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Global identification from the equilibrium manifold under incomplete markets*

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Abstract

We show that, even under incomplete markets, the equilibrium manifold identifies individual demands everywhere in their domains. For this, we assume conditions of smoothness, interiority and regularity, and avoid observational requirements at the individual level. It is crucial that there be date-zero consumption. As a by-product, we develop some duality theory under incomplete markets.

The transfer paradox, first pointed out by Leontief (1936), and generalized by Donsimoni and Polemarchakis (1994), illustrates the importance of identifying the fundamentals of an economy from observable data. Under the hypothesis of general equilibrium, the aggregate demand function cannot be assumed to be observed: at equilibrium prices aggregate demand is, by definition, equal to aggregate endowment. Demand, either individual or aggregate, cannot be observed for out-of-equilibrium prices. One can observe, however, equilibrium prices and individual incomes. In this paper we address the problem of identifying individual preferences from the equilibrium manifold of a dynamic economy with financial markets (even when the latter are incomplete).

For the case of complete markets, positive results have been obtained by Balasko [1999], Chiappori et al [2000] and [2004], and Matzkin [2003]. Balasko's result has been criticized for making very strong observational assumptions: that one can observe equilibrium prices in situations in which

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endowment is zero for all individuals but one. Under additional assumptions, Chiappori et al obtain local identification of individual demands using a constructive argument. Matzkin determines the largest class of fundamentals for which identification is possible. Her argument, however, is not constructive.

The case of incomplete markets is more cumbersome. Kubler et al [2000] and [2002] use the implicit function theorem to identify the aggregate demand function from the equilibrium manifold (hence they obtain a local identification of the aggregate demand function). They proceed to identify individual demands (locally) from the aggregate demand and then use Geanakoplos and Polemarchakis [1990] to identify preferences from individual demand functions. Therefore, they are able to obtain local identification of individual preferences when asset markets are incomplete.

When we have numeraire assets, we identify individual demands globally. For general real assets structures, we conjecture that our results hold generically in the space prices and endowments. We extend Balasko's idea on how to recover the aggregate demand function from the equilibrium manifold to the case of (possibly incomplete) asset markets, hence we avoid using the implicit function theorem. We then use a slightly different argument from Kubler et al.'s to identify individual demands from the aggregate demand function and we also avoid using Balasko's strong observational assumption pointed out before.

As a by-product, we develop some basic duality theory for incomplete markets.

1 Testability, identification and recoverability

Let $\mathcal{F}$ and X be nonempty sets. A model is a correspondence $\mathcal{M} : \mathcal{F} \rightrightarrows X$, in which $F \in \mathcal{F}$ is a vector of fundamentals and $x \in X$ is a vector of (exogenous and endogenous) variables. A data set is $D \subseteq X$.

A model $\mathcal{M}$ is falsifiable if

$$(\exists D \subseteq X)\,(\forall F \in \mathcal{F}) : D \cap (X \backslash \mathcal{M}\,(F)) \neq \varnothing$$

A data set is $D \subseteq \mathcal{P}\,(X) \setminus \{\varnothing\}$. Data set D is rationalizable if

$$(\exists F \in \mathcal{F}) : D \subseteq \mathcal{M}\,(F)$$

A test T of a model is a proposition such that

$$D \text{ is rationalizable} \implies T \text{ is true}$$

and T is strongest if

$$T \text{ is true} \implies D \text{ is rationalizable}$$

That is, a model is falsifiable if one can reject the hypothesis that a data set can be explained by the model and some fundamentals: there exist nonrationalizable tests. A test is a statement which is necessary for the existence of fundamentals that are able to explain a data set, while a test is strongest if any other test is implied by it. Obviously, interesting tests are nontautological propositions different from 'D is rationalizable,' and if a tautology is a strongest test of a model, then the model is unfalsifiable.

Definition 1 *A model $\mathcal{M}$ identifies fundamentals if*

$$(F, F' \in \mathcal{F}) \wedge (F \neq F') \Longrightarrow \mathcal{M}(F) \neq \mathcal{M}(F')$$

Definition 2 *A model $\mathcal{M}$ allows recoverability of fundamentals if it identifies fundamentals and for every $F \in \mathcal{F}$ there exists an algorithm with input $\mathcal{M}(F)$ and output F.*

Notice that falsifiability (and testability) is a question of existence only, while identification (and hence recoverability) presumes existence and studies uniqueness of fundamentals only. Identification occurs when any difference in fundamentals suffices to imply differences in the relation that the model imposes on its variables.

2 Abstract results

Fix $L, S \in \mathbb{N}$. Define $n = L(S+1)$ and $S_{++}^{n-1} = \left\{P \in \mathbb{R}_{++}^{n} : P_{0,1} = 1\right\}$. We write any $P \in S_{++}^{n-1}$ as $P = (P_0, P_{\mathbf{1}})$, where $P_{\mathbf{1}} = (P_1, ..., P_S) \in \mathbb{R}_{++}^{LS}$. Fix $J \in \mathbb{N}$ functions $\left(V^j : \mathbb{R}_{++}^{LS} \longrightarrow \mathbb{R}^S\right)_{j=1}^{J}$ such that

$$\left(\forall P_{\mathbf{1}} \in \mathbb{R}_{++}^{LS}\right) : rank\left(V\left(P_{\mathbf{1}}\right)\right) = J$$

where $V(P_{\mathbf{1}}) = \left[V^1(P_{\mathbf{1}}), ..., V^J(P_{\mathbf{1}})\right]$

Let $\mathfrak{f}$ be the class of functions $f : S_{++}^{n-1} \times \mathbb{R}_{+}^{n} \longrightarrow \mathbb{R}_{+}^{n}$ such that[1]

1. $\left(\forall P \in S_{++}^{n-1}\right)\left(\forall w \in \mathbb{R}_{+}^{n}\right)$:

$$P \cdot f(P, w) = P \cdot w \wedge P_{\mathbf{1}} \boxdot \left(f(P, w)_{\mathbf{1}} - w_{\mathbf{1}}\right) \in \langle V(P_1) \rangle$$

2. $\left(\forall P \in S_{++}^{n-1}\right)\left(\forall w, w' \in \mathbb{R}_{+}^{n}\right)$:

$$\left(P \cdot w = P \cdot w' \wedge P_{\mathbf{1}} \boxdot \left(w'_{\mathbf{1}} - w_{\mathbf{1}}\right) \in \langle V(P_1) \rangle\right) \Longrightarrow f(P, w) = f(P, w')$$

[1] For $(\rho, \gamma) = ((\rho_1, ..., \rho_S), (\gamma_1, ..., \gamma_S)) \in \mathbb{R}^{LS} \times \mathbb{R}^{LS}$, denote

$$\rho \boxdot \Delta = \begin{bmatrix} \rho_1 \cdot \Delta_1 \\ \vdots \\ \rho_S \cdot \Delta_S \end{bmatrix}$$

Let $\mathcal{F}_1$ be the class of functions $F : S^{n-1}_{++} \times \mathbb{R}^{nI}_{+} \longrightarrow \mathbb{R}^{n}_{+}$ for which there exists $\left(f^i\right)_{i=1}^{I} \in \mathfrak{f}^I$ such that

$$\left(\forall P \in S^{n-1}_{++}\right) \left(\forall \left(w^i\right)_{i=1}^{I} \in \mathbb{R}^{nI}_{+}\right) : F\left(P, \left(w^i\right)_{i=1}^{I}\right) = \sum_{i=1}^{I} f^i\left(P, w^i\right)$$

Define $\mathcal{M}_1 : \mathcal{F}_1 \rightrightarrows S^{n-1}_{++} \times \mathbb{R}^{nI}_{+}$ by

$$\mathcal{M}_1\left(F\right) = \left\{\left(P, \left(w^i\right)_{i=1}^{I}\right) \in S^{n-1}_{++} \times \mathbb{R}^{nI}_{+} : F\left(P, \left(w^i\right)_{i=1}^{I}\right) = \sum_{i=1}^{I} w^i\right\}$$

(In this model, unobserved fundamentals $X = \mathcal{F}_1$ are functions analogous to aggregate demands, and the model maps into sets analogous to equilibrium manifolds.)

Theorem 1 *Let $F \in \mathcal{F}_1$. For each $(P, w) \in S^{n-1}_{++} \times \mathbb{R}^{nI}_{+}$, there exists $\left(\widehat{w}^i\right)_{i=1}^{I} \in \mathbb{R}^{nI}_{+}$ such that*

1. $\left(P, \left(\widehat{w}^i\right)_{i=1}^{I}\right) \in \mathcal{M}_1\left(F\right)$
2. *For all i, $P_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) \in \left\langle V\left(P_{\mathbf{1}}\right)\right\rangle$ and $P \cdot \widehat{w}^i = P \cdot w^i$*

Moreover,

$$F\left(P, w\right) = \sum_{i=1}^{I} \widehat{w}^i$$

where $\left(\widehat{w}^i\right)_{i=1}^{I}$ is any one of the elements of $\mathbb{R}^{nI}_{+}$ that satisfy the previous two conditions.

Proof. Fix $\left(f^i\right)_{i=1}^{I} \in \mathfrak{f}^I$ such that

$$\left(\forall P \in S^{n-1}_{++}\right) \left(\forall \left(w^i\right)_{i=1}^{I} \in \mathbb{R}^{nI}_{+}\right) : F\left(P, \left(w^i\right)_{i=1}^{I}\right) = \sum_{i=1}^{I} f^i\left(P, w^i\right)$$

and define $\widehat{w}^i = f^i(P, w^i)$. Then $\left(P, \left(\widehat{w}^i\right)_{i=1}^{I}\right) \in \mathcal{M}_1\left(F\right)$, since by construction, for each i, $\widehat{w}^i = f^i\left(P, w^i\right) = f^i\left(P, \widehat{w}^i\right)$, which implies that

$$\begin{aligned} F\left(P, \left(\widehat{w}^i\right)_{i=1}^{I}\right) &= \sum_{i=1}^{I} f^i\left(P, \widehat{w}^i\right) \\ &= \sum_{i=1}^{I} \widehat{w}^i \end{aligned}$$

which proves condition 1. Condition 2 is straightforward.

Now, if $\left(P, \left(\widehat{w}^i\right)_{i=1}^I\right) \in \mathcal{M}_1(F)$ satisfies condition 2, then, by definition and construction,

$$\begin{aligned}\sum_{i=1}^I \widehat{w}^i &= \sum_{i=1}^I f^i(P, \widehat{w}^i) \\ &= \sum_{i=1}^I f^i(P, w^i) \\ &= F(P, w)\end{aligned}$$

■

Corollary 1 *Model $\mathcal{M}_1$ identifies fundamentals.*

Proof. Suppose that $F, F' \in \mathcal{F}_1$ and $\mathcal{M}_1(F) = \mathcal{M}_1(F')$. Let $(P, w) \in S_{++}^{n-1} \times \mathbb{R}_+^{nI}$. It follows from theorem 1 that for some $\left(\widehat{w}^i\right)_{i=1}^I \in \mathbb{R}_+^{nI}$,

$$\begin{aligned}\left(P, \left(\widehat{w}^i\right)_{i=1}^I\right) &\in \mathcal{M}_1(F) \\ (\forall i \in \{1, ..., I\}) &: P_{\mathbf{1}} \boxdot \left(\widehat{w}_{\mathbf{1}}^i - w_{\mathbf{1}}^i\right) \in \langle V(P_{\mathbf{1}})\rangle \wedge P \cdot \widehat{w}^i = P \cdot w^i \\ F(P, w) &= \sum_{i=1}^I \widehat{w}^i\end{aligned}$$

Since $\mathcal{M}_1(F) = \mathcal{M}_1(F')$, it follows from the fist two conditions and the second part of theorem 1 that $F'(P, w) = \sum_{i=1}^I \widehat{w}^i$ and then from the third condition that $F'(P, w) = F(P, w)$. ■

Let $\mathfrak{f}^0 \subseteq \mathfrak{f}$ be the class of functions $f : S_{++}^{n-1} \times \mathbb{R}_{++}^n \longrightarrow \mathbb{R}_{++}^n$ which are continuously differentiable and satisfy that:

1. $\left(\forall (P, w) \in S_{++}^{n-1} \times \mathbb{R}_{++}^L\right) (\forall (s, l), (s', l') \in (\{0, ..., S\} \times \{1, ..., L\}) \setminus \{(0, 1)\}) :$

$$\frac{\partial f_{s,l}^i}{\partial P_{s',l'}}(P, w) + \left(f_{s',l'}^i(P, w) - w_{s',l'}^i\right) \frac{\partial f_{s,l}^i}{\partial w_{0,1}^i}(P, w) = \frac{\partial f_{s',l'}^i}{\partial P_{s,l}}(P, w) + \left(f_{s,l}^i(P, w) - w_{s,l}^i\right) \frac{\partial f_{s',l'}^i}{\partial w_{0,1}^i}(P, w)$$

2. $\left(\forall P \in S_{++}^{n-1}\right) \left(\exists w \in \mathbb{R}_{++}^n\right) (\exists (s, l), (s', l') \in (\{0, ..., S\} \times \{1, ..., L\}) \setminus \{(0, 1)\}) :$

$$\begin{vmatrix} \frac{\partial^2 f_{s,l}^i}{\partial\left(w_{0,1}^i\right)^2}(P, w) & \frac{\partial^2 f_{s',l'}^i}{\partial\left(w_{0,1}^i\right)^2}(P, w) \\ \frac{\partial^3 f_{s,l}^i}{\partial\left(w_{0,1}^i\right)^3}(P, w) & \frac{\partial^3 f_{s',l'}^i}{\partial\left(w_{0,1}^i\right)^3}(P, w) \end{vmatrix} \neq 0$$

Also, denote $\mathcal{F}_2 = \left(\mathfrak{f}^0\right)^I$ and let $\mathcal{F}_1^0 \subseteq \mathcal{F}_1$ be the class of functions $F : S_{++}^{n-1} \times \mathbb{R}_{++}^{nI} \longrightarrow \mathbb{R}_{++}^n$ for which there exists $\left(f^i\right)_{i=1}^I \in \mathcal{F}_2$ such that

$$\left(\forall P \in S_{++}^{n-1}\right)\left(\forall \left(w^i\right)_{i=1}^I \in \mathbb{R}_{++}^{nI}\right) : F\left(P, \left(w^i\right)_{i=1}^I\right) = \sum_{i=1}^I f^i\left(P, w^i\right)$$

Define $\mathcal{M}_2 : \mathcal{F}_2 \rightrightarrows \mathcal{F}_1^0$ by

$$\mathcal{M}_2\left(\left(f^i\right)_{i=1}^I\right) = \left\{F \in \mathcal{F}_1^0 : \left(\forall P \in S_{++}^{n-1}\right)\left(\forall \left(w^i\right)_{i=1}^I \in \mathbb{R}_{++}^{LI}\right) : F\left(P, \left(w^i\right)_{i=1}^I\right) = \sum_{i=1}^I f^i\left(P, w^i\right)\right\}$$

which is a nonempty- and single-valued correspondence. (In this model, unobserved fundamentals $X = \mathcal{F}_2$ are analogous to profiles of individual demands, and the model maps into their aggregate.)

Theorem 2 $\mathcal{M}_2$ *identifies fundamentals.*

Proof. Let $\left(f^i\right)_{i=1}^I, \left(\widetilde{f}^i\right)_{i=1}^I \in \mathcal{F}_2$ and suppose that

$$\mathcal{M}_2\left(\left(f^i\right)_{i=1}^I\right) = \mathcal{M}_2\left(\left(\widetilde{f}^i\right)_{i=1}^I\right) = \{F\}$$

Fix $i \in \{1, ..., I\}$ and let $\varphi^i : S_{++}^{n-1} \times \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^n$, $\phi^i : S_{++}^{n-1} \longrightarrow \mathbb{R}_+^n$ and $\widetilde{\phi}^i : S_{++}^{n-1} \longrightarrow \mathbb{R}_+^n$ be defined by

$$\varphi^i(P, w) = F\left(P, (\mathbf{1}, \mathbf{1}, ..., w, ..., \mathbf{1})\right)$$

where w occupies the i^{th} position on its vector,

$$\phi^i(P) = -\sum_{j=1, j\neq i}^I f^j(P, \mathbf{1})$$

and

$$\widetilde{\phi}^i(P) = -\sum_{j=1, j\neq i}^I \widetilde{f}^j(P, \mathbf{1})$$

By definition,

$$\begin{aligned}
\left(\forall (P, w) \in S_{++}^{n-1} \times \mathbb{R}_+^n\right) &: \quad f^i(P, w) = \varphi^i(P, w) + \phi^i(P) \\
\left(\forall (P, w) \in S_{++}^{n-1} \times \mathbb{R}_+^n\right) &: \quad \widetilde{f}^i(P, w) = \varphi^i(P, w) + \widetilde{\phi}^i(P)
\end{aligned}$$

Since $f^i, \widetilde{f}^i \in \mathfrak{f}^0$, it follows that for every $(s, l), (s', l') \in (\{0, ..., S\} \times \{1, ..., L\}) \setminus \{(0, 1)\}$, everywhere in $S_{++}^{n-1} \times \mathbb{R}_{+}^{n}$ and ignoring the arguments,

$$\begin{aligned}
\frac{\partial f_{s,l}^{i}}{\partial P_{s',l'}} + \left(f_{s',l'}^{i} - w_{s',l'}^{i}\right) \frac{\partial f_{s,l}^{i}}{\partial w_{0,1}^{i}} &= \frac{\partial f_{s',l'}^{i}}{\partial P_{s,l}} + \left(f_{s,l}^{i} - w_{s,l}^{i}\right) \frac{\partial f_{s',l'}^{i}}{\partial w_{0,1}^{i}} \\
\frac{\partial \widetilde{f}_{s,l}^{i}}{\partial P_{s',l'}} + \left(\widetilde{f}_{s',l'}^{i} - w_{s',l'}^{i}\right) \frac{\partial \widetilde{f}_{s,l}^{i}}{\partial w_{0,1}^{i}} &= \frac{\partial \widetilde{f}_{s',l'}^{i}}{\partial P_{s,l}} + \left(\widetilde{f}_{s,l}^{i} - w_{s,l}^{i}\right) \frac{\partial \widetilde{f}_{s',l'}^{i}}{\partial w_{0,1}^{i}}
\end{aligned}$$

Fix $P \in S_{++}^{n-1}$. Substituting,

$$\frac{\partial \varphi_{s,l}^{i}}{\partial P_{s',l'}} + \frac{\partial \phi_{s,l}^{i}}{\partial P_{s',l'}} + \left(\varphi_{s',l'}^{i} + \phi_{s',l'}^{i} - w_{s',l'}^{i}\right) \frac{\partial \varphi_{s,l}^{i}}{\partial w_{0,1}^{i}} = \frac{\partial \varphi_{s',l'}^{i}}{\partial P_{s,l}} + \frac{\partial \phi_{s',l'}^{i}}{\partial P_{s,l}} + \left(\varphi_{s,l}^{i} + \phi_{s,l}^{i} - w_{s,l}^{i}\right) \frac{\partial \varphi_{s',l'}^{i}}{\partial w_{0,1}^{i}}$$

Taking that $(s,l) \neq (0,1)$ and $(s',l') \neq (0,1)$ and deriving once and twice with respect to income gives

$$\frac{\partial^2 \varphi_{s,l}^{i}}{\partial w_{0,1}^{i} \partial P_{s',l'}} + \left(\varphi_{s',l'}^{i} + \phi_{s',l'}^{i} - w_{s',l'}^{i}\right) \frac{\partial^2 \varphi_{s,l}^{i}}{\partial \left(w_{0,1}^{i}\right)^2} = \frac{\partial^2 \varphi_{s',l'}^{i}}{\partial w_{0,1}^{i} \partial P_{s,l}} + \left(\varphi_{s,l}^{i} + \phi_{s,l}^{i} - w_{s,l}^{i}\right) \frac{\partial^2 \varphi_{s',l'}^{i}}{\partial \left(w_{0,1}^{i}\right)^2}$$

and

$$\begin{aligned}
& \frac{\partial^3 \varphi_{s,l}^{i}}{\partial \left(w_{0,1}^{i}\right)^2 \partial P_{s',l'}} + \frac{\partial \varphi_{s',l'}^{i}}{\partial w_{0,1}^{i}} \frac{\partial^2 \varphi_{s,l}^{i}}{\partial \left(w_{0,1}^{i}\right)^2} + \left(\varphi_{s',l'}^{i} + \phi_{s',l'}^{i} - w_{s',l'}^{i}\right) \frac{\partial^3 \varphi_{s,l}^{i}}{\partial \left(w_{0,1}^{i}\right)^3} \\
= & \frac{\partial^3 \varphi_{s',l'}^{i}}{\partial \left(w_{0,1}^{i}\right)^2 \partial P_{s,l}} + \frac{\partial \varphi_{s,l}^{i}}{\partial w_{0,1}^{i}} \frac{\partial^2 \varphi_{s',l'}^{i}}{\partial \left(w_{0,1}^{i}\right)^2} + \left(\varphi_{s,l}^{i} + \phi_{s,l}^{i} - w_{s,l}^{i}\right) \frac{\partial^3 \varphi_{s',l'}^{i}}{\partial \left(w_{0,1}^{i}\right)^3}
\end{aligned}$$

Doing the same for $\widetilde{f}^{i}$, we can rewrite the resulting systems as

$$\begin{aligned}
\Delta_{(l,s),(l',s')}(w) \begin{bmatrix} \phi_{s',l'}^{i} \\ \phi_{s,l}^{i} \end{bmatrix} &= \Gamma_{(l,s),(l',s')}(w) \\
\Delta_{(l,s),(l',s')}(w) \begin{bmatrix} \widetilde{\phi}_{s',l'}^{i} \\ \widetilde{\phi}_{s,l}^{i} \end{bmatrix} &= \Gamma_{(l,s),(l',s')}(w)
\end{aligned}$$

where

$$\Delta_{(l,s),(l',s')}(w) = \begin{bmatrix} \frac{\partial^2 \varphi_{s,l}^{i}}{\partial \left(w_{0,1}^{i}\right)^2}(P,w) & -\frac{\partial^2 \varphi_{s',l'}^{i}}{\partial \left(w_{0,1}^{i}\right)^2}(P,w) \\ \frac{\partial^3 \varphi_{s,l}^{i}}{\partial \left(w_{0,1}^{i}\right)^3}(P,w) & -\frac{\partial^3 \varphi_{s',l'}^{i}}{\partial \left(w_{0,1}^{i}\right)^3}(P,w) \end{bmatrix}$$

and $\Gamma_{(l,s),(l',s')}(w)$ is a 2×1 matrix with first component

$$\frac{\partial^2 \varphi_{s',l'}^{i}}{\partial w_{0,1}^{i} \partial P_{s,l}} - \frac{\partial^2 \varphi_{s,l}^{i}}{\partial w_{0,1}^{i} \partial P_{s',l'}} + \left(\varphi_{s,l}^{i} - w_{s,l}^{i}\right) \frac{\partial^2 \varphi_{s',l'}^{i}}{\partial \left(w_{0,1}^{i}\right)^2} - \left(\varphi_{s',l'}^{i} - w_{s',l'}^{i}\right) \frac{\partial^2 \varphi_{s,l}^{i}}{\partial \left(w_{0,1}^{i}\right)^2}$$

and second component

$$\frac{\partial^3 \varphi^i_{s',l'}}{\partial \left(w^i_{0,1}\right)^2 \partial P_{s,l}} - \frac{\partial^3 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2 \partial P_{s',l'}} + \frac{\partial \varphi^i_{s,l}}{\partial w^i_{0,1}} \frac{\partial^2 \varphi^i_{s',l'}}{\partial \left(w^i_{0,1}\right)^2}$$
$$- \frac{\partial \varphi^i_{s',l'}}{\partial w^i_{0,1}} \frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2} + \left(\varphi^i_{s,l} - w^i_{s,l}\right) \frac{\partial^3 \varphi^i_{s',l'}}{\partial \left(w^i_{0,1}\right)^3} - \left(\varphi^i_{s',l'} - w^i_{s',l'}\right) \frac{\partial^3 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^3}$$

Moreover, since $f^i \in \mathfrak{f}^0$, for some $w \in \mathbb{R}^n_+$, $s, s' \in \{1, ..., S\}$ and $l, l' \in \{1, ..., L\}$, matrix $\Delta_{(l,s),(l',s')}(w)$ is invertible, which gives $\phi^i_{s,l}(P) = \widetilde{\phi}^i_{s,l}(P)$ and $\phi^i_{s',l'}(P) = \widetilde{\phi}^i_{s',l'}(P)$. For every other $(l'', s'') \in (\{1, ..., L\} \times \{0, ..., S\}) \setminus \{(0, 1)\}$,

$$\begin{aligned}
\phi^i_{l'',s''}(P) &= \frac{\frac{\partial^2 \varphi^i_{l'',s''}}{\partial w^i_{0,1} \partial P_{s,l}} - \frac{\partial^2 \varphi^i_{s,l}}{\partial w^i_{0,1} \partial P_{l'',s''}} + \left(\varphi^i_{s,l} + \phi^i_{s,l} - w^i_{s,l}\right) \frac{\partial^2 \varphi^i_{l'',s''}}{\partial \left(w^i_{0,1}\right)^2} - \left(\varphi^i_{l'',s''} - w^i_{l'',s''}\right) \frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2}}{\frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2}} \\
&= \frac{\frac{\partial^2 \varphi^i_{l'',s''}}{\partial w^i_{0,1} \partial P_{s,l}} - \frac{\partial^2 \varphi^i_{s,l}}{\partial w^i_{0,1} \partial P_{l'',s''}} + \left(\varphi^i_{s,l} + \widetilde{\phi}^i_{s,l} - w^i_{s,l}\right) \frac{\partial^2 \varphi^i_{l'',s''}}{\partial \left(w^i_{0,1}\right)^2} - \left(\varphi^i_{l'',s''} - w^i_{l'',s''}\right) \frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2}}{\frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2}} \\
&= \widetilde{\phi}^i_{l'',s''}(P)
\end{aligned}$$

where we have assumed, without loss of generality, that $\frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2} \neq 0$. Finally, it follows from Walras' law that $\phi^i_{0,1}(P) = \widetilde{\phi}^i_{0,1}(P)$.

By construction, then, $\phi^i = \widetilde{\phi}^i$ and $f^i = \widetilde{f}^i$.

It follows that $\left(f^i\right)^I_{i=1} = \left(\widetilde{f}^i\right)^I_{i=1}$. ■

3 The Incomplete Markets Model

Consider the canonical, two period, multigood, incomplete markets model with financial assets. There are $S+1$ states of nature, $s = 0, ..., S$,[2] I individuals, $i = 1, ..., I$, and $L \geqslant 2$ commodities available in each state, $l = 1, ..., L$. Again, denote $L(S+1)$ by n and define the commodity space as $\mathbb{R}^n_+$.

3.1 Financial Markets

A financial asset is a contract $v \in \mathbb{R}^S$ that promises delivery of an amount $v_s \in \mathbb{R}$ of the numeraire at state of nature $s = 1, ..., S$. Let good 1 be the numeraire and let $P \in \mathbb{R}^n_{++}$ denote the vector of spot prices, where $P_s = (P_{s,1}, .., P_{s,L}) \in \mathbb{R}^L_{++}$ and $P_{s,l}$ denotes the (current value of) price payable in state s for one unit of good l. Without loss of generality, normalize prices so that $P_{0,1} = 1$.

[2] State $s = 0$ is used to denote date zero.

Let $v^1, ..., v^J$ be $J \geq 1$ financial assets, define $V(P_{\mathbf{1}})$ as the matrix of income transfers:

$$\begin{aligned} V(P_{\mathbf{1}}) &= \begin{bmatrix} P_{1,1} & 0 & \cdots & 0 \\ 0 & P_{2,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{S,1} \end{bmatrix} V \\ &= \begin{bmatrix} V_1(P_{\mathbf{1}}) \\ V_2(P_{\mathbf{1}}) \\ \vdots \\ V_S(P_{\mathbf{1}}) \end{bmatrix} \end{aligned}$$

where

$$V = \begin{bmatrix} v_1^1 & \cdots & v_1^J \\ \vdots & \ddots & \vdots \\ v_S^1 & \cdots & v_S^J \end{bmatrix}$$

The space of income transfers is $\langle V(P_{\mathbf{1}}) \rangle$, the column span of $V(P_{\mathbf{1}})$:

$$\langle V(P_{\mathbf{1}}) \rangle = \left\{ t \in \mathbb{R}^S : \left(\exists z \in \mathbb{R}^J \right) : t = V(P_{\mathbf{1}}) z \right\}$$

In general, as $P_{\mathbf{1}}$ changes, $\langle V(P_{\mathbf{1}}) \rangle$ changes. By construction, however, for $P_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, the dimension of $\langle V(P_{\mathbf{1}}) \rangle$ is always equal to the rank of V.

Assume the following:

Condition 1 V *has full column rank.*

3.2 No-arbitrage equilibrium manifold

Let $P \in \mathbb{R}^n_{++}$ denote date-zero present value prices (see Magill and Shafer, p. 1534), where $P = (P_0, ..., P_S)$ and for every s, $P_s = (P_{s,1} ..., P_{s,L})$, and let $w \in \mathbb{R}^n_+$ represent an endowment of commodities.

For $P \in S^{n-1}_{++}$ and $w \in \mathbb{R}^n_+$, define the budget

$$B(P, w; V) = \left\{ x \in \mathbb{R}^n_+ : \sum_{s=0}^{S} P_s \cdot (x_s - w_s) \leq 0 \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}}) \rangle \right\}$$

Future consumption $x_{\mathbf{1}} \in \mathbb{R}^{LS}_+$ is financially feasible at future prices and endowments $(P_{\mathbf{1}}, w_{\mathbf{1}}) \in \mathbb{R}^{LS}_{++} \times \mathbb{R}^{LS}_+$ if the second condition in the definition of $B(P, w; V)$ is satisfied: there is a portfolio of assets, $z \in \mathbb{R}^J$, that delivers the transfers necessary to finance $x_{\mathbf{1}}$.[3]

[3] If $\dim \langle V(P_{\mathbf{1}}) \rangle = S$, or equivalently $\dim \langle V \rangle = S$, the second condition that defines $B(P, w; V)$ is nonbinding. This is the case of complete markets.

An individual is a (utility) function $u : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$. Let $\mathcal{U}$ be the class of individuals u that are continuous, monotone and strongly quasi-concave. Define the individual demand functional $\lambda(\bullet; V) : S^{n-1}_{++} \times \mathbb{R}^n_+ \times \mathcal{U} \to \mathbb{R}^n_+$ as:

$$\lambda(P, w, u; V) = \arg\max \{u(x) : x \in B(P, w; V)\}$$

Assume that there are $I \in \mathbb{N}$ individuals. Define the aggregate demand functional, $\Lambda(\bullet; V) : S^{n-1}_{++} \times \mathbb{R}^{nI}_+ \times \mathcal{U}^I \to \mathbb{R}^n_+$, as:

$$\Lambda(P, w, \left(u^i\right)_{i=1}^I; V) = \sum_{i=1}^I \lambda(P, w^i, u^i; V)$$

Both λ and Λ are well defined, since for $(P, w) \in S^{n-1}_{++} \times \mathbb{R}^n_+$, $B(P, w; V)$ is nonempty, compact and convex, and each $u \in \mathcal{U}$ is continuous and strongly quasi-concave.

Definition 3 *A no-arbitrage equilibrium for the economy* $E = \left(\left(u^i, w^i\right)_{i=1}^I, V\right)$ *is a pair* $(x, P) \in \mathbb{R}^{nI}_+ \times S^{n-1}_{++}$ *such that:*

1. *For every* i, $x^i = \lambda(P, w^i, u^i; V)$
2. $\Lambda(P, w, \left(u^i\right)_{i=1}^I; V) = \sum\limits_{i=1}^I w^i$

Notice that, since V has full column rank, in the previous definition we need not explicitly consider portfolios.

Definition 4 *Given* $\left(u^i\right)_{i=1}^I \in \mathcal{U}^I$ *and* V, *the no-arbitrage equilibrium manifold (for short, Equilibrium Manifold) is:*

$$M\left(\left(u^i\right)_{i=1}^I, V\right) = \left\{(P, w) \in S^{n-1}_{++} \times \mathbb{R}^{nI}_+ : \Lambda(P, w, \left(u^i\right)_{i=1}^I, V) = \sum_{i=1}^I w^i\right\}$$

4 Identification of fundamentals under incomplete markets

Let $\mathbf{f}$ be the class of class of functions $f : S^{n-1}_{++} \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+$ such that

$$(\exists u \in \mathcal{U}) \left(\forall (P, w) \in S^{n-1}_{++} \times \mathbb{R}^n_+\right) : f(P, w) = \arg\max \{u(x) : x \in B(P, w; V)\}$$

Let $\mathbf{F}_1$ be the class of functions $F : S^{n-1}_{++} \times \mathbb{R}^{nI}_+ \longrightarrow \mathbb{R}^n_+$ for which there exists $\left(f^i\right)_{i=1}^I \in \mathbf{f}^I$ such that

$$\left(\forall P \in S^{n-1}_{++}\right) \left(\forall \left(w^i\right)_{i=1}^I \in \mathbb{R}^{nI}_+\right) : F\left(P, \left(w^i\right)_{i=1}^I\right) = \sum_{i=1}^I f^i\left(P, w^i\right)$$

Define the Equilibrium Manifold Correspondence $M_1 : \mathbf{F}_1 \rightrightarrows S_{++}^{n-1} \times \mathbb{R}_+^{nI}$ by

$$M_1(F) = \left\{ \left(P, \left(w^i\right)_{i=1}^I\right) \in S_{++}^{n-1} \times \mathbb{R}_+^{nI} : F\left(P, \left(w^i\right)_{i=1}^I\right) = \sum_{i=1}^I w^i \right\}$$

The next theorem shows that one under no-arbitrage equilibrium, the aggregate demand function is uniquely determined from the equilibrium manifold, whenever individuals lie in the class $\mathcal{U}$.

Theorem 3 *The Equilibrium Manifold Correspondence identifies fundamentals.*

Proof. From corollary 1, it suffices to observe that $\mathbf{f} \subseteq \mathfrak{f}$ and that

$$(\forall F \in \mathbf{F}_1) : M_1(F) = \mathcal{M}_1(F)$$

■

Remark 1 *This is Balasko [1999] in incomplete markets. As in the complete markets case, one makes no use of any topological or differential property of the equilibrium manifold (strictly speaking, equilibrium set).*

If one is willing to assume that equilibrium prices are observable for situations in which the incomes of all individuals but one are zero, then it is straightforward that aggregate demand pins down individual demands: for all i, $f^i(P, w^i) = F(P, (\mathbf{0}, \mathbf{0}, ..., w^i, ..., \mathbf{0}))$. That is, when all agents different from i, have no income, the fact that prices are strictly positive implies no demand for agents different from i, and, therefore, that aggregate demand is agent i's individual demand. We now show that under some additional assumptions one can pin down an individual's demand without pegging everybody else's income at zero.

Let $\mathcal{U}_0$ be the subclass of individuals $u \in \mathcal{U}$ such that in the interior of the commodity space $\mathbb{R}_{++}^n$, u^i is differentiably strictly monotone and differentiably strongly quasiconcave, and for all $x \in \mathbb{R}_{++}^n$,

$$\left\{x' \in \mathbb{R}_+^n : u^i(x') \geq u^i(x)\right\} \subseteq \mathbb{R}_{++}^n.$$

Lemma 1 *Let $u \in \mathcal{U}_0$ and define $f : S_{++}^{n-1} \times \mathbb{R}_{++}^n \to \mathbb{R}_+^n$ as:*

$$f(P, w) = \arg\max \left\{u^i(x) : x \in B(P, w; V)\right\}$$

Then, for every $(P, w) \in S_{++}^{n-1} \times \mathbb{R}_{++}^n$, $f^i(P, w) \in \mathbb{R}_{++}^n$ and f^i is continuously differentiable.

Proof. For the first part, it suffices to notice that $w \in B(P, w; V)$ and that $\left\{x \in \mathbb{R}_+^n : u^i(x) \geq u^i(w)\right\} \subseteq \mathbb{R}_{++}^n$. Differentiability follows from Duffie and Shafer (1985, p. 293). ■

Theorem 4 *Let $u \in \mathcal{U}_0$ and define $f : S_{++}^{n-1} \times \mathbb{R}_{++}^n \to \mathbb{R}_+^n$ as:*

$$f(P, w) = \arg\max \{u(x) : x \in B(P, w; V)\}$$

Then, $\left(\forall (P,w) \in S_{++}^{n-1} \times \mathbb{R}_{++}^{L}\right) \left(\forall (s,l), (s',l') \in (\{0,...,S\} \times \{1,...,L\}) \setminus \{(0,1)\}\right):$

$$\frac{\partial f_{s,l}}{\partial P_{s',l'}}(P,w) + \left(f_{s',l'}(P,w) - w_{s',l'}\right) \frac{\partial f_{s,l}}{\partial w_{0,1}}(P,w) = \frac{\partial f_{s',l'}}{\partial P_{s,l}}(P,w) + \left(f_{s,l}(P,w) - w_{s,l}\right) \frac{\partial f_{s',l'}}{\partial w_{0,1}}(P,w)$$

Proof. This follows from theorems 5 and 7 in appendix 1. ∎

Let $\mathbf{f}^0$ be the class of functions $f : S_{++}^{n-1} \times \mathbb{R}_{++}^{n} \longrightarrow \mathbb{R}_{++}^{n}$ such that:[4]

1. $(\exists u \in \mathcal{U}_0)$

$$\left(\forall (P,w) \in S_{++}^{n-1} \times \mathbb{R}_{+}^{n}\right) : f^i(P,w) = \arg\max \{u(x) : x \in B(P,w;V)\}$$

2. $\left(\forall P \in S_{++}^{n-1}\right) \left(\exists w \in \mathbb{R}_{++}^{n}\right) \left(\exists (s,l), (s',l') \in (\{0,...,S\} \times \{1,...,L\}) \setminus \{(0,1)\}\right):$

$$\begin{vmatrix} \frac{\partial^2 f_{s,l}^i}{\partial \left(w_{0,1}^i\right)^2}(P,w) & \frac{\partial^2 f_{s',l'}^i}{\partial \left(w_{0,1}^i\right)^2}(P,w) \\ \frac{\partial^3 f_{s,l}^i}{\partial \left(w_{0,1}^i\right)^3}(P,w) & \frac{\partial^3 f_{s',l'}^i}{\partial \left(w_{0,1}^i\right)^3}(P,w) \end{vmatrix} \neq 0$$

Denote $\mathbf{F}_2 = \left(\mathbf{f}^0\right)^I$ and let $\mathbf{F}_1^0$ be the class of functions $F : S_{++}^{n-1} \times \mathbb{R}_{++}^{nI} \longrightarrow \mathbb{R}_{++}^{n}$ for which there exists $\left(f^i\right)_{i=1}^{I} \in \mathbf{F}_2$ such that

$$\left(\forall P \in S_{++}^{n-1}\right) \left(\forall \left(w^i\right)_{i=1}^{I} \in \mathbb{R}_{++}^{nI}\right) : F\left(P, \left(w^i\right)_{i=1}^{I}\right) = \sum_{i=1}^{I} f^i\left(P, w^i\right)$$

Define $\mathbf{M}_2 : \mathbf{F}_2 \rightrightarrows \mathbf{F}_1^0$ by

$$\mathbf{M}_2\left(\left(f^i\right)_{i=1}^{I}\right) = \left\{F \in \mathbf{F}_1^0 : \left(\forall P \in S_{++}^{n-1}\right) \left(\forall \left(w^i\right)_{i=1}^{I} \in \mathbb{R}_{++}^{LI}\right) : F\left(P, \left(w^i\right)_{i=1}^{I}\right) = \sum_{i=1}^{I} f^i\left(P, w^i\right)\right\}$$

which is a nonempty- and single-valued correspondence.

The next theorem shows that the profile of individual demands is uniquely determined from the aggregate demand function, whenever individuals lie in the class $\mathcal{U}_0$.

Theorem 5 $\mathbf{M}_2$ *identifies fundamentals.*

Proof. From lemma 1, theorem 4 and its definition, it follows that $\mathbf{F}_2 \subseteq \mathcal{F}_2$. The result then follows from theorem 2, since, by construction,

$$(\forall F \in \mathbf{F}_2) : M_2(F) = \mathcal{M}_2(F)$$

∎

[4] The second condition is usually referred to as Regularity. See appendix 2 for a demand system that satisfies it.

Remark 2 *Since V is of full rank by condition 1, the identification of individual assets demand is straightforward.*

4.1 Observability: Financial Markets Equilibrium Manifold

Identification results are useful when the model maps into observable data. In real life one does not observe equilibrium date zero present value prices but, instead, one observes (financial) equilibrium spot prices for commodities and assets.

Let $q \in \mathbb{R}^J$ be the price vector at which each assets can be bought at $s = 0$.

For $(p, q) \in \mathbb{R}^n_{++} \times \mathbb{R}^J$ and $w \in \mathbb{R}^n_+$, let

$$\mathbf{B}(p, q, w; V) = \left\{x \in \mathbb{R}^n_+ : \left(\exists z \in \mathbb{R}^J\right) : \; p_0 \cdot (x_0 - w_0) \leq -qz \wedge p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) = V\left(p_{\mathbf{1}}\right) z\right\}$$

Definition 5 *Given $p_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, $q \in \mathbb{R}^J$ is a no-arbitrage price vector if*

$$V\left(p_{\mathbf{1}}\right) z > 0 \Longrightarrow q \cdot z > 0$$

Let $\mathbf{S}$ denote the set of pairs $(p, q) \in S^{n-1}_{++} \times \mathbb{R}^J$ such that q is a no-arbitrage price vector, given $p_{\mathbf{1}}$. Define the individual demand functional in financial markets $\boldsymbol{\gamma}\left(\bullet; V\right) : \mathbf{S} \times \mathbb{R}^n_+ \times \mathcal{U} \to \mathbb{R}^n_+$, as:

$$\boldsymbol{\gamma}(p, q, w, u; V) = \arg\max\left\{u(x) : x \in \mathbf{B}(p, q, w; V)\right\}$$

which is well defined since $\mathbf{B}(p, q, w; V)$ is nonenpty, compact (because q is a no-arbitrage price vector, given $p_{\mathbf{1}}$) and convex, and u is continuous and strongly quasi-concave.

Define also the aggregate demand functional in financial markets, $\boldsymbol{\Gamma}\left(\bullet; V\right) : \mathbf{S} \times \mathbb{R}^{nI}_+ \times \mathcal{U}^I \to \mathbb{R}^n_+$, as:

$$\boldsymbol{\Gamma}(p, q, \left(w^i\right)_{i=1}^I, \left(u^i\right)_{i=1}^I; V) = \sum_{i=1}^I \boldsymbol{\gamma}(p, q, w^i, u^i; V)$$

A financial markets economy is: $E = \left(\left(u^i\right)_{i=1}^I, \left(w^i\right)_{i=1}^I, V\right)$

Definition 6 *A financial markets equilibrium for the economy $E = \left(\left(u^i\right)_{i=1}^I, \left(w^i\right)_{i=1}^I, V\right)$ is $(x, z, p, q) \in \mathbb{R}^{nI}_+ \times \mathbb{R}^J \times S^{n-1}_{++} \times \mathbb{R}^J$ such that:*

1. *For every i, $x^i = \boldsymbol{\gamma}(p, q, w^i, u^i; V)$, $p_0 \cdot (x^i_0 - w^i_0) = -qz^i$ and $p_{\mathbf{1}} \boxdot \left(x^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) = V(p_{\mathbf{1}})z^i$*

2. *$\boldsymbol{\Gamma}(p, q, \left(w^i\right)_{i=1}^I, \left(u^i\right)_{i=1}^I; V) = \sum_{i=1}^I w^i$ and $\sum_{i=1}^I z^i = 0$*

Since V is of full column rank, $\sum_{i=1}^I z^i = 0$ is redundant in the previous definition.

Definition 7 *Given* $\left(u^i\right)_{i=1}^I \in \mathcal{U}^I$ *and* V*, the financial markets equilibrium Manifold* $M_{FM}\left(\left(u^i\right)_{i=1}^I, V\right)$ *is:*

$$M_{FM}\left(\left(u^i\right)_{i=1}^I, V\right) = \left\{(p,q,w) \in S_{++}^{n-1} \times \mathbb{R}^J \times \mathbb{R}_+^{nI} : \mathbf{\Gamma}(p,q,\left(w^i\right)_{i=1}^I, \left(u^i\right)_{i=1}^I; V) = \sum_{i=1}^I w^i\right\}$$

What may be observed in the real world is $M_{FM}\left(\left(u^i\right)_{i=1}^I, V\right)$. We now show how to define $M\left(\left(u^i\right)_{i=1}^I, V\right)$ from $M_{FM}\left(\left(u^i\right)_{i=1}^I, V\right)$.

Proposition 1 *Let* $\left(u^i\right)_{i=1}^I \in \mathcal{U}^I$ *and* V *be fixed. Define the set*

$$\overline{M} = \{(P, \left(w^i\right)_{i=1}^I) \in S_{++}^{n-1} \times \mathbb{R}_+^{nI} : (P, \sum_{s=1}^S V_s\left(P_{\mathbf{1}}\right), \left(w^i\right)_{i=1}^I) \in M_{FM}\left(\left(u^i\right)_{i=1}^I, V\right)\}.$$

Then $\overline{M} = M\left(\left(u^i\right)_{i=1}^I, V\right)$.

Proof. For this, we first show that $\forall (P,w) \in S_{++}^{n-1} \times \mathbb{R}_+^n$

$$B(P,w;V) = \mathbf{B}(p, \sum_{s=1}^S V_s\left(P_{\mathbf{1}}\right), w; V)$$

Let $x \in B(P,w;V)$. By definition, $x \in \mathbb{R}_+^n$, $\sum_{s=0}^S P_s \cdot (x_s - w_s) \leq 0$ and $P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle$. Fix $z \in \mathbb{R}^J$ such that $p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) = V\left(p_{\mathbf{1}}\right) z$. then,

$$\begin{aligned} -\left(\sum_{s=1}^S V_s\left(P_{\mathbf{1}}\right)\right) z &= -\sum_{s=1}^S P_s \cdot (x_s - w_s) \\ &\geq P_0 \cdot (x_0 - w_0) \end{aligned}$$

Now, let $x \in \mathbf{B}(p, \sum_{s=1}^S V_s\left(P_{\mathbf{1}}\right), w; V)$. By construction, $\left(\exists z \in \mathbb{R}^J\right) : P_0 \cdot (x_0 - w_0) \leq -\left(\sum_{s=1}^S V_s\left(P_{\mathbf{1}}\right)\right) z$ and $P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) = V\left(P_{\mathbf{1}}\right) z$. That $P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle$ is immediate, whereas

$$\begin{aligned} \sum_{s=1}^S P_s \cdot (x_s - w_s) &= \left(\sum_{s=1}^S V_s\left(P_{\mathbf{1}}\right)\right) z \\ &\leq -P_0 \cdot (x_0 - w_0) \end{aligned}$$

Now, suppose that $(P, (w)_{i=1}^{I}) \in \overline{M}$. By definition,

$$\begin{aligned}
\sum_{i=1}^{I} w^i &= \mathbf{\Gamma}(P, \left(\sum_{s=1}^{S} V_s\left(P_{\mathbf{1}}\right)\right), \left(w^i\right)_{i=1}^{I}, \left(u^i\right)_{i=1}^{I}; V) \\
&= \sum_{i=1}^{I} \boldsymbol{\gamma}(P, \left(\sum_{s=1}^{S} V_s\left(P_{\mathbf{1}}\right)\right), w^i, u^i; V) \\
&= \sum_{i=1}^{I} \lambda(P, w^i, u^i; V) \\
&= \Lambda(P, \left(w^i\right)_{i=1}^{I}, \left(u^i\right)_{i=1}^{I}; V)
\end{aligned}$$

which implies that $(P, (w)_{i=1}^{I}) \in M\left(\left(u^i\right)_{i=1}^{I}, V\right)$.

If, on the other hand, $(P, (w)_{i=1}^{I}) \in M\left(\left(u^i\right)_{i=1}^{I}, V\right)$, then

$$\begin{aligned}
\sum_{i=1}^{I} w^i &= \Lambda(P, \left(w^i\right)_{i=1}^{I}, \left(u^i\right)_{i=1}^{I}; V) \\
&= \sum_{i=1}^{I} \lambda(P, w^i, u^i; V) \\
&= \sum_{i=1}^{I} \boldsymbol{\gamma}(P, \left(\sum_{s=1}^{S} V_s\left(P_{\mathbf{1}}\right)\right), w^i, u^i; V) \\
&= \mathbf{\Gamma}(P, \left(\sum_{s=1}^{S} V_s\left(P_{\mathbf{1}}\right)\right), \left(w^i\right)_{i=1}^{I}, \left(u^i\right)_{i=1}^{I}; V)
\end{aligned}$$

which implies that $(P, (w)_{i=1}^{I}) \in \overline{M}$. ■

This means that given two societies $\left(u^i\right)_{i=1}^{I}, \left(\widetilde{u}^i\right)_{i=1}^{I} \in \mathcal{U}^I$ and an asset structure V. If

$$M_{FM}\left(\left(u^i\right)_{i=1}^{I}, V\right) = M_{FM}\left(\left(\widetilde{u}^i\right)_{i=1}^{I}, V\right)$$

then

$$M\left(\left(u^i\right)_{i=1}^{I}, V\right) = M\left(\left(\widetilde{u}^i\right)_{i=1}^{I}, V\right)$$

Appendix 1: Duality in Incomplete Markets

Fix an individual $u \in \mathcal{U}_0$.

Define $U \subseteq \mathbb{R}$ as the image of $\mathbb{R}^n_{++}$ under u:

$$U = \left\{\mu \in \mathbb{R} : \left(\exists x \in \mathbb{R}^n_{++}\right) : u\left(x\right) = \mu\right\}$$

For each $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$, let $D\left(w_{\mathbf{1}}, \mu\right) \subseteq S^{n-1}_{++}$ be defined as follows:

$$D\left(w_{\mathbf{1}}, \mu\right) = \left\{P \in S^{n-1}_{++} : \left(\exists x \in \mathbb{R}^n_{++}\right) : u\left(x\right) = \mu \text{ and } P_{\mathbf{1}} \boxdot \left(x_{\mathbf{1}} - w_{\mathbf{1}}\right) \in \left\langle V\left(P_{\mathbf{1}}\right)\right\rangle\right\}$$

Proposition 2 *For each* $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$, $D\left(w_{\mathbf{1}}, \mu\right)$ *is diffeomorphic to*

$$\left\{\left(\left(P_{0,2}, ..., P_{0,L}\right), P_{\mathbf{1}}\right) \in \mathbb{R}^{n-1}_{++} : \left(\exists x \in \mathbb{R}^n_{++}\right) : u\left(x\right) = \mu \text{ and } P_{\mathbf{1}} \boxdot \left(x_{\mathbf{1}} - w_{\mathbf{1}}\right) \in \left\langle V\left(P_{\mathbf{1}}\right)\right\rangle\right\}$$

which is open.

Proof. Let D denote the latter set. That $D\left(w_{\mathbf{1}}, \mu\right)$ and D are diffeomorphic is straightforward. We now show that D is open. Let $P \in D$. By definition, for some $x \in \mathbb{R}^n_{++}$, $u\left(x\right) = \mu$ and $P_{\mathbf{1}} \boxdot \left(x_{\mathbf{1}} - w_{\mathbf{1}}\right) \in \left\langle V\left(P_{\mathbf{1}}\right)\right\rangle$, whereas using the implicit function theorem, since $\partial_{x_0}\left(u\left(x\right)\right) \in \mathbb{R}^L_{++}$, for some $\varepsilon > 0$, $B_\varepsilon\left(x_{\mathbf{1}}\right) \subseteq \mathbb{R}^{LS}_{++}$ and

$$\left(\forall \widetilde{x}_{\mathbf{1}} \in B_\varepsilon\left(x_{\mathbf{1}}\right)\right)\left(\exists \widetilde{x}_0 \in \mathbb{R}^L_{++}\right) : u\left(\widetilde{x}_0, \widetilde{x}_{\mathbf{1}}\right) = u(x)$$

Given that $\forall\left(s, l\right) \in \{1, ..., S\} \times \{1, ..., L\}$,

$$\lim_{\delta \longrightarrow 0} \frac{\delta\left(w_{s,l} - x_{s,l}\right)}{\frac{P_{s,l}}{P_{s,1}} + \delta} = 0$$

there exists $\overline{\delta}_{s,l} > 0$ such that

$$|\delta| < \overline{\delta}_{s,l} \Longrightarrow \frac{|\delta|\left|w_{s,l} - x_{s,l}\right|}{\left|\frac{P_{s,l}}{P_{s,1}} + \delta\right|} < \frac{\varepsilon}{\sqrt{LS}}$$

Define

$$\overline{\delta} = \min_{(s,l) \in \{1,...,S\} \times \{1,...,L\}} \left\{\overline{\delta}_{s,l}\right\}$$

and consider the function $h : \mathbb{R}^{n-1}_{++} \to \mathbb{R}^{n-1}_{++}$, $h(P) = \left(\left(P_{0,2}, ..., P_{0,L}\right), \frac{P_1}{P_{1,1}}, ... \frac{P_S}{P_{S,1}}\right)$. The function h is continuous, therefore there is a $\delta > 0$ such that for all $P' \in B_\delta\left(P\right)$, $\|h(P') - h(P)\| < \overline{\delta}$, in particular $\left|\frac{P'_{s,l}}{P'_{s,1}} - \frac{P_{s,l}}{P_{s,1}}\right| < \overline{\delta}$.

Define $x'_{\mathbf{1}} \in \mathbb{R}^{LS}$ as follows: $\forall (s,l) \in \{1,...,S\} \times \{1,...,L\}$,

$$x'_{s,l} = \frac{\frac{P_{s,l}}{P_{s,1}} x_{s,l} + \left(\frac{P'_{s,l}}{P'_{s,1}} - \frac{P_{s,l}}{P_{s,1}} \right) w_{s,l}}{\frac{P'_{s,l}}{P'_{s,1}}}$$

Then,

$$\left| x'_{s,l} - x_{s,l} \right| = \frac{\left| \frac{P'_{s,l}}{P'_{s,1}} - \frac{P_{s,l}}{P_{s,1}} \right| \left| w_{s,l} - x_{s,l} \right|}{\left| \frac{P'_{s,l}}{P'_{s,1}} \right|}$$

and, since $P' \in B_\delta (P)$, it follows that $\left| \frac{P'_{s,l}}{P'_{s,1}} - \frac{P_{s,l}}{P_{s,1}} \right| < \overline{\delta} \leqslant \overline{\delta}_{s,l}$, from where

$$\left| x'_{s,l} - x_{s,l} \right| < \frac{\varepsilon}{\sqrt{LS}}$$

and, hence $\| x'_{\mathbf{1}} - x_{\mathbf{1}} \| < \varepsilon$. This implies that $x'_{\mathbf{1}} \in B_\varepsilon (x_{\mathbf{1}})$ and, therefore, that there exists $x'_0 \in \mathbb{R}^L_{++}$ such that $u(x'_0, x'_{\mathbf{1}}) = u(x)$.

Finally, by construction, $(\frac{P'_1}{P'_{1,1}}, ... \frac{P'_S}{P'_{S,1}}) \boxdot (x'_{\mathbf{1}} - w_{\mathbf{1}}) = (\frac{P_1}{P_{1,1}}, ... \frac{P_S}{P_{S,1}}) \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V \rangle$, and, hence, $P' \in D$. ■

For each $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$ such that $D(w_{\mathbf{1}}, \mu) \neq \varnothing$, define the Hicksian demand function $h(\cdot; w_{\mathbf{1}}, \mu) : D(w_{\mathbf{1}}, \mu) \longrightarrow \mathbb{R}^n_{++}$, as:

$$h(P; w_{\mathbf{1}}, \mu) = \arg\min \left\{ \sum_{s=0}^{S} P_s \cdot x_s : u(x) \geq \mu \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}}) \rangle \right\}$$

and the expenditure function $e(\cdot; w_{\mathbf{1}}, \mu) : D(w_{\mathbf{1}}, \mu) \longrightarrow \mathbb{R}$ as:

$$e(P; w_{\mathbf{1}}, \mu) = P \cdot h(P; w_{\mathbf{1}}, \mu)$$

Since $u \in \mathcal{U}_0$, $h(P; w_{\mathbf{1}}, \mu)$ is well defined into $\mathbb{R}^n_{++}$.

Now, for each $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, define $\mathbf{D}(w_{\mathbf{1}}) \subseteq S^{n-1}_{++} \times \mathbb{R}_{++}$ as follows:

$$\mathbf{D}(w_{\mathbf{1}}) = \left\{ (P, m) \in S^{n-1}_{++} \times \mathbb{R}_{++} : \left(\exists x \in \mathbb{R}^n_{++} \right) : \sum_{s=0}^{S} P_s \cdot x_s \leqslant m \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}}) \rangle \right\}$$

Proposition 3 *For each $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, $\mathbf{D}(w_{\mathbf{1}})$ is diffeomorphic to*

$$\left\{ (((P_{0,2}, ..., P_{0,L}), P_{\mathbf{1}}), m) \in \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} : \left(\exists x \in \mathbb{R}^n_{++} \right) : \sum_{s=0}^{S} P_s \cdot x_s \leqslant m \textit{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}}) \rangle \right\}$$

which is nonempty and open.

Proof. This is straightforward. ■

For each $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, define the conditional individual demand function $\widetilde{f}(\cdot,\cdot;w_{\mathbf{1}}) : \mathbf{D}(w_{\mathbf{1}}) \longrightarrow \mathbb{R}^n_{++}$ as

$$\widetilde{f}(P,m;w_{\mathbf{1}}) = \arg\max\left\{u(x) : \sum_{s=0}^{S} P_s \cdot x_s \leq m \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle\right\}$$

If $u \in \mathcal{U}_0$, any solution to the maximization problem above lies in $\mathbb{R}^n_{++}$ and is unique. Obviously,

$$\widetilde{f}(P,\sum_{s=0}^{S} P_s \cdot w_s;w_{\mathbf{1}}) = f(P,w)$$

Following is the standard duality result, extended to the case of incomplete markets. It contains three parts:

1. Given endowments w, If x^* solves the utility maximization problem at prices and $P \in S^{n-1}_{++}$, then x^* solves the expenditure minimization problem at prices P and minimum utility $u^i(x^*)$.
2. Given endowments $w_{\mathbf{1}}$ and utility μ, if x^* solves the expenditure minimization problem at prices $P \in D(w_{\mathbf{1}},\mu)$, then x^* solves the utility maximization problem at prices P and endowments x^*.
3. Given endowments $w_{\mathbf{1}}$ and utility μ, if x^* solves the expenditure minimization problem at prices $P \in D(w_{\mathbf{1}},\mu)$, then x^* solves the conditional utility maximization problem at prices P and income $e^i(P,w,\mu)$.That is

Proposition 4 *1. For every $w = (w_0, w_{\mathbf{1}}) \in \mathbb{R}^n_{++}$ and every $P \in S^{n-1}_{++}$,*

$$u(f(P,w)) \in U$$

$$P \in D(w_{\mathbf{1}}, u(f(P,w)))$$

and

$$h(P;w_{\mathbf{1}}, u(f(P,w))) = f(P,w)$$

2. Given $(w_{\mathbf{1}},\mu) \in \mathbb{R}^{LS}_{++} \times U$, for every $P \in D(w_{\mathbf{1}},\mu)$,

$$f(P, h(P;w_{\mathbf{1}},\mu)) = h(P;w_{\mathbf{1}},\mu)$$

3. *Given* $(w_{\mathbf{1}},\mu)\in\mathbb{R}^{LS}_{++}\times U$, *for every* $P\in D(w_{\mathbf{1}},\mu)$,

$$(P,e(P,w,\mu))\in\mathbf{D}(w_{\mathbf{1}})$$

and

$$\widetilde{f}(P,e(P,w,\mu);w_{\mathbf{1}})=h(P;w_{\mathbf{1}},\mu)$$

Proof. Part (1) is straightforward given lemma 1, since $u\in\mathcal{U}^0$: argue by contradiction and use strict monotonicity of the utility function.

Given that u is continuous, for parts (2) and (3) it suffices to prove that $u(h(P;w_{\mathbf{1}},\mu))=\mu$. For this, suppose not: $u(h(P;w_{\mathbf{1}},\mu))>\mu$. Define $x=h(P;w_{\mathbf{1}},\mu)-(\varepsilon,0,...,0)$, where $\varepsilon\in\mathbb{R}_{++}$. By construction, $x_{\mathbf{1}}=h_{\mathbf{1}}(P;w_{\mathbf{1}},\mu)$, from where $P_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V(P_{\mathbf{1}})\rangle$, and $\sum_{s=0}^{S}P_s\cdot x_s<e(P;w_{\mathbf{1}},\mu)$, whereas since $h(P;w_{\mathbf{1}},\mu)\in\mathbb{R}^n_{++}$, for ε small enough $x\in\mathbb{R}^n_+$ and, by continuity, $u(x)\geqslant\mu$, which is a contradiction. ■

Proposition 5 (Shepard's Lemma) *For every* $(w_{\mathbf{1}},\mu)\in\mathbb{R}^{LS}_{++}\times U$, *the function* $e(\cdot;w_{\mathbf{1}},\mu):D(w_{\mathbf{1}},\mu)\longrightarrow\mathbb{R}_{++}$ *is differentiable and*

$$\partial_P(e(P;w_{\mathbf{1}},\mu))=h(P;w_{\mathbf{1}},\mu)$$

Proof. This is an immediate consequence of the Duality Theorem (see Mas-Colell et al, Proposition 3.F.1): let

$$K=\{x\in\mathbb{R}^n_+:u^i(x)\geq\mu\text{ and }P_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V(P_{\mathbf{1}})\rangle$$

Then, K is closed and $e(P;w_{\mathbf{1}},\mu)$ is the support function of K. ■

Proposition 6 *For every* $w_{\mathbf{1}}\in\mathbb{R}^{LS}_{++}$, *the function* $\widetilde{f}(\cdot,\cdot;w_{\mathbf{1}}):\mathbf{D}(w_{\mathbf{1}})\longrightarrow\mathbb{R}^n_{++}$ *is differentiable.*

Proof. This can be argued in the same way as fact 5 in Duffie and Shafer (1985). ■

Proposition 7 (Slutsky Equation in incomplete markets) . *Let* $(P,w)\in S^{n-1}_{++}\times\mathbb{R}^n_+$ *and* $\mu=u(f(P,w))$. *Then,* $h(\cdot;w_{\mathbf{1}},\mu):D(w_{\mathbf{1}},\mu)\longrightarrow\mathbb{R}^n_{++}$ *is differentiable and for all* $(s,l),(s',l')\in(\{0,...,S\}\times\{1,...,L\})\setminus\{(0,1)\}$, *we have:*

$$\frac{\partial h_{s,l}(P;w_{\mathbf{1}},\mu)}{\partial P_{s',l'}}=\frac{\partial f_{s,l}(P,w)}{\partial P_{s',l'}}+\frac{\partial f_{s,l}(P,w)}{\partial w_{0,1}}(f_{s',l'}(P,w)-w_{s',l'})$$

Proof. That $h(\cdot;w_{\mathbf{1}},\mu)$ is differentiable follows from propositions 4 and 6.

Also from proposition 4, we have that $h(P;w_{\mathbf{1}},\mu)=\widetilde{f}(P,e(P;w_{\mathbf{1}},\mu);w_{\mathbf{1}})$. Therefore,

$$\frac{\partial h_{s,l}(P;w_{\mathbf{1}},\mu)}{\partial P_{s',l'}}=\frac{\partial\widetilde{f}_{s,l}(P,e(P;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial P_{s',l'}}+\frac{\partial\widetilde{f}_{s,l}(P,e(P;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial m}\frac{\partial e(P;w_{\mathbf{1}},\mu)}{\partial P_{s',l'}}$$

By proposition 5, we have:

$$\frac{\partial h_{s,l}(P;w_{\mathbf{1}},\mu)}{\partial P_{s',l'}}=\frac{\partial \widetilde{f}_{s,l}\left(P,e\left(P;w_{\mathbf{1}},\mu\right);w_{\mathbf{1}}\right)}{\partial P_{s',l'}}+\frac{\partial \widetilde{f}_{s,l}\left(P,e\left(P;w_{\mathbf{1}},\mu\right);w_{\mathbf{1}}\right)}{\partial m}h_{s',l'}(P;w_{\mathbf{1}},\mu)$$

Now, since $f(P,w)=\widetilde{f}\left(P,\sum_{s=0}^{S}P_s\cdot w_s;w_{\mathbf{1}}\right)$, then

$$\frac{\partial f_{s,l}\left(P,w\right)}{\partial P_{s',l'}}=\frac{\partial \widetilde{f}_{s,l}\left(P,\sum_{s=0}^{S}P_s\cdot w_s;w_{\mathbf{1}}\right)}{\partial P_{s',l'}}+\frac{\partial \widetilde{f}_{s,l}\left(P,\sum_{s=0}^{S}P_s\cdot w_s;w_{\mathbf{1}}\right)}{\partial m}w_{s',l'}$$

Under monotonicity, at $\mu=u^i\left(f^i\left(P,w\right)\right)$, $e\left(P;w_{\mathbf{1}},\mu\right)=\sum_{s=0}^{S}P_s\cdot w_s$ and, hence,

$$\frac{\partial f_{s,l}\left(P,w\right)}{\partial P_{s',l'}}=\frac{\partial \widetilde{f}_{s,l}\left(P,e\left(P,w,\mu\right);w_{\mathbf{1}}\right)}{\partial P_{s',l'}}+\frac{\partial \widetilde{f}_{s,l}\left(P,e\left(P,w,\mu\right);w_{\mathbf{1}}\right)}{\partial m}w_{s',l'}$$

Solving for

$$\frac{\partial \widetilde{f}_{s,l}\left(P,e\left(P,w,\mu\right);w_{\mathbf{1}}\right)}{\partial P_{s',l'}}$$

and replacing gives us

$$\frac{\partial h_{s,l}(P;w_{\mathbf{1}},\mu)}{\partial P_{s',l'}}=\frac{\partial f_{s,l}\left(P,w\right)}{\partial P_{s',l'}}+\frac{\partial \widetilde{f}_{s,l}\left(P,e\left(P,w,\mu\right);w_{\mathbf{1}}\right)}{\partial m}\left(h_{s',l'}(P;w_{\mathbf{1}},\mu)-w_{s',l'}\right)$$

By proposition 4, since $\mu=u(f(P,w))$,

$$\frac{\partial h_{s,l}(P;w_{\mathbf{1}},\mu)}{\partial P_{s',l'}}=\frac{\partial f_{s,l}\left(P,w\right)}{\partial P_{s',l'}}+\frac{\partial \widetilde{f}_{s,l}\left(P,\sum_{s=0}^{S}P_s\cdot w_s;w_{\mathbf{1}}\right)}{\partial m}\left(f_{s',l'}\left(P,w\right)-w_{s',l'}\right)$$

Finally, notice that

$$\frac{\partial f_{s,l}\left(P,w\right)}{\partial w_{0,1}}=\frac{\partial \widetilde{f}_{s,l}\left(P,\sum_{s=0}^{S}P_s\cdot w_s;w_{\mathbf{1}}\right)}{\partial m}$$

Substitution gives us the desired result. ■

Appendix 2: Regularity

The regularity condition assumed here is

Condition 2 *For every individual i and every $P \in S^{n-1}_{++}$, there exist $w \in \mathbb{R}^n_+$, and $(s,l), (s',l') \in (\{0,...,S\} \times \{1,...,L\}) \setminus \{(0,1)\}$, such that:*

$$\begin{vmatrix} \frac{\partial^2 f^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2}(P,w) & \frac{\partial^2 f^i_{s',l'}}{\partial \left(w^i_{0,1}\right)^2}(P,w) \\ \frac{\partial^3 f^i_{s,l}}{\partial \left(w^i_{0,1}\right)^3}(P,w) & \frac{\partial^3 f^i_{s',l'}}{\partial \left(w^i_{0,1}\right)^3}(P,w) \end{vmatrix} \neq 0$$

The condition is inspired by but different from the regularity condition used by Kubler et al (2002): first, we write the condition in terms of present value prices (i.e. no-arbitrage individual demands); second, we do not use conditional demands or asset demands to write the regularity condition; third, in Kubler et al the condition is written for every state of the world, so they need at least three goods while, in theory at least, we only need one good and two states.

It follows from Lewbel (2001) and Banks, Blundell Lewbel (1997) that there exists $u_0 : \mathbb{R}^3_+ \longrightarrow \mathbb{R}$ such that

$$f_l(P,m) = \frac{m}{P_l}\left(A_l(P) + B_l(P)\log\left(\frac{m}{a(P)}\right) + C_l(P)\log\left(\frac{m}{a(P)}\right)^2\right), \ l = 0,1,2.$$

are the solutions to the problem

$$\max_x u_0(x) \ \text{ st. } \ P \cdot x = m$$

where A_l, B_l and C_l are homogeneous of degree zero in P and a is homogeneous of degree 1 in $P \in \mathbb{R}^3_{++}$ (these two conditions guarantee that f_l is homogeneous of degree zero in P and w) and

$$\sum_{l=1}^{3} A_l(P) + \sum_{l=1}^{3} B_l(P)\log\left(\frac{m}{a(P)}\right) + \sum_{l=1}^{3} C_l(P)\log\left(\frac{m}{a(P)}\right)^2 = 1$$

for all P and m.

Now, suppose that there are three commodities and two states of nature so that, for some $u_1 : \mathbb{R}^3_+ \longrightarrow \mathbb{R}$ and $u_2 : \mathbb{R}^3_+ \longrightarrow \mathbb{R}$,

$$u(x_0, x_1, x_2) = u_0(x_0) + \min\{u_1(x_1), u_2(x_2)\}$$

(Although the function is only weakly monotone and violates differentiability, it serves the illutrative purpose of our example.)

Suppose that there is only one asset and

$$V(P_{\mathbf{1}}) = \begin{bmatrix} P_{1,1} \\ -P_{2,1} \end{bmatrix}$$

define

$$\begin{aligned} v &: S^8_{++} \times \mathbb{R}^9_+ \longrightarrow \mathbb{R}; v(P,w) = \max_x u(x) \text{ st. } \left\{ \begin{array}{c} P \cdot x = P \cdot w \\ \left[\begin{array}{c} P_1 \cdot (x_1 - w_1) \\ P_2 \cdot (x_2 - w_2) \end{array} \right] \in \langle V(P_{\mathbf{1}}) \rangle \end{array} \right. \\ v_s &: \mathbb{R}^3_{++} \times \mathbb{R}_+ \longrightarrow \mathbb{R}; v_s(P,m) = \max_x u_s(x) \text{ st. } P \cdot x = m \end{aligned}$$

Claim 1 *For every $P = (P_0, P_1, P_2)$ and $w = (w_0, w_1, w_2)$*

$$\begin{aligned} v(P,w) &= \max_{m_0,m_1,m_2} \left(v_0(P_0,m_0) + \min\{v_1(P_1,m_1), v_2(P_2,m_2)\} \right) \\ & \quad st. \left\{ \begin{array}{c} m_0 + m_1 + m_2 = P \cdot w \\ \left[\begin{array}{c} m_1 - P_1 \cdot w_1 \\ m_2 - P_2 \cdot w_2 \end{array} \right] \in \langle V(P_{\mathbf{1}}) \rangle \end{array} \right. \end{aligned}$$

Proof. This follows by construction. ■

Claim 2 *Let $m_0(P,w)$, $m_1(P,w)$ and $m_2(P,w)$ denote the solution of*

$$\begin{aligned} & \max_{m_0,m_1,m_s} \left(v_0(P_0,m_0) + \min\{v_1(P_1,m_1), v_2(P_2,m_2)\} \right) \\ & st. \left\{ \begin{array}{c} m_0 + m_1 + m_2 = P \cdot w \\ \left[\begin{array}{c} m_1 - P_1 \cdot w_1 \\ m_2 - P_2 \cdot w_2 \end{array} \right] \in \langle V(P_{\mathbf{1}}) \rangle \end{array} \right. \end{aligned}$$

and let $(P, \widetilde{w})$ be such that

$$v_1(P_1, m_1(P,\widetilde{w})) = v_2(P_2, m_2(P,\widetilde{w}))$$

Then, for every $w_0 > \widetilde{w}_0$,

$$\frac{\partial m_0}{\partial w_{0,1}}(P,(w_0,\widetilde{w}_{\mathbf{1}})) = 1$$

Proof. Since $w_0 > \widetilde{w}_0$,

$$v_1(P_1, m_1(P,(w_0,\widetilde{w}_{\mathbf{1}}))) = v_2(P_2, m_2(P,(w_0,\widetilde{w}_{\mathbf{1}})))$$

Consider a perturbation $dw_{0,1}$ to $w_{0,1}$. Notice that by construction of $V(P_{\mathbf{1}})$,

$$dm_1 > 0 \Longrightarrow dm_2 < 0 \Longrightarrow (dv_1 > 0 \text{ and } dv_2 < 0)$$

whereas

$$dm_2 > 0 \Longrightarrow dm_1 < 0 \Longrightarrow (dv_1 < 0 \text{ and } dv_2 > 0)$$

which cannot be optimal, given that v_s is increasing in m. ■

Now, suppose that for every P, there exists $\widetilde{w}$ such that

$$v_1\left(P_1, m_1\left(P, \widetilde{w}\right)\right) = v_2\left(P_2, m_2\left(P, \widetilde{w}\right)\right)$$

and consider only w with $w_0 > \widetilde{w}_0$ and $w_{\mathbf{1}} = \widetilde{w}_{\mathbf{1}}$

Let $f_{s,l}\left(P, w\right)$ denote optimal demands. By the first claim, for all l,

$$f_{0,l}\left(P, w\right) = f_l(P_0, m_0\left(P, w\right))$$

so

$$\begin{aligned} \frac{\partial f_{0,l}}{\partial w_{0,1}}\left(P, w\right) &= \frac{\partial f_l}{\partial m}(P_0, m_0\left(P, w\right))\frac{\partial m_0}{\partial w_{0,1}}\left(P, w\right) \\ &= \frac{\partial f_l}{\partial m}(P_0, m_0\left(P, w\right)) \end{aligned}$$

where the second equality follows from the second claim. It then follows (using the second claim again) that

$$\begin{aligned} \frac{\partial^2 f_{0,l}}{\partial w_{0,1}^2}\left(P, w\right) &= \frac{\partial^2 f_l}{\partial m^2}(P_0, m_0\left(P, w\right)) \\ \frac{\partial^3 f_{0,l}}{\partial w_{0,1}^3}\left(P, w\right) &= \frac{\partial^3 f_l}{\partial m^3}(P_0, m_0\left(P, w\right)) \end{aligned}$$

Now, the rank of system

$$\begin{bmatrix} f_1\left(P_0, m\right) \\ f_2\left(P_0, m\right) \\ f_3\left(P_0, m\right) \end{bmatrix}$$

is, by definition, the rank of:

$$\begin{bmatrix} A_1(P_0) & B_1(P_0) & C_1(P_0) \\ A_2(P_0) & B_2(P_0) & C_2(P_0) \\ A_3(P_0) & B_3(P_0) & C_3(P_0) \end{bmatrix}$$

If B_l and C_l are zero then the system is of rank 1, the utility function is homothetic and clearly the regularity condition does not hold. If $B_2\left(P_0\right)C_3\left(P_0\right) - B_2\left(P_0\right)C_3\left(P_0\right) \neq 0$ the system has rank at least 2. Below, we prove that, for this case, the regularity condition holds.

Remark 3 *Not every rank* 2 *system satisfies the regularity condition, but every rank* 3 *system of this form does. See below.*

Set $P_{0,1} = 1$. Then,

$$\begin{aligned}
\frac{\partial f_l}{\partial m}(P_0, m) &= \frac{1}{P_{0,l}}\left(A_l(P_0) + B_l(P_0)\log\left(\frac{m}{a(P_0)}\right) + C_l(P_0)\log\left(\frac{m}{a(P_0)}\right)^2\right) + \\
&\quad \frac{1}{P_{0,l}}\left(B_l(P_0) + 2C_l(P_0)\log\left(\frac{m}{a(P_0)}\right)\right) \\
&\Rightarrow \\
\frac{\partial^2 f_l}{\partial m^2}(P_0, m) &= \frac{1}{P_{0,l}m}\left(B_l(P_0) + 2C_l(P_0)\left(\log\left(\frac{m}{a(P_0)}\right) + 1\right)\right) \\
&\Rightarrow \\
\frac{\partial^3 f_1}{\partial m^3}(P_0, m) &= -\frac{1}{P_{0,l}\,(m)^2}\left(B_l(P_0) + 2C_l(P_0)\log\left(\frac{m}{a(P_0)}\right)\right)
\end{aligned}$$

It follows that the regularity condition is satisfied if

$$\left|\begin{array}{cc}
\frac{1}{P_{0,2}m}\left(B_2(P_0) + 2C_2(P_0)\left(\log\left(\frac{m}{a(P_0)}\right) + 1\right)\right) & \frac{1}{P_{0,3}m}\left(B_3(P_0) + 2C_3(P_0)\left(\log\left(\frac{m}{a(P_0)}\right) + 1\right)\right) \\
-\frac{1}{P_{0,2}m^2}\left(B_2(P_0) + 2C_2(P_0)\log\left(\frac{m}{a(P_0)}\right)\right) & -\frac{1}{P_{0,3}m^2}\left(B_3(P_0) + 2C_3(P_0)\log\left(\frac{m}{a(P_0)}\right)\right)
\end{array}\right| \neq 0$$

$$\Leftrightarrow \left|\begin{array}{cc}
\left(B_2(P_0) + 2C_2(P_0)\left(\log\left(\frac{m}{a(P_0)}\right) + 1\right)\right) & \left(B_3(P_0) + 2C_3(P_0)\left(\log\left(\frac{m}{a(P_0)}\right) + 1\right)\right) \\
\left(B_2(P_0) + 2C_2(P_0)\log\left(\frac{m}{a(P_0)}\right)\right) & \left(B_3(P_0) + 2C_3(P_0)\log\left(\frac{m}{a(P_0)}\right)\right)
\end{array}\right| \neq 0$$

$$\Leftrightarrow \left|\begin{array}{cc}
C_2(P_0) & C_3(P_0) \\
\left(B_2(P_0) + 2C_2(P_0)\log\left(\frac{m}{a(P_0)}\right)\right) & \left(B_3(P_0) + 2C_3(P_0)\log\left(\frac{m}{a(P_0)}\right)\right)
\end{array}\right| \neq 0$$

$$\Leftrightarrow C_2(P_0)B_3(P_0) - C_3(P_0)B_2(P_0) \neq 0$$

A rank 3 system clearly satisfies this condition and, if the condition is satisfied, then the rank is at least 2.

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