Conjectural variations and evolutionary stability: A new rationale for consistency*

Wieland Müller†
Humboldt University
& New York University

Hans-Theo Normann‡
Royal Holloway,
University of London

June 24, 2002

Abstract

Adopting an evolutionary approach, we explain the conjectural variations firms may hold in duopoly. Given conjectures, firms play the market game rationally. Success in the market game determines fitness in the evolutionary game. We show that the unique conjectures which are evolutionarily stable are consistent in that they anticipate rivals' behavior correctly.

Keywords: consistent conjectures, Cournot duopoly, evolutionary stability, indirect evolutionary approach

JEL classification numbers: C72, D43, L13

---

*We thank Jörg Oechssler and Yossi Spiegel for helpful comments. The first author acknowledges financial support from the German Science Foundation, DFG. Furthermore he thanks the Center for Experimental Social Science (CESS) at New York University for its hospitality.

†Department of Economics, Humboldt-University Berlin, Spandauer Str. 1, 10178 Berlin, Germany, Fax +49 30 2093 5704, email: *wmueller@wiwi.hu–berlin.de.*

‡Department of Economics, Royal Holloway, Egham, Surrey TW20 0EX, UK, fax: +44 1784 439534, email: *hans.normann@rhul.ac.uk.*

1. Introduction

The predictions of oligopoly theory depend crucially on behavioral assumptions on how a firm conjectures about the way the other firms react to its own actions. Cournot made the assumption that firms maximize their profits, taking as given the quantity of the rival firms. That is, rivals are expected not to react at all to changes of a firm's own action. Later contributions by Bowley (1924), Stackelberg (1934), Hicks (1935) and Leontieff (1936) varied this assumption and proposed alternative solutions, initiating the conjectural variations literature.

Interest in conjectural variations grew with the analysis of the consistency criteria. Consistency, in addition to the individual rationality assumption underlying the notion of Nash equilibrium, requires that conjectures about rivals' behavior have to be correct. In Bresnahan (1981), the consistency of conjectures occurs whenever the slopes of firms' reaction functions are (locally) equal to the conjectural variations.[1] Applying this definition, Bresnahan (1981) shows, among other things, that a unique solution exists for duopoly with linear-quadratic costs.

The literature following Bresnahan (1981) pointed out two fundamental problems with the conjectural variations approach and the consistency criteria in particular: "The heart of the problem is the notion of a conjectural variation. This notion is *ad hoc* inasmuch as none of the models using a conjectural variation explains how it is formed or whence it came" (Daughety, 1985, p.246). The second problem is closely related to the first. Conjectures have been found very difficult to rationalize (Makowski, 1987). Theorists may find consistent conjectures appealing because of the parallel to rational-expectations theory. However, attempts to derive conjectures merely from rationality assumptions have not been successful. Conjectures are essentially "a-rational" (Makowski, 1987).

Recently, some authors addressed these problems by proposing explicitly dynamic models, usually repeated Cournot settings (Dockener, 1992; Sabourian, 1992; Cabral, 1995). These authors examine conditions under which the outcome of the repeated games equals the outcome of the static conjectural-variations model. For example, Cabral (1995) proposes

[1] This definition can actually be traced back to Leontieff (1936). See also Martin (2002).

an infinitely repeated game with minimax punishments in which, for each discount factor, there is a conjectural variation, such that, for any linear oligopoly structure, a firm's output in the optimal equilibrium is equal to the quantity of the conjectural variations solution. In this way, the conjectural variations models are justified as a "short cut" (Sabourian, 1992, p.236), mimicking the outcome of more complex dynamic games. However, note that only the conjectural variations outcome is justified but nothing is said about the origin and nature of the conjectures themselves.

In this paper, we propose an evolutionary approach to explain conjectures. We do not impose any rationality or consistency criterion on the conjectures firms may hold. However, given the conjectures, firms play the market game rationally. The link between market performance and conjectures is that profits in the duopoly game determine the success in an evolutionary game. So, what our model does is to impose evolutionary selection of conjectures followed by rational choice of actions in the basic market game. As a result, we show that the conjectures surviving the evolutionary process are the consistent conjectures proposed by Bresnahan (1981). That is, we do not only justify the market outcome implied by consistent conjectures, we also justify the conjectures themselves.

The evolutionary process we apply has successfully been applied to explain various economic phenomena. The concept was proposed by Güth and Yaari (1992) who labelled it "indirect evolutionary approach". As in our paper, the idea is that subjects act rationally in their market transactions but factors influencing the market game like preferences or beliefs are formed in an evolutionary process. This approach has been used to explain e.g. monopolistic competition (Güth and Huck, 1997), altruism (Bester and Güth, 1998) and behavior in the ultimatum game (Huck and Oechssler, 1999). Königstein and Müller (2000) propose a formal framework for the indirect evolutionary approach.

We proceed as follows. Section 2 briefly states the main modelling assumptions. In section 3, we first define the market before deriving the consistent conjectures equilibrium. In the second part of the section, we determine the evolutionarily stable conjectures. In section 4 we discuss our findings.

2. Modelling assumptions

We consider two firms $i = 1, 2$ in a heterogeneous-goods market. The strategy sets are $S_i = \{q_i \mid q_i \geq 0\}$, $i = 1, 2$, and the inverse demand functions are given by

$$p_i(q_i, q_j) = \widehat{a} - bq_i - \theta q_j, \; i, j = 1, 2; i \neq j \tag{2.1}$$

with $0 \leq \theta \leq 1$ and $b \geq \theta$. Cost functions have the form

$$c_i(q_i) = \widehat{c}q_i + c\,(q_i)^2 /2, \; i = 1, 2 \tag{2.2}$$

with $\widehat{c}, c \geq 0$. Thus, firm i's profit is given by

$$\begin{aligned} \pi_i(q_i, q_j) &= p_i(q_i, q_j)q_i - c_i(q_i) && (2.3) \\ &= (\widehat{a} - bq_i - \theta q_j)\, q_i - \widehat{c}q_i - \frac{c}{2}\,(q_i)^2 && (2.4) \\ &= (a - bq_i - \theta q_j)\, q_i - \frac{c}{2}\,(q_i)^2 && (2.5) \end{aligned}$$

where $a := \widehat{a} - \widehat{c}$. Note that our assumptions on demand and cost are equal to Assumptions 1 and 2 in Bresnahan (1981), except that we assume that firms are symmetric.[2] The case of constant marginal cost is obtained by setting $c = 0$.

3. Results

3.1. Consistent conjectures equilibrium

We start by reiterating Bresnahan's (1981) definition of a consistent conjectures equilibrium (CCE). Let $\rho_i = \rho_i(q_j)$, $i \neq j$, denote firm i's reaction function. From our assumptions, we know that a unique and *linear* CCE exists (Bresnahan, 1981, Theorem 1). We therefore restrict the attention to linear conjectures such that $r_i \in \mathbb{R}$, $i = 1, 2$, denotes firm i's' conjectures about firm j's reaction to q_i.

[2] With asymmetric demand and cost functions, the evolutionary analysis below is extremely messy and cumbersome. Bresnahan (1981) shows that the model may also allow for fixed costs which, from his Assumption 3, should not be too large.

Definition 1. *A consistent conjectures equilibrium is a pair of quantities, (q_1^*, q_2^*), and of conjectures, (r_1^*, r_2^*), such that*

$$q_1^* = \rho_1(q_2^*), \; q_2^* = \rho_2(q_1^*), \tag{3.1}$$

and

$$r_1^* = \frac{\partial \rho_2(q_1)}{\partial q_1}, \; r_2^* = \frac{\partial \rho_1(q_2)}{\partial q_2}. \tag{3.2}$$

That is, firms' quantities have to be a Nash equilibrium (conditions (3.1)), and a firm's conjecture about the other firm's behavior has to be equal to the slope of the other firm's reaction function (conditions (3.2)).

We now compute a closed-form solution of the consistent-conjectures equilibrium for the market defined above. From the first-order conditions of profit maximization

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = a - \theta q_j - q_i(2b + \theta r_i + c) = 0 \tag{3.3}$$

we derive firm i's reaction function

$$\rho_i(q_j) = \frac{a - \theta q_j}{2b + \theta r_i + c}. \tag{3.4}$$

The slope of firm i's reaction function is

$$\partial \rho_i(q_j)/\partial q_j = -\frac{\theta}{2b + \theta r_i + c}. \tag{3.5}$$

Thus, the consistent conjectures are the solution of the following system of two simultaneous equations

$$r_i = -\frac{\theta}{2b + \theta r_j + c}, \; i, j = 1, 2; \; i \neq j \tag{3.6}$$

whose two candidate solutions are given by

$$r := r_i = r_j = \frac{-2b - c \pm A}{2\theta} \tag{3.7}$$

with

$$A := \sqrt{(2b + c)^2 - 4\theta^2} > 0. \tag{3.8}$$

The equilibrium quantities, q_i^*, are the solution of the system of two simultaneous equations (3.3). This solution is

$$q_i^* = \frac{a}{c + \theta r + 2b + \theta}. \tag{3.9}$$

Note that, using the fact that $r_1 = r_2 = r$, the second-order condition for profit maximization is given by

$$\frac{\partial^2 \pi_i(q_i, q_j)}{\partial q_i^2} = -2(b + \theta r) - c < 0. \tag{3.10}$$

The equilibrium quantities, q_i^*, evaluated at $r = (-2b - c \pm A)/2\theta$ equal

$$q_i^* = \frac{2a}{c + 2b + 2\theta \pm A} \tag{3.11}$$

which are both strictly positive. However, the s.o.c. (3.10) reads $-2(b+\theta r) - c = \pm A(-1)$. That is, the s.o.c. is negative for the positive root and positive for the negative root, so, the positive root yields the maximum.

To summarize the consistent-conjecture equilibrium, the unique conjecture is

$$r^* = \frac{-2b - c + A}{2\theta} \tag{3.12}$$

implying equilibrium outputs of

$$q_i^* = \frac{2a}{2(b + \theta) + c + A} \tag{3.13}$$

and profits of

$$\pi^*(r^*, r^*) = \frac{2a^2 A}{(2(b + \theta) + c + A)^2} \tag{3.14}$$

with A as in (3.8).

3.2. Evolutionarily stable conjectures

In this section, instead of imposing a consistency condition as in Definition 1, we will make conjectures subject to evolutionary selection. We will first derive firms' outputs given their conjectures. Since conjectures determine profits, they also determine reproductive success and we can study the evolutionary selection of conjectures in a second step. The underlying assumption is that if firms differ in evolutionary success, the individual characteristics of more successful firms will spread within the population more quickly than the characteristics of the less successful ones. This leads to a dynamic process that determines the long-run distribution of individual characteristics within a "society".

Consider the two steps more formally. We will refer to firm i's (constant) conjecture, $r_i \in \mathbb{R}$, as to firm i's type (higher polynomial conjectures are analytically not tractable). Firms' types may be completely arbitrary and types are known whenever two firms compete against each other. We will derive firms' behavior given their types. Within strategic games this implies that the chosen strategy profile is a Nash equilibrium, denoted by ($q_i^*(r_i, r_j)$, $q_j^*(r_i, r_j)$). In the second step, the types (conjectures) are the strategies and the evolutionary success function, i.e., firm's profits

$$\pi_i^*(r_1, r_2) \equiv \pi_i(q_1^*(r_i, r_j), q_2^*(r_i, r_j)) \tag{3.15}$$

evaluated at equilibrium strategies, are the payoff functions. To find the types that survive in the long run, we apply the static concept of an evolutionarily stable strategy, ESS (Maynard Smith, 1982).

Definition 2. *An equilibrium with evolutionarily stable conjectures is a pair of quantities,* (q_1^*, q_2^*), *and of conjectures,* (r_1^*, r_2^*), *such that*

$$q_1^* = \rho_1(q_2^*), \;\; q_2^* = \rho_2(q_1^*), \tag{3.16}$$

and

$$\pi^*(r_i^*, r_j^*) \geq \pi^*(r, r_j^*) \text{ for all } r \text{ and } i, j = 1, 2; \;\; i \neq j \tag{3.17}$$

and

$$\pi^*(r_i^*, r) > \pi^*(r, r) \text{ for all } r \text{ and } i, j = 1, 2; \;\; i \neq j \text{ with } \pi^*(r_i^*, r_j^*) = \pi^*(r, r^*). \tag{3.18}$$

That is, an equilibrium with evolutionarily stable conjectures requires a Nash equilibrium in outputs given the types (3.16), and an evolutionarily stable preference type r^* which is a best reply against itself (3.17) and no r-mutant invading a society of r^*-players may be more successful than r^* (3.18).

We now solve for an equilibrium of this kind. Our system of first-order conditions (3.3) can be solved for equilibrium strategies[3]

$$q_i^*(r_i, r_j) = \frac{a\,(2b + \theta\,(r_j - 1) + c)}{4b\,(b + c) + \theta\,(2b + c)\,(r_i + r_j) + \theta^2\,(r_i r_j - 1) + c^2}. \tag{3.19}$$

Substituting $q_i^*(r_i, r_j)$ and $q_j^*(r_i, r_j)$ in $\pi_i(.)$ yields the evolutionary success $\pi_i^*(r_i, r_j)$ of type r_i, given that the opponent exhibits type r_j:[4]

$$\begin{aligned} \pi_i^*(r_i, r_j) &= \pi_i(q_i^*(r_i, r_j), q_j^*(r_i, r_j)) && (3.20)\\ &= \frac{(q_i^*)^2}{2}(c + 2\,(b + \theta r_i)). && (3.21) \end{aligned}$$

Note that the evolutionary success functions are symmetric (in the sense of $\pi_1^*(r_1, r_2) = \pi_2^*(r_2, r_1)$) and that the function $\pi_i^*(r_i, r_j)$ determines evolutionary success for all combinations of types. Therefore, we can simplify the notation and refer to $\pi^*(r, l)$ as type r's evolutionary success when paired with type l.

In order to satisfy stability requirement (3.17), we have to find an r^* that is a best reply against itself. Candidates can be found by considering the first-order condition

$$\frac{\partial}{\partial r}\pi^*(r, l) = 0 \tag{3.22}$$

which can be solved for $r = -\theta/(2b + \theta l + c)$. Setting $r = l = r^*$ and solving the resulting quadratic equation with respect to r^* results in two candidates for an ESS:

$$r^* = \frac{-2b - c \pm A}{2\theta} \tag{3.23}$$

where A is defined as in (3.8). We know already that the negative root violates the second-order condition for profit maximization with respect to output. Therefore, only the candidate $r^* = (-2b - c + A)\,/2\theta$ remains.

[3] We do not need to impose parameter restrictions that guarantee equilibrium quantities to be non-negative as economically meaningless behavior will be driven out by evolutionary forces (see below).

[4] Note that the game with types $\widehat{r}_i$, $\widehat{r}_j$ does not have an equilibrium if $4b\,(b + c) + \theta\,(2b + c)\,(\widehat{r}_i + \widehat{r}_j) + \theta^2\,(\widehat{r}_i\widehat{r}_j - 1) + c^2 = 0$. For such $\widehat{r}_i$, $\widehat{r}_j$ we proceed as in Possajennikov (2000) by extending the fitness function by continuity in the first argument in the sense of $\pi_i^*(\widehat{r}_i, \widehat{r}_j) = \lim_{r_i \to \widehat{r}_i} \lim_{r_j \to \widehat{r}_j} \pi_i^*(r_i, r_j)$. This limit does always exist on the extended real line $\mathbb{R} \cup \{\pm\infty\}$ and, as a result, the function $\pi_i^*(r_i, r_j)$ is differentiable with respect to the first argument at $r_j = r_i$.

To prove that r^* is the unique best preference parameter against itself, consider

$$r\left(r^*, r^*\right) - r\left(r, r^*\right) = \frac{4a^2\theta^2 f(r)}{\left(A^2 - 2\theta^2 + \theta r\left(2b + c\right) + \left(2b + \theta r + c\right)A\right)^2 \left(2(b+\theta) + c + A\right)^2} \quad (3.24)$$

where $f(r) = a_2 r^2 + a_1 r + a_0$ with

$$a_2 = 8b(b^2 - \theta^2) + c\left(4(3b^2 - \theta^2) + c(c + 6b)\right) + A\left(c^2 + 4bc + 4b^2 - 2\theta^2\right) \quad (3.25)$$

$$a_1 = -8\theta^3 + 8\theta b^2 + 8bc\theta + 2\theta c^2 + A\left(4\theta b + 2\theta c\right) \quad (3.26)$$

$$a_0 = 2\theta^2 A. \quad (3.27)$$

It is clear that the sign of (3.24) is determined by the sign of the function $f(r) = a_2 r^2 + a_1 r + a_0$. Note that $a_2 > 0$ for fixed b, c and θ. Thus, $f(r)$ is a U-shaped parabola for each fixed set of b, c and θ. Solving $\partial f(r)/\partial r = 0$ for r, shows that the minimum of the function $f(r)$ occurs at $r = -a_1/2a_2$. Now, note that $f(-a_1/2a_2) = 0$ and that $-a_1/2a_2 = r^*$. That is, the function f and thus the expression $r\left(r^*, r^*\right) - r\left(r, r^*\right)$ in (3.24) is 0 if and only if $r = r^*$ and otherwise it is positive. This implies that r^* is the unique evolutionarily stable type (conjecture).

Proposition 1. *The unique evolutionarily stable conjectures of the market game defined above are given by* $r^* = (-2b - c + A)/2\theta$ *and are equal to the consistent conjectures.*

Since the evolutionarily stable conjecture is equal to the consistent conjecture, also outputs and profits are as in (3.13) and (3.14) above.

4. Discussion

In this paper we propose an evolutionary process to select among conjectural variations. We first determine the unique equilibrium in quantities for all possible combinations of linear conjectures. For the evolutionary game with conjectures as mutants and reproductive success (a firm's profit) as the payoff functions, we study conjectures which are evolutionarily stable. It turns out that the equilibrium with evolutionarily stable conjectures is the same as Bresnahan's (1981) consistent-conjectures equilibrium. In this way, we justify both the outcome implied by consistent conjectures and the conjectures themselves.

Evolution favors firm types with better relative performance. In our model, a negative conjecture serves as a commitment device in the sense that it yields a relative profit improvement compared to a type with a larger conjecture. Therefore, evolution selects generally negative conjectures. With homogenous goods and constant marginal cost, for example, it yields the so-called Bertrand conjectures ($r^* = -1$). However, the result that the evolutionarily stable conjectures coincide with the consistent conjectures is surprising as there is no obvious analogy of the two concepts.

Our result may be positively interpreted as it provides support of consistent conjectures. The other side of the medal, the negative interpretation of our result, is that no other conjecture can be justified by arguments based on evolutionary selection. Many empirical researchers use the notion of conjectural variation as a useful shortcut to capture the degree of "competitiveness" which is not reflected in the number of firms, the extent of product differentiation, cost asymmetries, etc. The conjecture is supposed to capture something that can be thought of as conduct in the industry but that is hard to model explicitly (see, e.g., Kim and Vale, 2001). Our result indicates that conjectural variations can not be used to reflect any degree of competitiveness as only one specific conjecture is evolutionarily stable. This indicates that more research on the theoretical foundations of conjectural variations is needed.

References

[1] Bester, H. and W. Güth (1998): "Is altruism evolutionarily stable?", *Journal of Economic Behavior and Organization,* 34/2, 193-209.

[2] Bowley, A.L. (1924): *Mathematical Groundwork of Economics,* Oxford.

[3] Bresnahan, T.F. (1981): "Duopoly models with consistent conjectures", *American Economic Review*, 71, 934-945.

[4] Cabral, L.M.B. (1995): "Conjectural variations as a reduced form", *Economic Letters*, 49, 397-402.

[5] Daughety, A.F. (1985): "Reconsidering Cournot: the Cournot equilibrium is consistent", *RAND Journal of Economics,* 16(3), 368-379.

[6] Dockner, E.J. (1992): "A dynamic theory of conjectural variations", *Journal of Industrial Economics,* 40, 377-95.

[7] Güth, W. and S. Huck (1997): "A new justification of monopolistic competition", *Economics Letters*, 57, 177-182.

[8] Güth, W. and M. Yaari (1992): "An evolutionary approach to explain reciprocal behavior in a simple strategic game", in: *Explaining Process and Change - Approaches to Evolutionary Economics* (ed.: U. Witt), The University of Michigan Press, Ann Arbor, 23-34.

[9] Hicks, J.R. (1935), "Annual survey of economic theory: The theory of monopoly", *Econometrica,* 3, 1–20.

[10] Huck, S. and J. Oechssler (1999): "The indirect evolutionary approach to explaining fair allocations", *Games and Economic Behavior* 28, 13-24.

[11] Kim, M. and B. Vale (2001): "Non-price strategic behavior: the case of bank branches", *International Journal of Industrial Organization,* 19, 1583-1602.

[12] Königstein, M. and W. Müller (2000): "Combining rational choice and evolutionary dynamics: The indirect evolutionary approach", *Metroeconomica* 51(3), 235-256.

[13] Leontieff, W. (1936): "Stackelberg on monopolistic competition", *Journal of Political Economy,* 44, 554-559.

[14] Makowski, L. (1987): "Are 'rational conjectures' rational?", *Journal of Industrial Economics,* 36, 35-37.

[15] Martin, S. (2002): *Advanced Industrial Economics*, 2nd edition, Blackwell, Oxford UK & Cambridge USA.

[16] Maynard Smith, J. (1982): *Evolution and the Theory of Games,* Cambridge University Press.

[17] Possajennikov, A. (2000): "On the evolutionary stability of altruistic and spiteful preferences", *Journal of Economic Behavior and Organization* 42, 125-129.

[18] Sabourian, H. (1992): "Rational conjectural equilibrium and repeated games", in *Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn* (eds.: P. Dasgupta, D. Gale, O. Hart, and E. Maskin), Cambridge: MIT Press, 228-57.

[19] Stackelberg (1934): *Marktform und Gleichgewicht,* Springer, Vienna and Berlin.