worker succeeds in moving to a new firm with a job type $\phi(k(t_1))$. This move is equivalent to increasing tenure from t_1 to $t_2 = \tau(\phi(k(t_1)))$ on the

baseline schedule, where the function $\tau(.)$ is defined above. This move takes the worker from the job type $k(t_1)$ to $k(t_2) = \phi(k(t_1))$. After the move, the job type for the worker increases along the baseline curve until the worker succeeds again in moving to another job. The maximum effective tenure length below which a worker moves to another job at a positive rate is $T = \tau(k_T)$, where k_T is defined by (3.8). For all effective tenure lengths greater than or equal to T, the only cause of an increase in the job type is job upgrading.

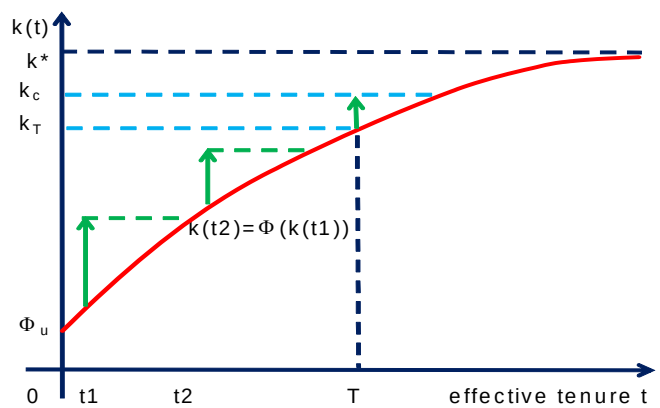

Figure 1. Efficient job upgrading and switching

4. Distribution of Workers

This section characterizes the distribution of workers in the steady state. Although this distribution does not affect the efficient allocation, it is necessary for calculating aggregate statistics such as the mean and the standard deviation of net output. Let n_u be the measure of unemployed workers and $n_e = 1 - n_u$ the measure of employed workers. Because an unemployed worker finds a job at the rate p_u and an employed worker separates from a job into unemployment at the rate δ, the measure of unemployed workers remains constant over time if and only if $n_u p_u = n_e \delta$. This equation solves

$$n_u = \frac{\delta}{\delta + p_u}.$$

Let t denote a worker's actual tenure. In contrast to the effective tenure, the actual tenure is reset to zero whenever a worker switches a job. Denote the cumulative distribution

of employed workers over (k,t) as $\Omega(k,t)$, and the corresponding density function as $\omega(k,t)$. For any $k \geq \phi_u$ and $t \geq 0$, the measure of workers with $\left(\tilde{k},\tilde{t}\right) \in (k,k^*] \times (t,\infty)$ is

$$\bar{\Omega}(k,t) \equiv \int_k^{k^*} \int_t^{\infty} \omega\left(\tilde{k},\tilde{t}\right) \mathrm{d}\tilde{t}\mathrm{d}\tilde{k}. \tag{4.1}$$

This is less than $1-\Omega(k,t)$, because there are workers with $(k,k^*]$ and $[0,t]$ and workers with $[\phi_u,k]$ and (t,∞). Denote the partial derivatives of $\bar{\Omega}$ as

$$\begin{array}{l} \bar{\Omega}_k(k,t) \equiv -\int_t^{\infty} \omega\left(k,\tilde{t}\right) \mathrm{d}\tilde{t} \\ \bar{\Omega}_t(k,t) \equiv -\int_k^{k^*} \omega\left(\tilde{k},t\right) \mathrm{d}\tilde{k}. \end{array} \tag{4.2}$$

The amount $-\bar{\Omega}_k(k,t)$ is the measure of workers employed at the job type k over all tenure greater than t, and the amount $-\bar{\Omega}_t(k,t)$ is the measure of workers employed at tenure t over all job types greater than k. Clearly, $\bar{\Omega}_{kt}(k,t) = \bar{\Omega}_{tk}(k,t) = \omega(k,t)$.

Let $G(k)$ be the cumulative distribution of employed workers with job types at or lower than k summed over all lengths of actual tenure, and let $g(k)$ be the corresponding density. Similarly, let $G_t(t)$ be the cumulative distribution of employed workers with tenure less than or equal to t, summed over all job types, and let $g_t(t)$ be the corresponding density function. The distribution of employed workers is characterized by the following proposition (see Appendix E for a proof):

Proposition 4.1. *Assume that the joint density of employed workers, $\omega(k,t)$, exists for all $(k,t) \in [\phi_u,k^*] \times [0,\infty)$. Then $\omega\left(\tilde{k},t\right) = 0$ for all $\tilde{k} < k(t)$ and $\omega\left(k,\tilde{t}\right) = 0$ for all $\tilde{t} > t(k)$, where $t(k)$ is the inverse function of $k(t)$. For all (k,t), the joint distribution of employed workers obeys:*

$$i(k)\,\bar{\Omega}_k(k,t) + \bar{\Omega}_t(k,t) = \int_k^{k^*} \left[\delta + \lambda p\left(\tilde{k}\right)\right] \bar{\Omega}_k\left(\tilde{k},t\right) d\tilde{k}. \tag{4.3}$$

For all $k \in [\phi_u,k^*]$, the marginal distribution of employed workers over k obeys:

$$i(k)\,g(k) = \delta\left[1-G(k)\right] + \bar{\Omega}_t(k,0) + \int_k^{k^*} \lambda p\left(\tilde{k}\right) g\left(\tilde{k}\right) d\tilde{k}. \tag{4.4}$$

For all $t \geq 0$, the marginal distribution of employed workers over t obeys:

$$g_t(t) = \delta\left[1-G_t(t)\right] - \int_{\phi_u}^{k^*} \lambda p\left(\tilde{k}\right) \bar{\Omega}_k\left(\tilde{k},t\right) d\tilde{k}. \tag{4.5}$$

The marginal densities have the following properties: (i) $g_t(t)$ is strictly decreasing and differentiable for all $t \in [0, \infty)$; (ii) $g(k)$ is differentiable for all $k \in [\phi_u, k^*)$; (iii) For all $k \in (k_c, k^*)$, $\frac{d}{dk}[i(k)g(k)] = -\delta g(k) < 0$, and $g(k)$ is decreasing if and only if $f'(k) < \delta i c''(i) + (r+\delta)c'(i)$ where $i = i(k)$.

If a worker never switches jobs, the job type reached at tenure t by job upgrading is $k(t)$. Because the job type can also increase as a result of OJS, no worker at tenure t has a job type lower than $k(t)$. That is, $\omega\left(\tilde{k}, t\right) = 0$ for all $\tilde{k} < k(t)$. Similarly, $\omega\left(k, \tilde{t}\right) = 0$ for all $\tilde{t} > t(k)$, where $t(k)$ is the inverse function of $k(t)$.

Equation (4.3) is the steady state equation of the measure of workers with $\left(\tilde{k}, \tilde{t}\right) \in (k, k^*] \times (t, \infty)$. The amount $-i(k)\bar{\Omega}_k(k,t)\Delta$ is the inflow of workers employed slightly below k whose jobs are upgraded above k in a small time interval Δ. The amount $-\bar{\Omega}_t(k,t)\Delta$ is the inflow of workers with job types above k whose tenure increases above t in a small time interval Δ. The outflow is $-\Delta$ times the right-hand side of (4.3), which is generated by exogenous and endogenous separation from the jobs.

The marginal density of employed workers over job types obeys (4.4). To explain this equation, it is useful to rewrite it to involve only the marginal distribution of k. Note that the measure of workers with tenure 0 and job types above k is $-\bar{\Omega}_t(k,0) > 0$. These workers are the ones who just moved to job types above k, whose tenure is reset to 0. Of these movers, a subgroup of workers were employed at or above k before the job switch, and their measure is given by the integral on the right-hand side of (4.4). Thus, the sum of the last two terms in (4.4) is equal to the negative of the measure of the workers who just moved to job types at or above k from jobs below k. Calculating this measure, I can rewrite (4.4) solely in terms of the distribution of k as

$$i(k)g(k) = \delta[1 - G(k)] - \int_{\max\{\phi^{-1}(k), \phi_u\}}^{k} \lambda p\left(\tilde{k}\right) \mathrm{d}G\left(\tilde{k}\right). \tag{4.6}$$

The lower bound on the integral uses the fact that no worker is employed below ϕ_u.

Properties (i)-(iii) in Proposition 4.1 are intuitive. Because of job upgrading, all employed workers at any state (k,t) exit the state in an arbitrarily small interval of time.

Thus, the distribution of employed workers has no mass point and the marginal densities are differentiable for all $t < \infty$ and $k < k^*$, as described by (i) and (ii). The only possible exception is the final state $(k, t) = (k^*, \infty)$, because job upgrading stops at $k = k^*$. To explain (iii), note that for all $k \in (k_c, k^*)$, job upgrading is the only process by which an employed worker can reach a higher job type. However, a worker at such a job type can exit the job type by either job upgrading or exogenous separation. For the density of workers at such a job type to be stationary, the rate of the outflow through job upgrading must decline over k in order to balance the exogenous separation. That is, $\frac{\mathrm{d}}{\mathrm{d}k}\left[i(k)\, g(k)\right] = -\delta g(k) < 0$. Since $i(k)$ is also decreasing at such job types, $g(k)$ is decreasing if and only if $-i'(k) < \delta$, which is equivalent to $f'(k) < \delta i c''(i) + (r + \delta)\, c'(i)$ where $i = i(k)$. Therefore, the distribution of employed workers can be decreasing at high job types when the marginal productivity diminishes sufficiently quickly.

5. Quantitative Analysis

5.1. Calibration

To calibrate the model, I use the following functional forms of gross output, $f_a(k)$, the vacancy cost, $\psi(k)$, the upgrading cost, $c(i)$, and the tightness, $\theta(p)$:

$$
\begin{aligned}
&f_a(k) = f_0 k^{\alpha}, \quad f_0 > 0,\ \alpha \in (0,1);\\
&\psi(k) = \psi_0 k^{\psi_1}, \quad \psi_0 > 0,\ \psi_1 \geq 1;\\
&c(i) = c_1 i^2, \quad c_1 > 0;\\
&\theta(p) = p_0 \left[p^{-\rho} - 1\right]^{-1/\rho}, \quad p_0, \rho \in (0, \infty).
\end{aligned}
$$

Net output is $f(k) = f_a(k) - rk$. The parameter ψ_1 governs the convexity of the vacancy cost in the job type. Since the relative convexity of the vacancy cost to the upgrading cost is important, I set the latter as a quadratic function. The function $\theta(p)$ comes from the generalized telephone matching function in Example 2.1. The model is calibrated monthly. Table 1 lists the parameters, their values and the calibration targets.

Several aspects of the calibration are worth noting. First, the capital share in output is an increasing function of the job type. Since the capital share is equal to α at the final job type k^*, it is smaller than α at all job types lower than k^*. With $\alpha = 0.55$ and other

calibration targets, the capital share at the average employed job type is 0.34, which is close to the value used in macro models.

Second, an unemployed worker's monthly job-finding rate is set at 0.374 and an unemployed worker's home production at 31% of the *highest* net output in the market. The target on the job-finding rate is equivalent to setting the unemployment rate to 0.065, and the target on an unemployed worker's home production amounts to 40% of average net output of employed workers. These two targets are the same as in Hornstein et al. (2011). The value of home production is also close to the one used by Shimer (2005) in the study of the labor market in the business cycle, but lower than the one used by Hagedorn and Manovskii (2008). As shown by Hornstein et al. (2011), setting home production as 0.4 of the average market production already puts a strong discipline on how much frictional wage inequality that a search model can generate. If home production is set to a much higher value, say 0.85, the model is a non-starter to generate frictional dispersion.

Table 1. Identification of the parameters

Parameter	Value	Target
r	4.15×10^{-3}	quarterly interest rate $= 0.0125$
α	0.55	capital share at k^* is 0.55
f_0	0.0599	normalize $k^* = 100$
δ	0.026	monthly EU rate in CPS
λ	1	benchmark
ρ	0.5	elasticity $\frac{\mathrm{d}\ln p_u}{\mathrm{d}\ln \theta_u} = 0.39$
k_u	3.5279	$\frac{f(k_u)}{f(k^*)} = 0.31$
ψ_0	7.37×10^{-8}	$\frac{\psi(\phi_u)}{f(\phi_u)} = 0.10$
ψ_1	6.4101	(ψ_1, p_0, c_1) minimize the distance
p_0	1.1151	between $\left(\frac{u}{0.065}, \frac{f(\phi_u)}{0.42 f(k^*)}\right)$ and
c_1	0.0368	$(1, 1)$

Third, the value $\rho = 0.5$ implies that the elasticity of the job-finding rate with respect to the tightness of the submarket for unemployed workers is $\varepsilon_u \equiv \frac{\mathrm{d}\ln p_u}{\mathrm{d}\ln \theta(p_u)} = 0.39$. There is no empirical estimate of this elasticity. Instead, the estimate is on the elasticity of an unemployed worker's job-finding rate with respect to the tightness $\bar{\theta} = \bar{v}/u$, where $\bar{v}$ is the total number of vacancies. This estimate is around 0.27 (see Shimer, 2005), although the value 0.5 is also used sometimes (e.g., Petrongolo and Pissarides, 2001). Menzio and Shi

(2011) explain that the empirical estimate is lower than ε_u because there is OJS and search is directed into submarkets. Their calibration yields $\varepsilon_u = 0.60$. The choice of ρ, together with the target on the job-finding rate $p_u = 0.374$, yields an elasticity between these two numbers in the literature.

Fourth, at the lowest employed job type ϕ_u, net output is set to be 42% of the highest output and the vacancy cost to 10% of net output. The target on $f(\phi_u)$ is equivalent to targeting the capital share at ϕ_u to 0.163. Together with $\alpha = 0.55$, this target also yields the capital share at the average employed job type to be 0.34, which is reasonable. The vacancy cost is comparable to the ones in Silva and Toledo (2009).[15] Note that these targets on $f(\phi_u)$ and $\psi(\phi_u)$ require $\phi_u = 6.679$ and $\psi_0(\phi_u)^{\psi_1} = 0.0143$, and so $\psi_0 = 0.0143 \times 6.679^{-\psi_1}$.

The parameters, (ψ_1, p_0, c_1), are chosen to minimize the distance between the ratios of $\left(u, \frac{f(\phi_u)}{f(k^*)}\right)$ to their targets $(0.065, 0.42)$ from $(1, 1)$. The resulting values of (ψ_1, p_0, c_1) yield $u = 0.0649$ and $f(\phi_u)/f(k^*) = 0.424$, which are close to the targets. The implied value of ϕ_u is 6.827. Appendix F provides further details of the calibration, the computation of the value and policy functions, and the simulation of the steady state distribution.

5.2. Worker Mobility and Job Upgrading

In the computed model, job switching stops at $k_T = 8.087$ and the highest job type that can be reached by job switching is $k_c = 9.232$. Although these job types seem low relative to the highest job type $k^* = 100$, net output levels at these job types are significant. As fractions of the highest net output, net output is 42.4% at the lowest job type, 45.8% at the job type k_T, and 48.7% at the job type k_c. Thus, OJS can yield 15% of gain in net output. Figure 2 depicts the policy functions of the efficient allocation.[16] In order to

[15] They estimate that it takes 17.2 days on average to fill a vacancy and, during this time, the average time spent on recruiting by hiring supervisors is 13.5 hours. This resource spent in hiring amounts to 13% of the monthly wage of a newly hired worker. I approximate this vacancy cost as a fraction of a new worker's output instead of wages.

[16] In the computation, the search problem in (2.3) and the job upgrading problem in (2.4) are solved with grid search. The resulting policy functions $(\phi(k), p(k), i(k))$ are single-valued for each k, thus validating the assumption in Proposition 3.4.

exhibit the search sequence clearly, the upper panel plots $f(\phi(k))$ against $f(k)$, instead of the function $\phi(k)$. The solid line in the upper panel in Figure 2 is net output at the search target, $f(\phi(k))$, as a function of a worker's current net output. Confirming Proposition 3.4, this function is strictly increasing for all $k < k_T$, and it coincides with $f(k)$ for all $k > k_T$. The flat segment of the function $f(\phi(k))$ illustrates the features that a worker at k_T searches for k_c and that job search stops for $k > k_T$.[17] For all $k > k_c$, $f(\phi(k)) = f(k)$.

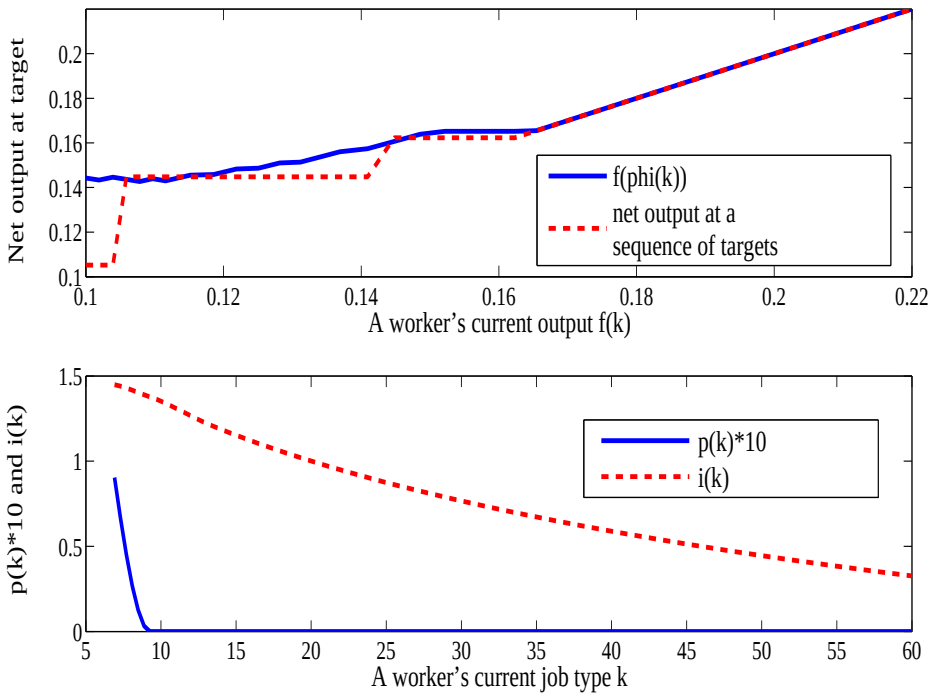

Figure 2. The policy functions $f(\phi(k)), p(k)$ and $i(k)$

The dashed steps in the upper panel in Figure 2 are the sequence of net output levels searched by a worker who starts in unemployment and succeeds in every job search. The first step is an unemployed worker's home production, $f(k_u) = 0.105$, the second step is net output of the lowest job, $f(\phi_u) = 0.144$, and the third step is net output of the next job, $f(\phi(\phi_u)) = 0.157$. Since $f(k_T) = 0.156$, then $\phi(\phi_u) > k_T$. That is, OJS stops after one job switch. Note that the depicted job path occurs with probability zero, because the probability of getting another job immediately after a job switch is zero. Almost surely,

[17] The flat segment can start at any $f(k) \in [f(\phi^{-1}(k_T)), f(k_T))$ and end at the corresponding target $f(\phi(k)) \in [f(k_T), f(k_c))$. These flat segments are not depicted.

a worker will stay with a job for a positive length of time before succeeding in getting another job. During this time, the worker's job type is upgraded. If the tenure in the job is sufficiently short, the worker will search for another job and experience a job switch with positive probability. If the tenure in the job is so long that the job type is upgraded beyond k_T, the worker will stop searching. If a worker becomes unemployed, the worker will start the process of job search and job upgrading anew.

The lower panel in Figure 2 depicts the policy functions of the job switching rate, $p(k)$, and the upgrading intensity, $i(k)$. The job switching rate is multiplied by 10 in order to fit into the figure. This function is strictly decreasing for all $k < k_T$, reaches zero at $k_T = 8.087$ and stays at zero thereafter. The job upgrading rate is positive for all $k < k^*$ and declines over the job type. The possibility described in Proposition 3.4 that the job upgrading rate can increase at low job types does not arise with the particular parameter values. The reason is that the job switching rate is small at all job types. Since an upgraded job faces only a low probability of being destroyed by job switching, the gain from front-loading job upgrading dominates the gain from delaying job upgrading.

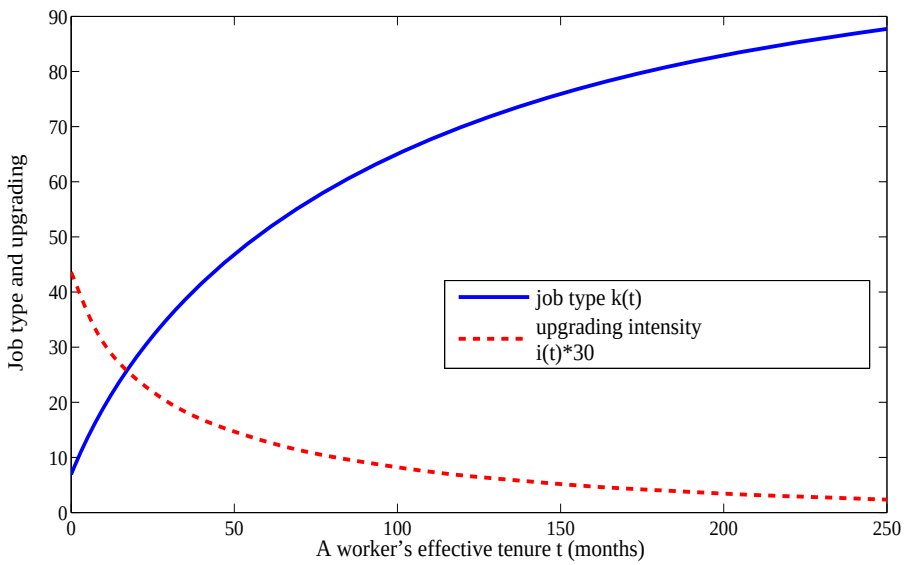

Figure 3. The job type and upgrading as functions of effective tenure

Figure 3 depicts the job type and the upgrading rate as functions of a worker's effective tenure. The solid line is the baseline schedule of job types. As described earlier, this is

the path of the job types of a worker who never succeeds in OJS. The dashed line is the upgrading intensity for such a worker, multiplied by 30. As the effective tenure increases, the job type increases and approaches the final level k^* asymptotically.

5.3. Worker Distribution and Frictional Dispersion

Using the policy functions, I simulate the economy to obtain the steady state. For the simulation, time is discretized into small intervals $\mathrm{d}t$ and the job type is discretized according to the function $k(t)$ (see Appendix F). The top panel in Figure 4 depicts the frequency function of unemployed workers over the unemployment duration, where duration is expressed as $\frac{n}{3}$ quarters with $n = 0, 1, 2, \ldots$. The vertical bar on a number $\frac{n}{3}$ is the frequency of workers whose unemployment duration lies in $[\frac{n}{3}, \frac{n+1}{3})$ quarters. The frequency of unemployed workers declines sharply over the unemployment duration. About 80 percent of unemployed workers become employed in one quarter.

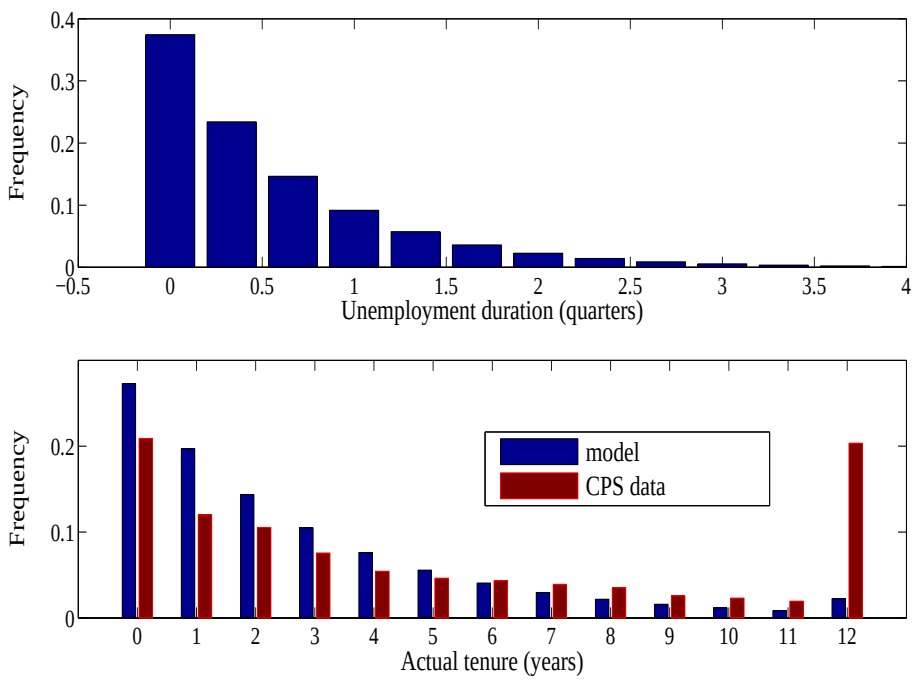

Figure 4. Frequency of unemployment duration and employed tenure

The lower panel in Figure 4 depicts the frequency function of employed workers over actual tenure (in years) in the model and contrasts it with the CPS data documented by Diebold et al. (1997). A bar over a tenure length $t < 12$ is the frequency of workers whose

actual tenure lies in $[t, t+1)$ years. All employed workers with tenure greater than or equal to 12 years are lumped together at $t = 12$. The frequency of workers decreases in tenure. In the model, 27% of employed workers have tenure less than one year, but only 20% of employed workers have tenure between one and two years. Exogenous separation contributes to this large drop in the frequency of employed workers from less than one year to above one year of tenure, but it is not the only cause. Another cause is that tenure is reset to zero when a worker switches jobs, which increases the flow of workers into tenure less than one year. Since there is no such inflow of workers from longer tenure into two years of tenure, there is a large drop in the measure of workers when actual tenure increases from less than one year to above one year. Notice that the frequency of tenure in the CPS data also has a large drop from $t = 0$ to $t = 1$. However, in contrast to the model, the CPS data shows that the frequency of employed workers at short tenure lengths is smaller, and the frequency at medium or long tenure lengths declines less sharply, than in the model. A possible cause of these discrepancies is that workers or jobs in the data are heterogeneous in the exogenous separation rate, where workers selected into longer tenure are less likely to separate into unemployment (see Jarosch, 2015).

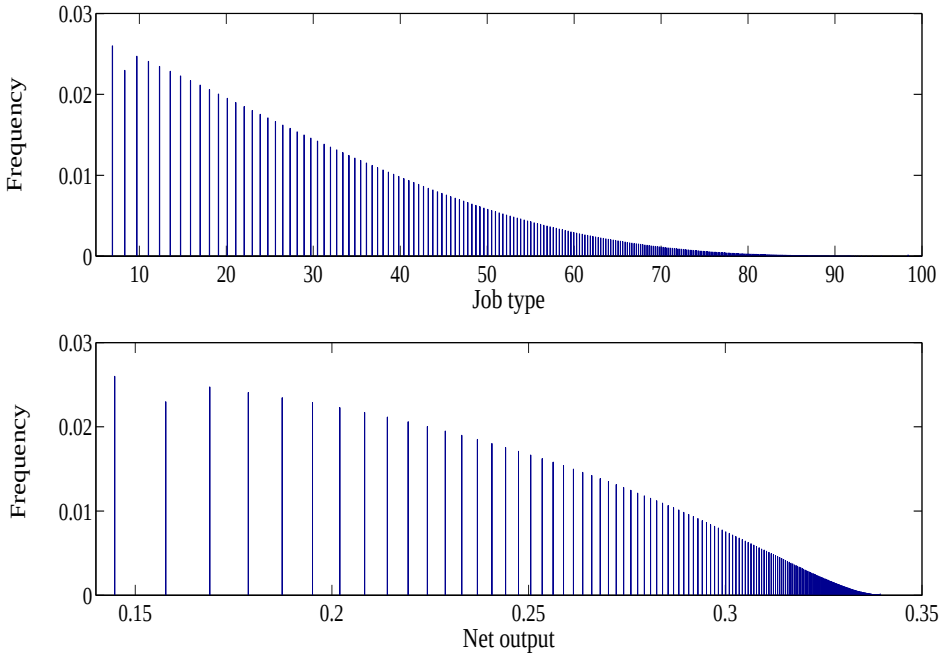

Figure 5. Frequency of employed workers' job types and net output

Figure 5 depicts the distribution of workers over job types (the upper panel) and net output (the lower panel). Since the job type is discretized according to the function $k(t)$ given the grid of t, the frequency is positive only at a finite number of job types. Because the slope of $k(t)$ declines as t increases (see Figure 3), the gap between two adjacent job types with positive frequency becomes smaller when the job type increases. The frequency function of workers decreases in the job type. Relative to the frequency of job types, the frequency of net output (in the lower panel) has wider gaps at low levels of net output and is bunched more closely at high levels of net output. This pattern is caused by concavity of the net output function $f(k)$, which makes the gap between two adjacent levels of net output with positive frequency shrink quickly as net output increases.

Table 2 lists the statistics of the job type, gross output $f_a(k)$ and net output $f(k)$. The dispersion in each of the three variables is large. The mean-min ratio of the job type is 4.98. This translates into a mean-min ratio of 2.34 in gross output and 1.81 in net output. I consider three other measures of dispersion: the coefficient of variation, the 90-10 percentile ratio, and the 50-10 percentile ratio. The coefficient of variation is 0.53 in job types, 0.31 in gross output and 0.20 in net output. The 90-10 percentile ratio is 4.97 in the job type, 2.42 in gross output and 1.72 in net output. The 50-10 percentile ratio is 2.65 in job types, 1.71 in gross output and 1.45 in net output. All these numbers indicate significant frictional dispersion. The large 90-10 and 50-10 percentile ratios show that the large frictional dispersion is not caused by a small measure of workers employed at the lowest net output. Moreover, the mean-min ratio is significantly larger than the 50-10 percentile ratio, which implies that net output is significantly lower at the lowest employed level than at the 10 percentile.

Table 2. Statistics of job types, gross and net output

	mean	std	$\frac{\text{std}}{\text{mean}}$	min	max	$\frac{90}{10}$	$\frac{50}{10}$	$\frac{\text{mean}}{\text{min}}$
k	34.86	18.46	0.53	6.999	99.67	4.97	2.65	4.98
$f_a(k)$	0.407	0.126	0.31	0.174	0.753	2.42	1.71	2.34
$f(k)$	0.262	0.051	0.20	0.145	0.339	1.72	1.45	1.81

To relate the dispersion in Table 2 to observed dispersion in wage rates requires an assumption on how wages are determined. A reasonable assumption is that the wage rate

is a constant fraction of net output. Under this assumption, the mean-min ratio, the coefficient of variation, the 50/10 percentile ratio and the 90/10 percentile ratio are the same in wages as in net output. In particular, the mean-min ratio in the wage rate is 1.81. Thus, the model can capture most of wage dispersion among homogeneous workers observed in the data (see Hornstein, et al., 2011). After quantifying several versions of search models, those authors find that the models can generate no more than 1.05 as the mean-min ratio in wage rates, where the empirical estimate of the mean-min ratio is between 1.7 and 2.

As explained well by Hornstein et al. (2011), wage dispersion is small in most search models because of two empirical restrictions: The monthly job-finding rate is relatively high and the value of an unemployed workers' home production is significantly positive. For the job-finding rate to be high, an unemployed worker must be eager to take up a job rather than continue to search. That is, the option value of staying unemployed must be low, which consists of the value of home production and the future value of obtaining high offers after search. Since the value of home production is significantly positive, the option value of continuing to search for higher offers must be low in order to explain the high job-finding rate. This implies that dispersion of wages is small. To generate significant wage dispersion and an empirically reasonable job-finding rate, Hornstein et al. (2011) find that most search models would need the value of home production to be negative.

The calibration above satisfies both empirical restrictions in Hornstein et al. (2011). Thus, the large wage dispersion in the current model must be caused by the new ingredients in the model: the convex vacancy cost, job upgrading, and one the job search. Section 5.5 will investigate the quantitative role of each of these ingredients. It is important to repeat that net output dispersion in this model is entirely caused by search frictions. In particular, the role of job upgrading should be attributed to search frictions. If the vacancy cost were small or independent of the job type, job upgrading would not be necessary and, hence, would not contribute to net output dispersion in the efficient allocation.

5.4. Return to Tenure and the Cost of Job Loss

The endogenous job ladder also sheds new light on two issues that have been examined extensively in the literature: the return to tenure and the cost of job loss. On both issues, empirical work has struggled with the difficulty of controlling for unobserved heterogeneity among workers and endogenous selection of workers into tenure or job loss. The current model eliminates this difficulty by assuming that all workers are identical. Thus, the model is able to isolate the role of the endogenous job ladder induced by search frictions. Moreover, by construction of the model, both the return to tenure and the cost of job loss are associated with the socially efficient allocation.

On the return to tenure, Mincer and Jovanovic (1981) have found a strong relationship between tenure and wage rates in the estimation with cross sectional and time series data. This estimate has been questioned for its robustness to factors such as unobserved heterogeneity among workers and endogeneity of tenure. Using instrumental variables to partially control for these factors, some authors (e.g., Altonji and Shakotko, 1987) have revised the return to tenure downward significantly while others have confirmed the large return to tenure (e.g., Topel, 1991). In particular, using a two-stage estimation procedure, Topel (1991) finds that increasing tenure by 10 years increases wages by 0.246 log points. Using the model simulated data, I run the OLS regression of log net output on years of tenure. The coefficient of tenure in the regression is 0.05, which is larger than in Topel (1991). More precisely, I measure the return to t years of tenure by $[\ln(f_t) - \ln(f(\phi_u))]$, where $\ln(f_t)$ is the average of log net output among workers with tenure in $[t, t+1)$ years in the model simulated data. This return to tenure is depicted in the upper panel of Figure 6. The majority of the return occurs in the first five years of tenure.

On the cost of job loss, the empirical literature calculates a displaced worker's loss in wages or earnings relative to a comparison worker who was not displaced. Jacobson et al. (1993) find that a displaced worker in Pennsylvania in the early 1980s had 50% lower earnings in the near term and 30% lower earnings five years after displacement. Although subsequent studies using different datasets have obtained smaller estimates of the cost of

job loss, these estimates remain large (see Davis and von Wachter, 2011, for a survey). In particular, Davis and von Wachter (2011) estimate earnings losses of displaced workers relative to workers in the control group who were not displaced in the year or next 2 years. They find that the earnings loss is 39% in the first year for displacements that occur in recessions and 25% for displacements that occur in expansions. The losses are also long lasting, ranging from 15 to 20% from 10 to 20 years out for displacement that occur in recessions and about 10% for those that occur in expansions.

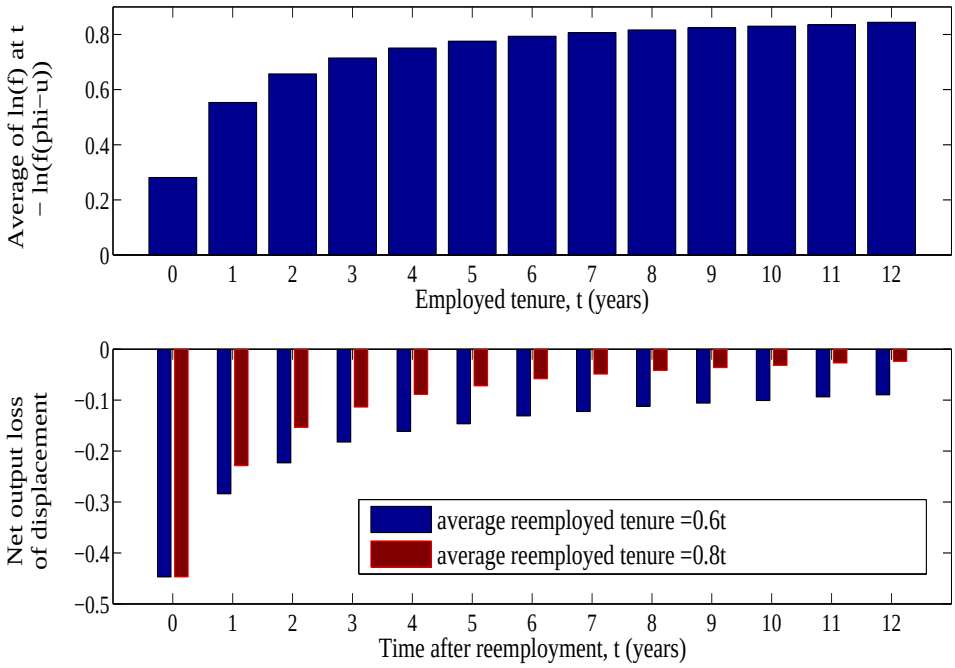

Figure 6. Return to tenure and net output loss due to displacement

I calculate the cost of job loss as the percentage loss in net output caused by a one-time displacement. The study group is a worker who is hit by the exogenous separation shock, and the control group is a worker who is continuously employed. Since the displacement is one time, the displaced worker under the study is assumed to stay employed forever once the Worker regains employment. The output loss is traced out over the years after reemployment, $t = 0, 1, 2, ...$ For the control group, net output at each t is set to average net output of employed workers whose tenure is at least t. For the study group, $t = 0$ is the time at which the displaced worker just regained employment. At this time, the worker's net output is equal to $f(\phi_u)$, which is lower than net output of employed workers

with tenure 0. At any $t \geq 1$, the reemployed worker's tenure is smaller than t, because job-to-job transitions reset tenure to 0. I approximate the reemployed worker's net output at $t \geq 1$ by average net output of employed workers with tenure in the interval $[at, t)$, where a is set to 0.2 and 0.6 in turn. The two values of a imply that a worker's tenure after t years of reemployment is approximately $0.6t$ and $0.8t$, respectively.

The lower panel of Figure 6 depicts the percentage loss in net output caused by a one-time displacement, traced out over time after a displaced worker is reemployed. The loss is equal to 45% at reemployment ($t = 0$), which is equal to $\left(\frac{\text{min}}{\text{mean}} - 1\right)$ in net output. With $a = 0.2$, the loss is persistent. The loss is equal to 18% five years after reemployment and 12% ten years after reemployment. With $a = 0.6$, the loss is less persistent: it is equal to 10% five years after reemployment and 3.5% ten years after reemployment.

Caution is needed for interpreting the above findings on the return to tenure and the cost of job loss. The model assumes that workers at all tenure experience the exogenous separation shock at the same rate δ, which is unlikely to be true in the data. Rather, workers with shorter tenure are more likely to separate into unemployment than workers with longer tenure. If the model incorporates this realistic feature, then an increase in tenure from a short tenure is likely to increase net output by a smaller amount than that depicted in the upper panel of Figure 6. When calculating the cost of job loss, the control group should sample workers with short tenure more heavily because they are closer to a displaced worker. As a result, the cost of job loss is likely to be smaller in the near time after a displaced worker is reemployed than that depicted in the lower panel of Figure 6. Nevertheless, the cost of job loss can still be persistent.

5.5. Counterfactual Experiments

The convexity of the vacancy cost, job upgrading and OJS all increase net output dispersion. The convexity of the vacancy cost makes it socially efficient to set the starting job low. Job upgrading and OJS induce unemployed workers to accept a low job type as the starting job and provide net output growth during employment. To investigate the quantitative importance of the three ingredients, I conduct counterfactual experiments by

eliminating each of these ingredients in turn. Table 3 lists the recalibrated parameters, the values of (p_u, ϕ_u) and dispersion in net output.

In the experiment on the convexity of the vacancy cost, I set $\psi_1 = 1$ so that the vacancy cost is linear in the job type. The model is recalibrated to identify (ψ_0, p_0, c_1). The parameter ψ_0 needs to be recalibrated in order to make the experiment non-trivial. If ψ_0 is kept at the baseline value, the reduction in ψ_1 from the baseline value to 1 will reduce the vacancy cost by so much that net output dispersion will be minuscule. I set ψ_0 so that the vacancy cost at the lowest employed job type in the baseline economy is the same as under $\psi_1 = 1$. The parameters (p_0, c_1) are recalibrated to minimize the distance between the target on p_u and the model. The target on $f(\phi_u)/f(k^*)$ is dropped because the lowest employed job type is an important channel through which the convexity of the vacancy cost affects net output dispersion. The recalibrated values of $(\psi_1, p_0, c_1, \psi_0)$ are listed in Table 3, while other parameters are kept at their baseline values.

Table 3. Counterfactual experiments

	ψ_1	p_0	c_1	ψ_0	u	$\frac{f(\phi_u)}{f(k^*)}$	$\frac{50}{10}$	$\frac{\text{mean}}{\text{min}}$ in f
baseline	6.410	1.115	0.037	7.37×10^{-8}	0.0649	0.424	1.45	1.81
$\psi_1 = 1$	1	0.372	0.368	0.0026	0.0694	0.998	1.0001	1.0001
$c_1 = 10^6$	5.128	0.372	10^6	7.37×10^{-8}	0.0654	0.580	1.37	1.37
$\lambda = 0$	7.772	2.071	0.040	7.37×10^{-8}	0.0656	0.341	1.53	2.12

With linear vacancy costs, the efficient allocation calls for the starting job type to be very high. Net output at the starting job is 99.8% of that at the final job type. With such a high starting job, there is no gain to creating vacancies for OJS and very little room for job upgrading. The 50-10 percentile ratio and the mean/min ratio in output are close to 1. Thus, the convexity of the vacancy cost in the job type is critical for the model to generate realistic frictional dispersion in net output.

In the counterfactual experiment on job upgrading, I set $c_1 = 10^6$ so that job upgrading is almost impossible. The parameters (ψ_1, p_0) are recalibrated to minimize the distance between the model and the target on u. Other parameters remain at their baseline values in Table 1. The target on $f(\phi_u)/f(k^*)$ is dropped again because one purpose of the experiment is to check how the the absence of job upgrading affects the lowest employed

job type. When job upgrading is almost impossible, the efficient allocation features a higher starting job type than in the baseline economy. Net output at the starting job is 58% of the highest net output. With this high starting job type, the vacancy cost must be less convex than in the baseline economy in order to yield a similar unemployment rate, as indicated by the lower value of ψ_1. As a result, dispersion in net output in the economy without job upgrading is lower than in the baseline economy. The 50-10 percentile ratio and the mean-min ratio in net output are 1.37.

Frictional dispersion is still large when the model with only OJS is compared with a similar model by Burdett and Mortensen (1998). After calibrating the latter model, Hornstein et al. (2011) find that the mean-min ratio in wages is between 1.16 and 1.27 if the job-to-job transition rate is not unrealistically high. In the above model with only OJS, the monthly job-to-job transition rate is about 2%, which is realistic. The significantly higher mean-min ratio can be traced to the convex vacancy cost. As explained above, a higher convexity of the vacancy cost in the job type makes it socially efficient to reduce the starting job type, thus increasing net output dispersion.

Finally, I set $\lambda = 0$ to shut down OJS and recalibrate (ψ_1, p_0, c_1) to minimize the distance of the model from the targets on $(u, f(\phi_u)/f(k^*))$. Other parameters are fixed at their values in the benchmark calibration. The recalibration yields $u = 0.656$ and $f(\phi_u)/f(k^*) = 0.341$. The 50-10 percentile ratio in net output is 1.53 and the mean/min ratio is 2.12, which are *larger* than in the baseline model. This result arises from the recalibration. The absence of OJS reduces the appeal of employment to unemployed workers. To generate a realistic unemployment rate, the starting job in the economy without OJS must have a higher matching rate than in the baseline economy. This requires the starting job to be lower and, hence, the vacancy cost to be more convex than in the baseline economy. Indeed, in the recalibrated economy without OJS, net output of the starting job is close to an unemployed worker's home production. This low starting job increases the room for job upgrading and, hence, frictional dispersion.

6. Conclusion

A worker's job can be improved through on-the-job search (OJS) and job upgrading. Incorporating these external and internal job dynamics into a directed search model, this paper analytically characterizes the socially efficient allocation and quantitatively evaluates the model. The analysis shows that efficient job upgrading continues throughout a worker's career and may be hump-shaped over tenure. In contrast, OJS is front-loaded in a worker's career and stops after a finite number of job switches. Job upgrading and OJS induce a job ladder even though all workers are identical. This endogenous job ladder implies large frictional dispersion in net output, the return to tenure and persistently large costs of job loss. The analysis also reveals the importance of the calibrated feature that the vacancy cost is convex in the job type. If the marginal cost of a vacancy is non-increasing in the job type, the social planner will start all jobs at a high type and leave very little room for job upgrading or OJS.

The formal characterization and the quantitative evaluation in this paper should both be useful for future research. To focus on frictional dispersion, I have deliberately abstracted from heterogeneity among workers and firms. It is useful to incorporate such heterogeneity and, especially, match-specific heterogeneity. In the calibrated model, the job-to-job transition is unrealistically low and OJS stops after one job switch. If match-specific productivity is introduced, a match may yield productivity lower than expected, in which case the efficient allocation is to move the worker to a better match. This may increase the amount and the duration of OJS, delay job upgrading, and enable the model to match the empirical tenure distribution better than in the model here. Another useful extension is to study how the market may succeed or fail to internalize the externalities created by OJS and job upgrading.

Appendix

A. Proof of Lemma 3.1

The domain of k is $[0, k^*]$, where $f'(k^*) = 0$. Define $V^* \equiv \frac{f(k^*)}{r}$. Since $s = 0 = \upsilon$ when job switching and job upgrading come to an end, (2.2) shows that $V \leq V^*$. Thus, it suffices to consider $V \in [0, V^*]$. Because V is bounded, s and υ must also be bounded. This implies that the choices p and p_u must be bounded by some number $\bar{p} < \infty$.

For $\lambda \neq 1$, the functional equation of V_u differs from that of $V(k)$. Denote $\Delta V(k, V_u) \equiv V(k) - V_u$. I first fix V_u to be any arbitrary value in $[0, V^*]$ and prove that the function $\Delta V(k, V_u)$ exists, is unique, and is continuous. Then I prove that a unique number V_u exists. To derive the functional equation of ΔV, use the fact that V_u is a constant to rewrite (2.2), (2.3) and (2.4) as

$$(r + \delta)\,\Delta V(k, V_u) = f(k) - rV_u + \lambda s(k) + \upsilon(k) \tag{A.1}$$

$$s(k) = \max_{(\phi,p)} \{p\,[\Delta V(\phi, V_u) - \Delta V(k, V_u)] - \psi(\phi)\,\theta(p)\} \tag{A.2}$$

$$\upsilon(k) = \max_{i} \left[\frac{\mathrm{d}\Delta V(k, V_u)}{\mathrm{d}t} - c(i)\right] \tag{A.3}$$

Since $\upsilon(k)$ involves the derivative of ΔV, so does the right-hand side of (A.1). The existence of this derivative is yet to be proven. To resolve this problem, I express the efficient allocation alternatively as a dynamic optimization problem of choosing the paths of (ϕ, p, i) and show that this problem yields a unique value function. Because the efficient allocation is time consistent, the value function generated by the dynamic optimization problem is also the unique fixed point of the Bellman equation, (2.2).

The choices in the dynamic optimization are functions of the effective tenure. I add the symbol $\hat{}$ to these functions to distinguish them from the policy functions. Let t denote the tenure of a worker in a firm. Consider a worker who has tenure $t_0 \geq 0$ in a firm and a job type $k(t_0)$. Given $k(t_0)$, the planner chooses a time path of job upgrading, $\{\hat{\imath}(t)\}_{t \geq t_0}$, and a time path of OJS, $\{\hat{\phi}(t), \hat{p}(t)\}_{t \geq t_0}$. These paths are operative only when the worker stays in the firm. If the worker moves to a new match, the planner chooses new paths of job upgrading and search. Suppose that the path, $\{\hat{\imath}(t), \hat{\phi}(t), \hat{p}(t)\}_{t \geq t_0}$, is efficient. Let $\{k(t)\}_{t \geq t_0}$ be the path of job types induced by the path of $\hat{\imath}$ according to (2.1), where the symbol $\hat{}$ on k is suppressed. Substitute s and υ from (A.2) and (A.3) into (A.1). For the moment, omit the two maximization operators. Denote the effective discount factor

between t_0 and t as

$$D(t,t_0) = e^{-\int_{t_0}^{t}[r+\delta+\lambda\hat{p}(\tau)]\mathrm{d}\tau}. \tag{A.4}$$

For any $t_0 \geq 0$ and $k(t_0) \geq 0$, integrating (A.1) yields

$$\begin{aligned} &\Delta V(k(t_0), V_u) \\ &= \int_{t_0}^{\infty} \left[f(k) - c(\hat{\imath}) - rV_u + \lambda\hat{p}\Delta V(\hat{\phi}, V_u) - \lambda\psi\left(\hat{\phi}\right)\theta(\hat{p}) \right] D(t,t_0)\,\mathrm{d}t, \end{aligned} \tag{A.5}$$

where the dependence of the integrand on t is suppressed. Given $k(t_0)$, the path of efficient choices from t_0 onward must maximize the right-hand side of (A.5). That is, the path, $\{\hat{\imath}(t), \hat{\phi}(t), \hat{p}(t)\}_{t\geq t_0}$, solves:

$$\begin{aligned} &\Delta V(k(t_0), V_u) \\ &= \max \int_{t_0}^{\infty} \left[f(k) - c(\hat{\imath}) - rV_u + \lambda\hat{p}\Delta V(\hat{\phi}, V_u) - \lambda\psi\left(\hat{\phi}\right)\theta(\hat{p}) \right] D(t,t_0)\,\mathrm{d}t, \end{aligned} \tag{A.6}$$

subject to (2.1), where $k(t_0)$ is taken as given.

Denote the right-hand side of (A.6) as $(\mathcal{T}\Delta V)(k(t_0), V_u)$, where $\mathcal{T}$ is the implied mapping on ΔV. Let $\mathcal{C}$ be the space that contains all continuous functions defined on $[0, k^*] \times [0, V^*]$, with the sup-norm. Then, $\mathcal{T}$ is a self-map on $\mathcal{C}$. I prove that $\mathcal{T}$ satisfies Blackwell's sufficient conditions for a monotone contraction; i.e., it is monotone and has discounting (see Stokey et al., 1989). Thus, there is a unique fixed point of $\mathcal{T}$ in $\mathcal{C}$, which is $\Delta V(k, V_u)$.

Since $\lambda p \geq 0$, it is clear that $\mathcal{T}$ is an increasing mapping. That is, for any ΔV_A and ΔV_B in $\mathcal{C}$, with $\Delta V_A(k, V_u) \geq \Delta V_B(k, V_u)$ for all (k, V_u), $\mathcal{T}\Delta V_A(k, V_u) \geq \mathcal{T}\Delta V_B(k, V_u)$ for all (k, V_u). To verify that $\mathcal{T}$ has discounting, let $a \in (0, \infty)$ be an arbitrary constant and consider any $\Delta V \in \mathcal{C}$. Define the function $(\Delta V + a)$ by $(\Delta V + a)(k, V_u) = \Delta V(k, V_u) + a$ for all (k, V_u). The optimal choices under the value function $(\Delta V + a)$ are denoted with a subscript a, and the optimal choices under the value function ΔV are denoted without the subscript a. For any given $k(t_0)$, I have

$$\begin{aligned} &\mathcal{T}(\Delta V + a)(k(t_0), V_u) \\ = \;& \int_{t_0}^{\infty} \left[f(k_a) - c(\hat{\imath}_a) - rV_u + \lambda\hat{p}_a\Delta V(\hat{\phi}_a, V_u) - \lambda\psi\left(\hat{\phi}_a\right)\theta(\hat{p}_a) \right] D_a(t,t_0)\,\mathrm{d}t \\ & + a\int_{t_0}^{\infty} \lambda\hat{p}_a D_a(t,t_0)\,\mathrm{d}t \\ \leq \;& \mathcal{T}\Delta V(k(t_0), V_u) + a\int_{t_0}^{\infty} \lambda\hat{p}_a D_a(t,t_0)\,\mathrm{d}t. \end{aligned}$$

The inequality follows from the definition that $\mathcal{T}\Delta V(k(t_0), V_u)$ is the maximized value of the first integral. Since $p \leq \bar{p} < \infty$, then

$$\int_{t_0}^{\infty} D_a(t,t_0)\,\mathrm{d}t \geq \int_{t_0}^{\infty} e^{-(r+\delta+\lambda\bar{p})(t-t_0)}dt = \frac{1}{r+\delta+\lambda\bar{p}}.$$

Using this result and the definition in (A.4), I derive:

$$\begin{aligned}\int_{t_0}^{\infty} \lambda\hat{p} D(t,t_0)\,\mathrm{d}t &= \int_{t_0}^{\infty} (r+\delta+\lambda\hat{p}) D(t,t_0)\,\mathrm{d}t - (r+\delta)\int_{t_0}^{\infty} D(t,t_0)\,\mathrm{d}t \\ &= -\int_{t_0}^{\infty} \mathrm{d}D(t,t_0) - (r+\delta)\int_{t_0}^{\infty} D(t,t_0)\,\mathrm{d}t \\ &= 1-(r+\delta)\int_{t_0}^{\infty} D(t,t_0)\,\mathrm{d}t \leq \tfrac{\lambda\bar{p}}{r+\delta+\lambda\bar{p}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{T}\left(\tilde{V}+a\right)(k(t_0),V_u) &\leq \mathcal{T}\Delta V(k(t_0),V_u) + a\int_{t_0}^{\infty} \lambda\hat{p}_a D_a(t,t_0)\,\mathrm{d}t \\ &\leq \mathcal{T}\Delta V(k(t_0),V_u) + \tfrac{\lambda\bar{p}}{r+\delta+\lambda\bar{p}}a.\end{aligned}$$

Because $\lambda\bar{p} < \infty$ and $r+\delta > 0$, then $\frac{\lambda\bar{p}}{r+\delta+\lambda\bar{p}} \in (0,1)$. This establishes that $\mathcal{T}$ has discounting, and so $\mathcal{T}$ has a unique fixed point $\Delta V(k,V_u)$ in $\mathcal{C}$.

For any $\Delta V \in \mathcal{C}$, it is straightforward to verify that $\mathcal{T}$ has the following properties: (i) If $\Delta V(k,V_u)$ is (weakly) increasing in k for $k<k^*$, then $\mathcal{T}\Delta V(k,V_u)$ is strictly increasing in k for $k<k^*$; (ii) If $\Delta V(k,V_u)$ is (weakly) decreasing in V_u, then $\mathcal{T}\Delta V(k,V_u)$ is strictly decreasing in V_u; (iii) If $\Delta V(k,0)\geq 0$, then $\mathcal{T}\Delta V(k,0)\geq 0$, with strict inequality for $k>0$; (iv) If $\Delta V(k,V^*)\leq 0$, then $\mathcal{T}\Delta V(k,V^*)\leq 0$, with strict inequality for $k<k^*$. With the contraction mapping argument (see Stokey et al., 1989), these properties imply that the fixed point $\Delta V(k,V_u)$ is increasing in k (strictly so for $k<k^*$), strictly decreasing in V_u, satisfying $\Delta V(k,0)\geq 0$ (with strict inequality for all $k>0$), and satisfying $\Delta V(k,V^*)\leq 0$ (with strict inequality for all $k<k^*$).

The number V_u satisfies (2.6), which can be rewritten as

$$rV_u = f(k_u) + \max_{(\phi,p)}\left[p\Delta V(\phi,V_u) - \psi(\phi)\theta(p)\right].$$

In the maximization problem above, the choices (ϕ,p) lie in the compact set $[0,k^*]\times[0,\bar{p}]$. Since $\Delta V(\phi,V_u)$ is continuous on $[0,k^*]\times[0,V^*]$, the Theorem of the Maximum implies that the right-hand side of the above equation is continuous in V_u. It is also decreasing in V_u, because $\Delta V(\phi,V_u)$ is so. Since $\Delta V(k,0)\geq 0$ and $\Delta V(k,V^*)\leq 0$ for all k, and since $k_u<k^*$, the right-hand side is strictly positive at $V_u=0$ and strictly less than rV^* at $V_u=V^*$. Therefore, there is a unique $V_u\in(0,V^*)$ that satisfies the above equation.

To verify the properties of V in Lemma 3.1, return to the problem in (2.2), where the efficient choices are policy functions of k. Because $V(k)$ is continuous and the domain of k is compact, V is bounded. For $V(k)$ to be bounded, the right-hand side of (2.2) must be unbounded and, in particular, $\upsilon(k)$ must be bounded. Since $\frac{\mathrm{d}V(k)}{\mathrm{d}t}$ appears on the right-hand side through $\upsilon(k)$ (see (2.4)), this derivative must exist and be bounded. Note that $\frac{\mathrm{d}V(k)}{\mathrm{d}t} = V'(k(t))\,\hat{\imath}(t)$, where (2.1) is used. Because $c(\infty)=\infty$, the optimal choice i is bounded. Existence and boundedness of $\frac{\mathrm{d}V(k)}{\mathrm{d}t}$ imply that $V'(k)$ exists and is bounded.

Then, the job upgrading problem becomes (2.5). Given k, the objective function in (2.5) is strictly concave in i. The optimal choice satisfies

$$V'(k) - c'(i) \leq 0 \text{ and } i \geq 0,$$

where the two inequalities hold with complementary slackness. If $i > 0$, then $V'(k) = c'(i) > 0$, as in (3.1), in which case $V(k)$ is strictly increasing. If $i = 0$, then $V'(k) \leq c'(0) = 0$. However, since $V(k)$ is (weakly) increasing, $V'(k) \geq 0$. Thus, if $i = 0$, then $V'(k) = 0 = c'(0)$. That is, (3.1) holds for both $i > 0$ and $i = 0$. Finally, whenever $\frac{\mathrm{d}i}{i\,\mathrm{d}t}$ exists, differentiating (3.1) with respect to t and using (2.1) yields (3.2). Because $c''(i) > 0$ for all $i \geq 0$, then $V''(k) < 0$ if and only if $\frac{\mathrm{d}i}{i\,\mathrm{d}t} < 0$. **QED**

B. Proof of Proposition 3.2

Differentiating (3.1) with respect to t yields $V''(k)\,i = c''(i)\frac{\mathrm{d}i}{\mathrm{d}t}$, where $k = k(t)$ and $i = i(k(t))$. This shows that $V''(k)\,i(k)$ exists if $\frac{\mathrm{d}i}{\mathrm{d}t}$ exists. Because $i(k)$ solves (3.1) and $c''(i) > 0$ for all $i \geq 0$, the function $i(k)$ is continuous. To prove that $\upsilon'(k)$ exists and satisfies the envelope condition for (2.5), I calculate the one-sided derivatives of υ. Let k be an arbitrary job type and $\varepsilon > 0$ an arbitrarily small number. Temporarily denote $F(i,k) = V'(k)\,i - c(i)$. The derivative of υ on the left side of k is

$$\upsilon'(k_-) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[F(i(k),k) - F(i(k-\varepsilon),k-\varepsilon)\right].$$

Because $i(k-\varepsilon)$ is the optimal when the job type is $(k-\varepsilon)$, then $F(i(k-\varepsilon),k-\varepsilon) \geq F(i(k),k-\varepsilon)$. This inequality implies

$$\upsilon'(k_-) \leq \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[F(i(k),k) - F(i(k),k-\varepsilon)\right] = V''(k)\,i(k).$$

On the other hand, because $i(k)$ is optimal when the job type is k, then $F(i(k),k) \geq F(i(k-\varepsilon),k)$, which implies

$$\upsilon'(k_-) \geq \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[F(i(k-\varepsilon),k) - F(i(k-\varepsilon),k-\varepsilon)\right] = V''(k)\,i(k).$$

Notice that the last equality uses continuity of $i(k)$. Thus, $\upsilon'(k_-) = V''(k)\,i(k)$. A similar procedure proves that the derivative of υ on the right side of k is $\upsilon'(k_+) = V''(k)\,i(k)$. Thus, $\upsilon'(k)$ exists and is given as $\upsilon'(k) = V''(k)\,i(k)$.

To establish that $s'(k)$ exists, consider (2.2). The left-hand side of (2.2) is differentiable because $V'(k)$ exists. Since $f'(k)$ and $\upsilon'(k)$ exist, $s'(k)$ must exist; otherwise, the right-hand side of (2.2) would fail to be differentiable. To prove that $s'(k)$ satisfies the envelope condition for (2.3), I calculate the one-sided derivatives of s. In contrast to the proof that

$v'(k)$ satisfies the envelope condition, I cannot use continuity of the policy functions of job search, since this continuity is yet to be established. Instead, I use the result that $s'(k)$ exists, which was just proven. Consider an arbitrary k and an arbitrarily small number $\varepsilon > 0$. Temporarily denote the objective function in (2.3) as $F(\phi, p, k)$. Let $(\phi(k), p(k))$ be the policy functions of the efficient choices of (ϕ, p). Because

$$F(\phi(k-\varepsilon), p(k-\varepsilon), k-\varepsilon) \geq F(\phi(k), p(k), k-\varepsilon),$$

then the left side derivative of $s(k)$ satisfies:

$$\begin{aligned} s'(k_-) &= \lim_{\varepsilon \downarrow 0} \tfrac{1}{\varepsilon} \left[F(\phi(k), p(k), k) - F(\phi(k-\varepsilon), p(k-\varepsilon), k-\varepsilon)\right] \\ &\leq \lim_{\varepsilon \downarrow 0} \tfrac{1}{\varepsilon} \left[F(\phi(k), p(k), k) - F(\phi(k), p(k), k-\varepsilon)\right] \\ &= -p(k) V'(k). \end{aligned}$$

The last expression is the partial derivative of $F(\phi, p, k)$ with respect to k. This partial derivative exists, as shown in Lemma 3.1. Similarly, because

$$F(\phi(k+\varepsilon), p(k+\varepsilon), k+\varepsilon) \geq F(\phi(k), p(k), k+\varepsilon),$$

the right side derivative of $s(k)$ satisfies:

$$\begin{aligned} s'(k_+) &= \lim_{\varepsilon \downarrow 0} \tfrac{1}{\varepsilon} \left[F(\phi(k+\varepsilon), p(k+\varepsilon), k+\varepsilon) - F(\phi(k), p(k), k)\right] \\ &\geq \lim_{\varepsilon \downarrow 0} \tfrac{1}{\varepsilon} \left[F(\phi(k), p(k), k+\varepsilon) - F(\phi(k), p(k), k)\right] \\ &= -p(k) V'(k). \end{aligned}$$

Since $s'(k)$ exists, as shown above, these results on the one-sided derivatives imply

$$-p(k) V'(k) \leq s'(k_+) = s'(k) = s'(k_-) \leq -p(k) V'(k).$$

Therefore, $s'(k) = -p(k) V'(k)$.

Differentiate the Bellman equation (2.2) with respect to V. Substituting the results for $v'(k)$ and $s'(k)$ yields (3.3). Substituting $V'(k) = c'(i)$ and $V''(k) i = c''(i) \frac{\mathrm{d}i}{\mathrm{d}t}$ yields (3.4). **QED**

C. Proof of Lemma 3.3

For (i), consider an arbitrary $k \in (0, k^*)$ such that $p(k) > 0$. Given $p > 0$, the optimal choice $\phi(k)$ must satisfy $\phi(k) > k$, because $\phi \leq k$ implies that the return on search is $-\psi(\phi)\theta(p) < 0$. Also, given p, a marginal increase in the search target ϕ increases the expected social value by

$$\Delta(\phi, p) \equiv V'(\phi) - \psi'(\phi) \frac{\theta(p)}{p}.$$

The optimal choice ϕ must satisfy $\phi(k) < k^*$: If $\phi(k) \geq k^*$, then $V'(\phi) \leq 0$ and $\Delta(\phi) < 0$, in which case the return on search can be increased by reducing the search target. Thus, $k < \phi(k) < k^*$. Since $\phi(k)$ is interior, it satisfies the first-order condition, $\Delta(\phi(k)) = 0$, which is equivalent to (3.5).

For (ii), given the efficient choice ϕ, the objective function of job search in (2.3) is strictly concave in p. Thus, the efficient choice $p(k)$ satisfies the complementary slackness condition, (3.6). Given ϕ_u, the efficient choice p_u for an unemployed worker satisfies a similar condition.

For (iii), consider the case where $p(k) > 0$. I prove that $\Delta'(\phi(k)) \geq 0$ is inconsistent with optimality, where Δ is defined above with p being suppressed. If $\Delta'(\phi(k)) > 0$, then $\phi(k)$ achieves a local minimum instead of a maximum, which is clearly not optimal. Suppose $\Delta'(\phi(k)) = 0$. Consider the choice $\left(\tilde{\phi}, \tilde{p}\right)$, where $\tilde{p} = p(k) + \varepsilon_p$ and $\tilde{\phi} = \phi(k) - \varepsilon_\phi$. Let $\varepsilon_p > 0$ be a sufficiently small number and $\varepsilon_\phi > \frac{\psi(\phi)p\theta''(p)\varepsilon_p}{2\psi'(\phi)[p\theta'(p)-\theta(p)]}$, where $\phi = \phi(k)$ and $p = p(k)$. Because $\theta'' > 0$ and $p\theta' > \theta$ for all $p > 0$ by assumption, then $\varepsilon_\phi > 0$, and $\varepsilon_\phi \to 0$ as $\varepsilon_p \to 0$. Since $p(k) > 0$, the first-order condition of p, (3.6), holds as equality. Thus, the change from the choice $(\phi(k), p(k))$ to $\left(\tilde{\phi}, \tilde{p}\right)$ has no first-order effect on the return on search. Since $\Delta'(\phi(k)) = 0$, the second-order effect of this change is

$$-\frac{1}{2}\psi\theta''\varepsilon_p^2 + \left(\theta' - \frac{\theta}{p}\right)\psi'\varepsilon_p\varepsilon_\phi > 0,$$

where I substituted $V' = \psi'\frac{\theta}{p}$ and used the inequality between ε_ϕ and ε_p. Since this result contradicts the optimality of $(\phi(k), p(k))$, then $\Delta'(\phi(k)) < 0$ must hold if $p(k) > 0$. If ψ is linear, $\Delta'(\phi(k)) < 0$ implies $V''(\phi(k)) < 0$. **QED**

D. Proof of Proposition 3.4

Assume that $p(k)$ exists and is unique for every k. For (i) of the proposition, consider any $k \in (0, k^*)$ such that $p(k) > 0$. Define $\Phi(p)$ by $V'(\Phi(p)) = \psi'(\Phi(p))\frac{\theta(p)}{p}$. By (iii) of Lemma 3.3, $\Phi(p)$ is well-defined and single valued, with $\phi(k) = \Phi(p(k))$. Since $\frac{\theta(p)}{p}$ is strictly increasing in p, then $\Phi'(p) < 0$. Thus, $\phi'(k) > 0$ if and only if $p'(k) < 0$. To prove $p'(k) < 0$, denote the marginal contribution of the matching rate to the return on search at $\phi = \Phi(p)$ as

$$M(p, k) = V(\Phi(p)) - V(k) - \psi(\Phi(p))\theta'(p).$$

It is clear from (3.6) that $M(p(k), k) = 0$, since $p(k) > 0$ in the case considered here. Moreover, the optimality of $p(k)$ requires the return on search to be concave in p at

$p = p(k)$; i.e., the partial derivative of M with respect to p satisfies $M_1(p(k), k) \leq 0$. If $M_1(p(k), k) = 0$, then for an arbitrarily small $\varepsilon > 0$,

$$M(p(k-\varepsilon), k-\varepsilon) - M(p(k), k) \approx -M_2\varepsilon = V'(k)\varepsilon > 0,$$

where I have used the fact that $V'(k) = c'(i) > 0$. This implies $M(p(k-\varepsilon), k-\varepsilon) > M(p(k), k) = 0$, which contradicts the optimality of $p(k-\varepsilon)$ when the job type is $k-\varepsilon$. Thus, $M_1(p(k), k) < 0$. Differentiating the equation $M(p(k), k) = 0$ yields

$$p'(k) = \frac{V'(k)}{M_1(p(k), k)} < 0.$$

For (ii), define $k_a = \sup\{k : p(k) > 0\}$. Because $p'(k) < 0$ for all k such that $p(k) > 0$, it is clear that $p(k) > 0$ if and only if $k < k_a$. By (3.5), k_a satisfies

$$V'(\phi(k_a)) = \psi'(\phi(k_a)) \lim_{p \downarrow 0} \frac{\theta(p)}{p} = \psi'(\phi(k_a))\theta'(0).$$

With k_c defined by (3.7), this result shows $k_c = \phi(k_a)$. Since $p(k) > 0$ for all $k < k_a$, the first-order condition for $p(k)$ in (3.6) is satisfied with equality for all $k < k_a$. In the limit $k \uparrow k_a$, this first-order condition becomes:

$$V(k_c) - V(k_a) - \psi(k_c)\theta'(0) = 0.$$

With k_T defined by (3.8), this equation implies $k_a = k_T$ and so $k_c = \phi(k_T)$. Because $\theta'(0) > 0$ by assumption, (3.7) implies $V'(k_c) > 0 = V'(k^*)$, and so, $k_c < k^*$. Similarly, (3.8) implies $V(k_c) > V(k_T)$, and so $k_c > k_T$.

For (iii), I prove that $\frac{\mathrm{d}i(k)}{\mathrm{d}t} < 0$ for all $k \in [k_T, k^*)$. This result further implies $V''(k) < 0$ for all $k \in [k_T, k^*)$, because $\frac{\mathrm{d}i}{\mathrm{d}t} < 0$ if and only if $V(k)$ is strictly concave (see Lemma 3.1). For all $k \in [k_T, k^*)$, $p(k) = 0$, and so (3.4) yields

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{c''(i)}\left[(r+\delta)c'(i) - f'(k)\right].$$

This equation and (2.1) form a system of differential equations for (i, k). Define

$$I(k) = c'^{-1}\left(\frac{f'(k)}{r+\delta}\right).$$

If $i(k) < I(k)$ for all $k \in [k_T, k^*)$, then $\frac{\mathrm{d}i}{\mathrm{d}t} < 0$ for all such k. Suppose, to the contrary, that $i(k(t_0)) \geq I(k(t_0))$ for some $t_0 < \infty$ and $k(t_0) \in [k_T, k^*)$. Then $\frac{\mathrm{d}i(k(t))}{\mathrm{d}t}|_{t=t_0} \geq 0$ and $\frac{\mathrm{d}k(t)}{\mathrm{d}t}|_{t=t_0} = i(k(t_0)) \geq I(k(t_0)) > 0$, where the last strict inequality comes from $k(t_0) < k^*$. Re-use the notation $\hat{\imath}(t) = i(k(t))$. For a sufficiently small $\varepsilon > 0$, $\hat{\imath}(t_0+\varepsilon) \geq \hat{\imath}(t_0)$ and $k(t_0+\varepsilon) > k(t_0)$, where the inequality on k is strict because $\frac{\mathrm{d}k(t)}{\mathrm{d}t}|_{t=t_0} > 0$. Then,

$$c'(\hat{\imath}(t_0+\varepsilon)) \geq c'(\hat{\imath}(t_0)) \geq \frac{f'(k(t_0))}{r+\delta} > \frac{f'(k(t_0+\varepsilon))}{r+\delta}.$$

The first inequality comes from $\hat{\imath}(t_0+\varepsilon) \geq \hat{\imath}(t_0)$ and convexity of c. The second inequality comes from the hypothesis $\hat{\imath}(t_0) \geq I(k(t_0))$. The third (strict) inequality comes from $k(t_0) < k(t_0+\varepsilon)$ and strict concavity of f. This result implies $\hat{\imath}(t_0+\varepsilon) > I(k(t_0+\varepsilon))$, which in turn implies $\frac{\mathrm{d}\hat{\imath}(t)}{\mathrm{d}t}|_{t=t_0+\varepsilon} > 0$ and $\frac{\mathrm{d}k(t)}{\mathrm{d}t}|_{t=t_0+\varepsilon} = \hat{\imath}(t_0+\varepsilon) > 0$. By induction, $\hat{\imath}(t) > I(k(t))$, $\frac{\mathrm{d}\hat{\imath}(t)}{\mathrm{d}t} > 0$ and $\frac{\mathrm{d}k(t)}{\mathrm{d}t} > 0$ for all $t \in (t_0, \infty)$. Since $\hat{\imath}(t)$ increases in t, then $\lim_{t\to\infty} \frac{\mathrm{d}k(t)}{\mathrm{d}t} = \lim_{t\to\infty} \hat{\imath}(t) \geq \hat{\imath}(t_0) \geq I(k(t_0)) > 0$. From $k(t_0)$, the path of the job type will surpass k^* in a finite length of time and will keep increasing thereafter. Because $f'(k) < 0$ for all $k > k^*$, this path cannot be socially efficient. Therefore, $\hat{\imath}(t) < I(k(t))$ and so $\frac{\mathrm{d}\hat{\imath}(t)}{\mathrm{d}t} < 0$ for all $k(t) \in [k_T, k^*)$. Similarly, $\hat{\imath}(t) > 0$ for all $k(t) \in [k_T, k^*)$. The convergence of (i, k) to the final state $(0, k^*)$ is asymptotic.

For (iv), first consider the case where ψ is linear. Then, (iii) of Lemma 3.3 implies that $V(\phi(k))$ is strictly concave in ϕ whenever $p(k) > 0$. Be the definition of k_T, $p(k) > 0$ for all $k < k_T$. Because $\phi(k_u) = \phi_u$ and $\phi(k_T) = k_c$, then $V(k)$ is strictly concave for all $k \in [\phi_u, k_c)$. With this result, (3.1) implies $c'(i) = V'(k) > V'(k_c) > 0$ and, hence, $i(k) > 0$ for all $k \in [\phi_u, k_c)$. Furthermore, (3.2) implies $\frac{\mathrm{d}i(k)}{\mathrm{d}t} = \frac{V''(k)i(k)}{c''(i(k))} < 0$ for all $k \in [\phi_u, k_c)$. Because $k_c > k_T$, these results clearly hold for $k \in [\phi_u, k_T]$.

If ψ is sufficiently convex, it is possible that $V''(k) > 0$ for some $k \in [\phi_u, k_T]$, and so $\frac{\mathrm{d}i(k)}{\mathrm{d}t} > 0$ for such k. This completes the proof of Proposition 3.4.

The remainder of this proof specifies a sufficient condition for $p(k)$ to be unique for every k. Because $p(k) = 0$ for all $k \geq k_T$, it suffices to consider only $k < k_T$. The proof of Proposition 3.4 above has shown that $p(k)$ is given by the solution to $M(p(k), k) = 0$, where $M_1(p(k), k) < 0$. The proof of Proposition 3.4 has shown that $p(k)$ is given by the solution to $M(p(k), k) = 0$, where $M_1(p(k), k) < 0$. For any $k < k_T$, the sufficient condition for $p(k)$ to exist is $M(0, k) > 0 > M(\tilde{p}, k)$ for some $\tilde{p} \geq 0$. Use the function $\Phi(p)$ defined in the proof of Proposition 3.4. The definition of k_c in (3.7) implies $\Phi(0) = k_c$. Thus, $M(0, k) > 0$ if and only if $V(k) < V(k_c) - \psi(k_c)\theta'(0)$. This condition is equivalent to $k < k_T$. Define $\tilde{p} = \Phi^{-1}(k) \geq 0$. Then, $M(\tilde{p}, k) = -\psi(k)\theta'(\tilde{p}) < 0$.

Now turn to uniqueness of $p(k)$. Since $M(p(k), k) = 0$ and $M_1(p(k), k) < 0$, the necessary and sufficient condition for $p(k)$ to be single valued is that $M_1(p, k) < 0$ for all p such that $M(p, k) = 0$. Let p be a solution to $M(p, k) = 0$. Then,

$$\Phi'(p) = \frac{-(\theta' - \theta/p)\psi'}{\psi''\theta - V''p},$$

where the argument in ψ and V is $\Phi(p)$, which is suppressed. Substituting this derivative and $V' = \psi'\theta/p$, I can calculate

$$M_1(p, k) = \frac{[(\theta' - \theta/p)\psi']^2}{\psi''\theta - V''p} - \psi\theta''.$$

The possibility of $V'' > 0$ makes it difficult to obtain the general condition for $M_1(p,k) < 0$ for all p such that $M(p,k) = 0$. If $V(k)$ is (weakly) concave for all $k \geq \phi_u$, I can provide sufficient conditions for $p(k)$ to be unique. If $V'' \leq 0$, then

$$M_1(p,k) \leq \frac{[(\theta' - \theta/p)\psi']^2}{\psi''\theta} - \psi\theta''.$$

Then, a sufficient condition for $M_1(p,k) < 0$ is

$$\frac{\psi\psi''}{(\psi')^2} > \frac{(\theta' - \theta/p)^2}{\theta\theta''}.$$

Using $P(\theta)$ as the inverse of $\theta(p)$, I can write this condition as

$$\frac{\psi\psi''}{(\psi')^2} > \frac{(1 - \theta P'/P)^2}{-\theta P''/P'}. \tag{D.1}$$

If V is strictly concave, the above condition can be relaxed to a weak inequality.

Consider the example $\psi(k) = \psi_0 k^{\psi_1}$, where $\psi_0 > 0$ and $\psi_1 \geq 1$, and the matching functions in Example 2.1. With the urn-ball matching function, (D.1) becomes

$$1 - \frac{1}{\psi_1} > \frac{e^{1/\theta} - 1 - \frac{1}{\theta}}{(e^{1/\theta} - 1)^2}.$$

It can be verified that the right-hand side is a decreasing function of $1/\theta$ for all $1/\theta \geq 0$, and so its maximum is achieved at $1/\theta \to 0$, which is $1/2$. Thus, a sufficient condition for $p(k)$ to be unique under the urn-ball matching function is $\psi_1 \geq 2$. If the matching function is the generalized telephone matching function in Example 2.1, then (D.1) becomes

$$1 - \frac{1}{\psi_1} > \left\{(1+\rho)\left[\left(\frac{p_0}{\theta}\right)^{\rho} + 1\right]\right\}^{-1}.$$

Since the right-hand side of this inequality is maximized at $\theta \to \infty$, a sufficient condition for $p(k)$ to be unique is $1 - \frac{1}{\psi_1} \geq \frac{1}{1+\rho}$, i.e., $\psi_1 \geq 1 + \frac{1}{\rho}$. **QED**

E. Proof of Proposition 4.1

Assume that the joint density $\omega(k,t)$ exists for all $(k,t) \in [\phi_u, k^*] \times [0,\infty)$. Recall that for any tenure $t \geq 0$, the job type $k(t)$ is reached by a worker who stays in the same firm all the way to t ever since being employed. If a worker succeeds in moving to another job, the job type reached by the worker at actual tenure t is higher than $k(t)$. Since $k(t)$ is the lowest job type reached by a worker with actual tenure t, then $\omega\left(\tilde{k}, t\right) = 0$ for all $\tilde{k} < k(t)$. Similarly, for any $k \in [\phi_u, k^*]$, the longest tenure needed to reach k is $t(k)$, where $t(k)$ is the inverse function of $k(t)$. Thus, $\omega\left(k, \tilde{t}\right) = 0$ for all $\tilde{t} > t(k)$.

To characterize the joint distribution of employed workers, consider employed workers in $(k, k^*] \times \{t\}$, i.e., the group of employed workers whose job types are higher than k and whose tenure is t. The density of these workers is $-\bar{\Omega}_t(k,t)$. In a small interval of time Δ, all of these workers exit the group because they either separate which resets tenure to zero, or stay in the same firm which increases tenure to $t+\Delta$. Thus, the outflow of workers from the group is $-\bar{\Omega}_t(k,t)$. The inflow is the group of workers employed in $L \cup (k, k^*]$ and $t-\Delta$ who do not separate from their jobs, where $L = \{\tilde{k} \leq k : \tilde{k} + i\left(\tilde{k}\right)\Delta > k\}$. For these workers, the passing of time increases tenure to t. If their job types were in L, which is outside $(k, k^*]$, job upgrading increases their job types into $(k, k^*]$. If their job types were already in $(k, k^*]$, they remain in this interval. The density of this inflow of workers is

$$\begin{aligned} & \textstyle\int_{\tilde{k} \in L \cup (k,k^*]} \omega\left(\tilde{k}, t-\Delta\right)\left[1-\delta\Delta-\lambda p\left(\tilde{k}\right)\Delta\right] \mathrm{d}\tilde{k} \\ = \; & -\bar{\Omega}_t(k, t-\Delta) + \textstyle\int_{\tilde{k} \in L} \omega\left(\tilde{k}, t-\Delta\right) \mathrm{d}\tilde{k} \\ & -\Delta \textstyle\int_{\tilde{k} \in L \cup (k,k^*]} \omega\left(\tilde{k}, t-\Delta\right)\left[\delta+\lambda p\left(\tilde{k}\right)\right] \mathrm{d}\tilde{k}. \end{aligned}$$

Notice that p is inside the integral because the endogenous separation rate depends on $\tilde{k}$.

The measure of the set L in the domain of k goes to 0 in the limit $\Delta \to 0$. Moreover, the ratio of the measure of L to Δ goes to $i(k)$ in the limit $\Delta \to 0$. Thus,

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\tilde{k} \in L} \omega\left(\tilde{k}, t-\Delta\right) \mathrm{d}\tilde{k} = i(k)\,\omega(k,t).$$

Equating the inflow to the outflow from the group $(k, k^*] \times \{t\}$, dividing by Δ and taking $\Delta \to 0$, I get:

$$i(k)\,\omega(k,t) + \bar{\Omega}_{tt}(k,t) = \int_k^{k^*} \left[\delta + \lambda p\left(\tilde{k}\right)\right] \omega\left(\tilde{k}, t\right) \mathrm{d}\tilde{k}. \tag{E.1}$$

Since $\omega(k,t) = \bar{\Omega}_{kt}(k,t)$ and $\bar{\Omega}_k(k,\infty) = \bar{\Omega}_t(k,\infty) = \bar{\Omega}(k,\infty) = 0$, integrating (E.1) over tenure in (t, ∞) yields (4.3).

The marginal distribution of employed workers over k is $G(k) = 1 - \bar{\Omega}(k,0)$, and the corresponding density is $g(k) = -\bar{\Omega}_k(k,0)$. Setting $t = 0$ in (4.3) yields (4.4). Similarly, since the lowest employed job type is ϕ_u, the marginal distribution of employed workers over tenure t is $G_t(t) = 1 - \bar{\Omega}(\phi_u, t)$, and the corresponding density is $g_t(t) = -\bar{\Omega}_t(\phi_u, t)$. Because all workers employed at $k = \phi_u$ just came out of unemployment, their actual tenure is 0. Thus, $\omega(\phi_u, t) = 0$ for all $t > 0$, and so $\bar{\Omega}_k(\phi_u, t) = 0$ for all $t \geq 0$.[18] Using this fact and setting $k = \phi_u$ in (4.3), I get (4.5) for all $t \geq 0$.

Differentiating (4.5) with respect to t yields:

$$g_t'(t) = -\delta g_t(t) - \int_{\phi_u}^{k^*} \lambda p\left(\tilde{k}\right) \omega\left(\tilde{k}, t\right) \mathrm{d}\tilde{k}.$$

[18] This is an implication of the earlier result that $\omega\left(\tilde{k}, t\right) = 0$ for all $\tilde{k} < k(t)$, because $k(t) > k(0) = \phi_u$ for all $t > 0$.

Because $\omega(k,t)$ exists for all (k,t), this equation shows that $g_t'(t)$ exists and $g_t'(t) < 0$ for all $t \in [0,\infty)$, as stated in (i). Similarly, differentiating (4.3) with respect to k yields:

$$\frac{\mathrm{d}}{\mathrm{d}k}\left[i(k)\,g(k)\right] = -\left[\delta + \lambda p(k)\right] g(k) + \omega(k,0).$$

Thus, $[i(k)\,g(k)]$ is differentiable for all k. Since this derivative is also equal to $i'(k)\,g(k) + i(k)\,g'(k)$, then $g'(k)$ exists if and only if $i'(k)$ exists and $i(k) \neq 0$. By Proposition 3.4, $i(k) > 0$ for all $k < k^*$. Note that $i'(k) = \frac{\mathrm{d}i(k)}{i(k)\mathrm{d}t}$. Because $\frac{\mathrm{d}i(k)}{\mathrm{d}t}$ exists for all $t \in [0,\infty)$ (see (3.4)), then $i'(k)$ exists if and only if $i(k) \neq 0$, i.e., if and only if $k < k^*$. Thus, $g'(k)$ exists for all $k \in [\phi_u, k^*)$. This result is (ii). Finally, to verify (iii), consider any $k \in (k_c, k^*)$. Since k_c is the highest job type that can be reached by job switching, an employed worker who has $k > k_c$ must have experienced job upgrading immediately before reaching such a job type and, hence, must have strictly positive tenure. That is, $\omega(k,0) = 0$ for all $k \in (k_c, k^*)$. Because $p(k) = 0$ for these job types, then $\frac{\mathrm{d}}{\mathrm{d}k}\left[i(k)\,g(k)\right] = -\delta g(k) < 0$. This result also shows that $g'(k) < 0$ if and only if $-i'(k) < \delta$. Since $i'(k) = \frac{\mathrm{d}i(k)}{i(k)\mathrm{d}t}$ and $p(k) = 0$, (3.4) implies that $g'(k) < 0$ if and only if $f'(k) < \delta i c''(i) + (r+\delta)\,c'(i)$ where $i = i(k)$. **QED**

F. Procedures of Calibration and Computation

F.1. Calibration Procedure

The model is calibrated at the monthly frequency. The discount rate r is determined by the target, $(1+r)^3 = 1.0125$. The parameter α in the output function is determined by the capital share at k^*, $\frac{rk^*}{f_a(k^*)} = 0.55$. Since $f_a'(k^*) = r$, then $\frac{rk^*}{f_a(k^*)} = \frac{k^* f_a'(k^*)}{f_a(k^*)} = \alpha$. The condition, $f_a'(k^*) = r$, implies $f_0 = \frac{r}{\alpha}(k^*)^{1-\alpha}$, which determines f_0 under the normalization $k^* = 100$. The calibration target on home production yields $f(k_u) = 0.31 f(k^*)$, which determines k_u. The monthly transition rate from employment to unemployment is $\delta = 0.026$, which is taken from the Current Population Survey (CPS). The value $\lambda = 1$ is chosen as the benchmark. With the target $u = 0.065$, the steady state yields $p_u = \delta(1-u)/u = 0.374$. Because $\varepsilon_u \equiv \frac{\mathrm{d}\ln p_u}{\mathrm{d}\ln\theta(p_u)} = 1 - (p_u)^\rho$, the target $\varepsilon_u = 0.39$ yields $\rho = 0.5$. The targets, $\frac{f(\phi_u)}{f(k^*)} = 0.42$ and $\frac{\psi(\phi_u)}{f(\phi_u)} = 0.1$, imply $\phi_u = 6.679$ and $\psi_0 = 0.0143 \times 6.679^{-\psi_1}$. Once ψ_1 is determined, ψ_0 is determined. The remaining parameters, (ψ_1, p_0, c_1), minimize the distance *drest*, where

$$drest\ (\%) \equiv \left[\left(\left(\frac{u}{0.065} - 1\right) \times 10\right)^2 + \left(\frac{f(\phi_u)}{0.42 f(k^*)} - 1\right)^2\right]^{1/2} \times 100\%.$$

The distance of $\frac{u}{0.065}$ from the target is multiplied by 10 in order to match the target on the unemployment rate closely. When ψ_1 varies, ψ_0 varies according to the relationship determined above: $\psi_0 = 0.0143 \times 6.679^{-\psi_1}$. Grid search is used to solve the above problem.

To construct the grid of (ψ_1, p_0, c_1), I approximate the optimality conditions of (ϕ_u, p_u) to get the pre-estimates of the parameters. Start with the envelope condition of ϕ_u:

$$V'(\phi_u) = \frac{f'(\phi_u) + V''(\phi_u)\, i(\phi_u)}{r + \delta + \lambda p(\phi_u)}.$$

Ignoring the effects of (V''', i', p') on V'', I get

$$V''(\phi_u) \approx \frac{f''(\phi_u)}{r + \delta + \lambda p(\phi_u)}.$$

This approximation is likely to over-estimate $[-V''(\phi_u)]$ by ignoring $i' < 0$ and $p' < 0$. Substituting this approximation into the envelope condition of ϕ_u yields:

$$V'(\phi_u) \approx \frac{f'(\phi_u)}{r + \delta + \lambda p(\phi_u)} + \frac{f''(\phi_u)\, i(\phi_u)}{[r + \delta + \lambda p(\phi_u)]^2}.$$

The first order condition of ϕ_u implies

$$\theta(p_u) \approx \frac{p_u}{\psi'(\phi_u)} V'(\phi_u).$$

Suppose that $V_u = V(k_0)$ for some k_0. Approximate $[V(k_0) - V(\phi_u)]$ near ϕ_u to the second order:

$$V(k_0) - V(\phi_u) \approx V'(\phi_u)(k_0 - \phi_u) + \frac{V''(\phi_u)}{2}(k_0 - \phi_u)^2.$$

Substituting this approximation into the first order condition of p_u and dividing the result by the first order condition of ϕ_u, I get:

$$\phi_u - k_0 - \frac{V''(\phi_u)}{2V'(\phi_u)}(\phi_u - k_0)^2 \approx \frac{\psi(\phi_u)}{\varepsilon_u \psi'(\phi_u)},$$

where I have used $\frac{p_u \theta'(p_u)}{\theta(p_u)} = \frac{1}{\varepsilon_u}$. To use this equation to determine ψ_1, I need the values of k_0, $i(\phi_u)$ and $p(\phi_u)$, which are endogenous. From the definition of k_0, $V(k_0) = V_u$. Note that the equivalent job type of an unemployed worker is k_u. An employed worker has the opportunity of getting the job type upgraded but an unemployed worker does not. This difference is likely to induce $V(k_0) < V(k_u)$. I set $k_0 = \frac{2}{3\lambda} k_u$. Also, set $i(\phi_u) = 0.25\phi_u$ and $\lambda p(\phi_u) = 0.70 p_u$ in the above approximation. Then, the above approximation yields the pre-estimate ψ_1. The first-order condition of ϕ_u yields $\theta(p_u)$. The matching function implies the pre-estimate of p_0 as $p_0 = \theta(p_u)\left[(p_u)^{-\rho} - 1\right]^{1/\rho} = 0.374$. The first-order condition of i yields the pre-estimate of c_1 as $c_1 = V'(\phi_u) / [2i(\phi_u)]$, where the above approximations for $V'(\phi_u)$ and $i(\phi_u) = 0.25\phi_u$ are used.

A grid is chosen for each parameter in (ψ_1, p_0, c_1) around the pre-estimates. The distance *drest* is minimized at $u = 0.0649$ and $f(\phi_u)/f(k^*) = 0.424$. The values of (ψ_1, p_0, c_1)

that minimize *drest* are reported in Table 1. Figure F.1 depicts the distance *drest* over the grid of (ψ_1, p_0) in the upper panel and of (p_0, c_1) in the lower panel. Notice that ψ_1 and p_0 are identified well, since moving away from the identified values increases *drest* significantly. The parameter c_1 is also identified well, despite that the distance *drest* depends on c_1 r

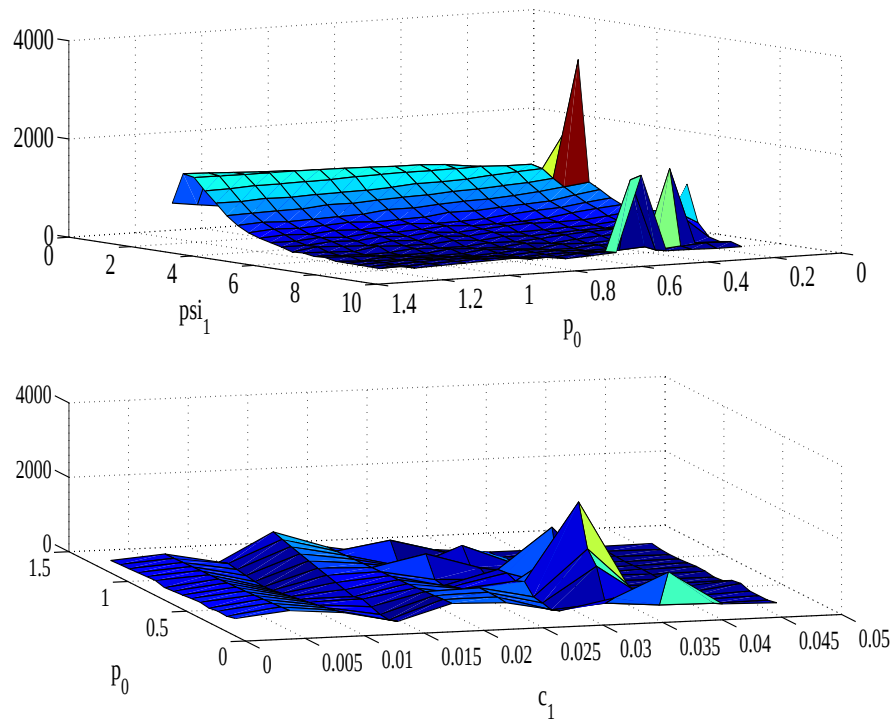

Figure F.1 The distance between $\left(\frac{u}{0.065}, \frac{f(\phi_u)}{0.42 f(k^*)}\right)$ and $(1, 1)$

F.2. Computation Procedure

The following procedure computes the efficient allocation first and then the distributions:

Step 1. Specify the parameter values, the forms of exogenous functions, and the domain of k. Compute k^*. Construct the Chebyshev basis for the value function.

Step 2. Compute the social value function and its derivative with Chebyshev projection. The same step yields the policy functions of efficient job upgrading and job search. Use grid search to solve the maximization problems in (2)-(4) below.

(1) Specify an initial function $V(k)$ and a number V_u. If $V(k)$ is not weakly increasing, modify it so. Compute the derivative $V'(k)$.

(2) Solve the efficient job upgrading problem (2.5) for the policy function $i(k)$ and the implied return on upgrading $v(k)$.

(3) Solve the efficient job search problem (2.3) for the policy functions $(\phi(k), p(k))$ and the implied return on search $s(k)$.

(4) Solve the efficient job search problem of an unemployed worker for the efficient allocation (ϕ_u, p_u) and the implied value of unemployment V_u.

(5) Update the function $V(k)$ by (2.2) and the number V_u by (2.6). Repeat (1)-(4) until convergence.

Step 3. Compute k_T and k_c. Use Chebyshev projection to spline the functions $V(k)$, $i(k)$, $\phi(k)$ and $p(k)$. Ensure $i(k) \geq 0$, $\phi(k) \geq k$, $p(k) \geq 0$, and $p(k) = 0$ for all $k \geq k_T$. Also compute the projection for the function $\frac{1}{i(k)}$, to be used later. Compute V' and ensure $V'(k) \geq 0$.

Step 4. Compute (k, i, ϕ, p) as functions of tenure, i.e., $\left(k(t), \hat{\imath}(t), \hat{\phi}(t), \hat{p}(t)\right)$. The function $k(t)$ is given by the differential equation $\frac{\mathrm{d}k(t)}{\mathrm{d}t} = i(k(t))$. However, numerically integrating this equation can be complicated. Instead, compute the inverse of $k(t)$ which can be approximated well by using the differential equation: $t'(k) = \frac{1}{i(k)}$. Since $t(\phi_u) = 0$, then

$$t(k) = \int_{\phi_u}^{k} \frac{1}{i(z)} \mathrm{d}z.$$

Since the projection for the function $\frac{1}{i(k)}$ was computed above, one can directly integrate the basis in the projection. Numerically invert $t(k)$ to get $k(t)$. Note that there is no iteration here. Once the function $k(t)$ is computed, evaluating the splines of $i(k)$, $\phi(k)$ and $p(k)$ on $k(t)$ yields:

$$\hat{\imath}(t) = i(k(t)), \ \hat{\phi}(t) = \phi(k(t)), \ \hat{p}(t) = p(k(t)).$$

Step 5. Compute the steady state distribution. The partial differential equation, (4.3), characterizes the distribution of employed workers over (k, t). Rather than solving this equation numerically, which yields significant imprecision, I use the policy functions to simulate the joint distribution by the Monte Carlo method. The procedure is as follows:

(1) Discretize the values of the unemployment duration as $A = \{0, \mathrm{d}t, 2\mathrm{d}t, ..., t_M \mathrm{d}t\}$, where $\mathrm{d}t > 0$ is a small number and t_M is a large integer. Tenure is also discretized as set A. Job types are discretized as the set $k(A) = \{k_0, k_1, k_2, ..., k_M\}$, where $k_0 = \phi_u$ and $k_j = k(j\mathrm{d}t)$. A generic bin for a worker is (m, j) where k_m is the job type and $j\mathrm{d}t$ is the actual tenure.

(2) Set the number of individuals to be simulated as a large number N, say, $N = 10^6$. Divide them into unemployed $N_u = n_u N$ and employed $N_e = (1 - n_u) N$. Specify the initial distribution of employed workers over the grid $A \times k(A)$ and the initial distribution of unemployed workers over the grid A. Use these distributions as the initial cumulative average of the distributions across simulations.

(3) Take each worker from the cumulative average distribution across previous simulations and simulate the worker's transition in one period. For each worker employed at

$(k_m, j\mathrm{d}t)$, make a random draw on job switching according to the probability $\lambda p(k_m)\mathrm{d}t$ and a random draw of exogenous separation according to the probability $\delta\mathrm{d}t$. If the worker separates into unemployment, set the unemployment duration to 0. If the worker switches jobs, set the worker's tenure to 0 and the new job type to the level on the grid $k(A)$ that is closest to $\phi(k_m)$. If the worker stays in the firm, increase the worker's tenure to $(j+1)\mathrm{d}t$ and the job type to k_{m+1}. Similarly, for an unemployed worker, make a random draw on job-finding according to the probability $p_u\mathrm{d}t$. If the worker finds a job, set the worker's tenure to 0 and the job type to $k_0 = \phi_u$.

(4) Calculate the distribution of employed workers over $A \times k(A)$ and the distribution of unemployed workers over A.

(5) Combine the newly simulated distributions in (4) with the previous cumulative average distribution of workers to update the cumulative average distribution of workers across simulations.

(6) Repeat (3)-(5) until convergence of the cumulative average distribution of workers.

(7) With the distribution obtained in (6), compute the distribution of employed workers over (k, t) and unemployed workers over the unemployment duration. Compute other statistics from the distribution.

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