What learning models tell us to expect in Three by Three Bimatrix games. *

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Abstract

This paper extends several existing learning models to investigate their fixed points (their long run predictions of play). The fixed points of the model are not necessarily at the Nash equilibria of the payoff matrices but are a function of both the Nash equilibria and the parameters of the model. The stability of these fixed points also depends on both the characteristics of payoff matrix used and the parameters of the model. These new findings indicate that behaviour previously thought to be inconsistent with theory may not necessarily be so.

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1 Introduction

All of economics deals with notions of equilibrium. These notions serve us well in a static world, but as soon as we begin to apply these models we encounter unanswered questions: how does one get to the equilibrium? how long will it take? and where will we actually end up?

Nash equilibrium theory, applicable in some cases, is incomplete in the case of multiple equilibria. Bayesian updating has been used to model people's adjustments over time but its reliance on priors (and its assumptions on the rationality of the players) precludes its use in all but the most basic applications. What is needed, is a model of adjusting behaviour that can predict where we will end up, how we will get there and how fast. This could then be used in conjunction with models of equilibria to more accurately predict out-of-equilibrium behaviour. Learning models may well provide such an adjustment mechanism.

The models I will be considering have roots in the behavioral school of psychology that was dominant in the first half of the century. Much of the psychology research on learning ended in the 1950's as the behavioral school fell into disfavour (and was replaced by cognitive approaches). Recently, there has been renewed interest in learning in psychology including work by Friedman, Massaro, Kitzis & Cohen (1995) and Kitzis, Kelley, Berg, Massaro & Friedman (1998).

Out of the pre-1950 tradition came the rote learning model (also known as reinforcement learning, stimulus response or the law of effect). In these models, successful strategies are reinforced and are more likely to be used again. First formalized by Bush & Mosteller (1955); recent work on reinforcement models has come from Harley (1981), Cross (1983), Börgers & Sarin (1996), Roth & Erev (1995),Erev & Roth (1998), and Tang (1996).

Belief models are another type of learning mechanism. Here, players form beliefs about the state of the world next period and optimally respond

to these beliefs and individuals are allowed to take into account things they have not personally experienced. Examples of the simplest of this type of learning include Cournot best response (Cournot 1863) and fictitious play (Brown 1951). As mentioned previously, in Cournot learning, what happened last period is assumed to be the best predictor of what will happen next period. In fictitious play, the opponent's average play over time is used as the best predictor of their action next period. Other models of this type include Cheung & Friedman (1997) and Fudenberg & Levine (1998) which both allow for the weighting of past periods. Still more sophisticated are the "sophisticated belief" models that provide complex models of opponent behavior (Selten 1991, Stahl 1993, Stahl 1996, Stahl 1999*a*, Stahl 1999*b*). Most recently, Camerer & Ho (1998*c*) show that the previously competing simple belief and rote learning models can be nested in their Experience Weighted Average (EWA) model, a hybrid model with belief and rote learning as special cases.

The goal of this paper is to present a model that allows the investigation of asymptotic stability in learning models and to investigate how these models describe actual learning behaviour. The paper focuses on three by three bimatrix games because they provide a crucial stepping stone from the two by two bimatrix world to the more flexible multi-choice world.

Learning models model the behaviour of individuals. In order to do this, they build propensities for each possible action. As the relative propensity for a particular action increases, the probability of playing that action increases. Propensities are built out of the stream of payoffs perceived by a player. Reinforcement learning rules give weight to payoffs from actions experienced while belief learning rules incorporate all possible payoffs.

Empirical work so far has concentrated on comparing different models in terms of their fit and predictive ability with experimental data. Camerer & Ho (1998*b*) and (1998*a*) find that their hybrid model performs better

than either of the other two approaches. Erev & Roth (1998) find that reinforcement models perform better and Feltovich (1998) finds that specific model performance depends both on the design of the experiment and on the comparison criterion.

The work to date has generally neglected two fundamental issues: the implications of these models on the location, and the stability of the fixed points. If players were acting in a way consistent with the model, what long run behaviour would we expect to see? Fudenberg & Levine (1998) and Ellison & Fudenberg (2000) have begun to look at these issues for special cases. However, the most common models are lacking this analysis.

This paper proposes a model that combines insights from Fudenberg & Levine (1995) and Camerer & Ho (1998*c*) which in turn was built on Cheung & Friedman (1998) and Roth & Erev (1995). The questions of fixed points and stability are examined.

2 The Model

The games I will be considering in this dissertation have payoffs that depend on both your own action and the action of your opponent (your opponent can be a single person or you can be playing against a group of opponents). These games are repeated for a number of periods. Every period you and your opponent simultaneously make choices among the available actions. Everyone gets to see the payoff consequences of these actions and you get to make your next period's choice[1]. The payoffs are used to build the propensities for each action which are then used to predict probabilities of play. The propensity $P^i_{t,j}$ of individual i at time t for each action j is:

[1]The model can be applied more generally than this discussion indicates. The structure of this game is introduced here for illustrative purposes only.

$$P^i_{t,j} = \frac{\int_0^t (\phi^i)^{(t-\tau)} \pi^i_{\tau,j}(\delta^i, c^i_\tau, s^i_\tau)\, \mathrm{d}\tau}{\int_0^t (\phi^i)^\tau \, \mathrm{d}\tau} \ . \tag{1}$$

The numerator is a discounted sum of the stream of payoffs over time and $\phi^i \in (0,1]$ is the discount. The payoff at time t to action j is $\pi^i_{t,j}(\cdot)$. The denominator normalizes the propensity so that the propensities at different times can be directly compared. A discount near zero, would indicate a Cournot type player who only uses payoff information from the last period. A player with a discount $\phi^i = 1$, would have a propensity:

$$P^i_{t,j} = \frac{\int_0^t 1^{(t-\tau)} \pi^i_{\tau,j}(\delta^i, c^i_\tau, s^i_\tau)\, \mathrm{d}\tau}{\int_0^t 1^\tau \, \mathrm{d}\tau} \tag{2}$$

$$= \frac{\int_0^t \pi^i_{\tau,j}(\delta^i, c^i_\tau, s^i_\tau)\, \mathrm{d}\tau}{t} \tag{3}$$

which is the average of the payoffs to each choice over time (this is also known as a Fictitious Play player).

The payoff, or profit, to action j is the jth element of the vector $\pi^i_t(\delta^i, c^i_t, s^i_t)$ and is a function of δ^i, a weight on the importance of the payoffs to actions not chosen relative to that chosen. As δ approaches 0, actions not chosen have a smaller and smaller weight (they are no longer used) which corresponds to reinforcement learning. At $\delta = 1$ we get pure belief learning and the propensities are updated with all the payoff information. Intermediate values of δ can be thought of as weak belief learning where you don't put as much weight on actions you have not experienced. The action chosen by i at time t is a vector c^i_t (and is assumed to be a pure strategy vector[2]). The state faced by i (what i's opponents are doing) at time t is s^i_t. Note that in both c^i_t and s^i_t, the elements sum to one since they represent distributions of play over possible actions.

The actual payoff function used in this analysis and in the experiments is a function of a payoff matrix M (with elements m_{kl}). For a three choice game, M is a 3x3 matrix and the payoff function is:

[2]This assumption is not necessary but simplifies the presentation of the model.

$$\pi_t^i(\delta^i, c_t^i, s_t^i) = \begin{bmatrix} \pi_{t,1}^i(\delta^i, c_t^i, s_t^i) \\ \pi_{t,2}^i(\delta^i, c_t^i, s_t^i) \\ \pi_{t,3}^i(\delta^i, c_t^i, s_t^i) \end{bmatrix} \tag{4}$$

$$= \left(diag \begin{bmatrix} c_1^i + \delta^i c_2^i + \delta^i c_3^i \\ \delta^i c_1^i + c_2^i + \delta^i c_3^i \\ \delta^i c_1^i + \delta^i c_2^i + c_3^i \end{bmatrix} \right) \cdot M \cdot \begin{bmatrix} s_1^i \\ s_2^i \\ s_3^i \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} (c_1^i + \delta^i c_2^i + \delta^i c_3^i)(m_{11} s_1^i + m_{12} s_2^i + m_{13} s_3^i) \\ (\delta^i c_1^i + c_2^i + \delta^i c_3^i)(m_{21} s_1^i + m_{22} s_2^i + m_{23} s_3^i) \\ (\delta^i c_1^i + \delta^i c_2^i + c_3^i)(m_{31} s_1^i + m_{32} s_2^i + m_{33} s_3^i) \end{bmatrix} \tag{6}$$

If an individual i had played $c_2^i = [1, 0, 0]'$ period 2 and had been faced by as state $s_2^i = [\frac{1}{3}, \frac{1}{2}, \frac{1}{6}]'$, her payoffs that period would be:

$$\pi_t^i(\delta^i, c_t^i, s_t^i) = \begin{bmatrix} (1 + \delta^i 0 + \delta^i 0)(m_{11}\,\frac{1}{3} + m_{12}\,\frac{1}{2} + m_{13}\,\frac{1}{6}) \\ (\delta^i 1 + 0 + \delta^i 0)(m_{21}\,\frac{1}{3} + m_{22}\,\frac{1}{2} + m_{23}\,\frac{1}{6}) \\ (\delta^i 1 + \delta^i 0 + 0)(m_{31}\,\frac{1}{3} + m_{32}\,\frac{1}{2} + m_{33}\,\frac{1}{6}) \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 1\,(m_{11}\,\frac{1}{3} + m_{12}\,\frac{1}{2} + m_{13}\,\frac{1}{6}) \\ \delta^i\,(m_{21}\,\frac{1}{3} + m_{22}\,\frac{1}{2} + m_{23}\,\frac{1}{6}) \\ \delta^i\,(m_{31}\,\frac{1}{3} + m_{32}\,\frac{1}{2} + m_{33}\,\frac{1}{6}) \end{bmatrix} \tag{8}$$

If $\delta = 1$, this model is equivalent to a belief model where all possible payoffs are weighted equally. If $\delta = 0$, we have the reinforcement learning model where only the action chosen is used in the propensity. Relaxing the assumption that the discount values in the numerator and denominator are equal makes this model asymptotically equivalent to a continuous time version of the EWA model[3].

The model is a probabilistic model, the probability of an individual choosing an action increases as the propensity for that action increases. To

[3]The EWA model, as presented in Camerer and Ho (1998a,b,c), consists of the observation equivalent of past experience N(t) and the propensity $P_i^j(t)$ for action j at time t.

map the propensities on to actions, the Logit probability response function is used so that the probability of i's jth action $c^i_{t,j}$ at time t is:

$$Prob(c^i_{t,j} = j) = \frac{e^{\lambda^i \cdot P^i_{t,j}}}{\sum_j e^{\lambda^i \cdot P^i_{t,j}}} \tag{9}$$

where λ^i is the parameter. A λ^i near zero would indicate a misspecified model while a large λ^i would indicate that the propensities do a good job of explain the observed play. Negative λ^i values would suggest misspecification, possibly due to higher order reasoning and anticipatory play in the sense of Selten (1991).

The Logit has been used widely used in the literature on Learning (Mookherjee & Sopher 1994, McKelvey & Palfrey 1995, Fudenberg & Levine 1998, Camerer & Ho 1998*b*). Other mapping functions are possible, but work by Camerer & Ho (1998*c*) has shown little difference between the Logit and power response functions. Tang (1996) had similar results looking only at reinforcement models. The Logit has the additional benefit of allowing negative payoffs.

The initial N(0) and P(0) are updated after period 0 so that:

$$N(t) = \rho \cdot N(t-1) + 1, t \geq 1$$

and,

$$P^j_i(t) = \frac{\phi \cdot N(t-1) \cdot P^j_i(t-1) + [\delta + (1-\delta) \cdot I(s^j_i, s_i(t))] \cdot \pi_i(s^j_i, s_{-i}(t))}{N(t)}, t \geq 1.$$

This is equivalent to

$$N(t) = \rho^t \cdot N(0) + \sum_{\tau=0}^{t-1} \rho^\tau$$

and,

$$P^j_i(t) = \frac{\phi^t N(0) P(0) + \sum_{\tau=1}^t \phi^{t-\tau}[\delta + (1-\delta) \cdot I(s^j_i, s_i(t))] \cdot \pi_i(s^j_i, s_{-i}(t))}{\rho^t \cdot N(0) + \sum_{\tau=1}^t \rho^{\tau-1}}.$$

For $0 < \phi, \rho < 1$, this is asymptotically equivalent to

$$P^j_i(t) = \frac{\sum_{\tau=1}^t \phi^{t-\tau}[\delta + (1-\delta) \cdot I(s^j_i, s_i(t))] \cdot \pi_i(s^j_i, s_{-i}(t))}{\sum_{\tau=1}^t \rho^{\tau-1}}.$$

The Logit is invariant to an additive constant on the propensities. In order to estimate the model the probabilities are normalized relative to one of the actions. The use of the Logit assumes the independence of irrelevant alternatives: that the ratio of the probabilities of any two actions j and k, P_j/P_k, be independent of the remaining probabilities (Green 1993). This implies that adding an alternative to the model, or changing the characteristics of another alternative that is already included, will not change the odds between actions j and k (Davidson & Mackinnon 1993), a plausible constraint in this class of games.

It is important (but neglected in the literature so far) to look at this model in terms of two characteristics: the fixed points of the model in relation to the Nash equilibrium of the payoff matrix, and the model stability around each fixed point. Fudenberg & Levine (1998) argue that learning models can suggest useful ways to evaluate and modify the traditional equilibrium concepts (including Nash Equilibria). Also, in order to apply learning models in anything but the shortest time frames, it is necessary to have some idea of the long run predictions of the models. I consider the fixed points and stability of this model in turn.

3 Fixed points

The fixed points for this model depend both on the Nash equilibria of the payoff matrix used and on the parameters of the model.

A fixed point is a set of probabilities ($Q^\star$) that map back on to themselves. In other words, players facing the state defined by a fixed point choose actions with probabilities that again yield the same fixed point. Assuming identical parameter across individuals the propensities satisfy:

$$P_t^\star = \frac{\int_0^t \phi^{(t-\tau)} \pi_j(\delta, c = Q^\star, s = Q^\star)\, \mathrm{d}\tau}{\int_0^t \phi^\tau \, \mathrm{d}\tau} \tag{10}$$

$$= \frac{\pi(\delta, Q^\star) \int_0^t \phi^{t-\tau} \, \mathrm{d}\tau}{\int_0^t \phi^\tau \, \mathrm{d}\tau} \tag{11}$$

$$= \pi(\delta, Q^\star). \tag{12}$$

Thus, propensities are static when play remains at $(Q^\star)$.

The fixed point(s) $(Q^\star)$ is defined by the equation:

$$Q^\star = F\left(\lambda \, diag \begin{bmatrix} q_1^\star + \delta q_2^\star + \delta q_3^\star \\ \delta q_1^\star + q_2^\star + \delta q_3^\star \\ \delta q_1^\star + \delta q_2^\star + q_3^\star \end{bmatrix} \cdot M \cdot Q^\star\right) \tag{13}$$

where $F(\cdot)$ is the Logit function and $Q^\star = [q_1^\star, q_2^\star, q_3^\star]'$ is the probability of play at the fixed point.

This is a transcendental equation and not analytically solvable. The smaller the Logit parameter (λ), the more there is a tendency towards the center of the simplex (1/3,1/3,1/3) so that:

$$\lambda = 0 \quad \Rightarrow \quad Q^* = (1/3, 1/3, 1/3) \tag{14}$$

$$\lambda \to \infty \quad \Rightarrow \quad Q^* = NE. \tag{15}$$

and continuous between. As mentioned previously, λ is assumed non negative and small values would indicate a misspecification of the model.

The parameter δ (the weight on actions not chosen relative to that chosen) is bounded between 0 and 1. Larger values equally weight all actions by their potential payoffs and should slow down the movement away from the center of the simplex as λ becomes large. The effects of the interactions of the parameters and how they interact with the payoff matrices (and therefore the NE of the payoff matrices) are difficult to quantify. As a result, we will rely on numerical simulations.

Tables 1 and 2 and Figures 1 and 2 show how the fixed points change as the parameters (δ, λ and ϕ) change using two different payoff matrices.

Figure 1: Matrix 2- Distribution of fixed points using different parameters ($\delta \in [0, 1]$ by 0.01 and $\lambda \in [0, 2]$ by 0.1)

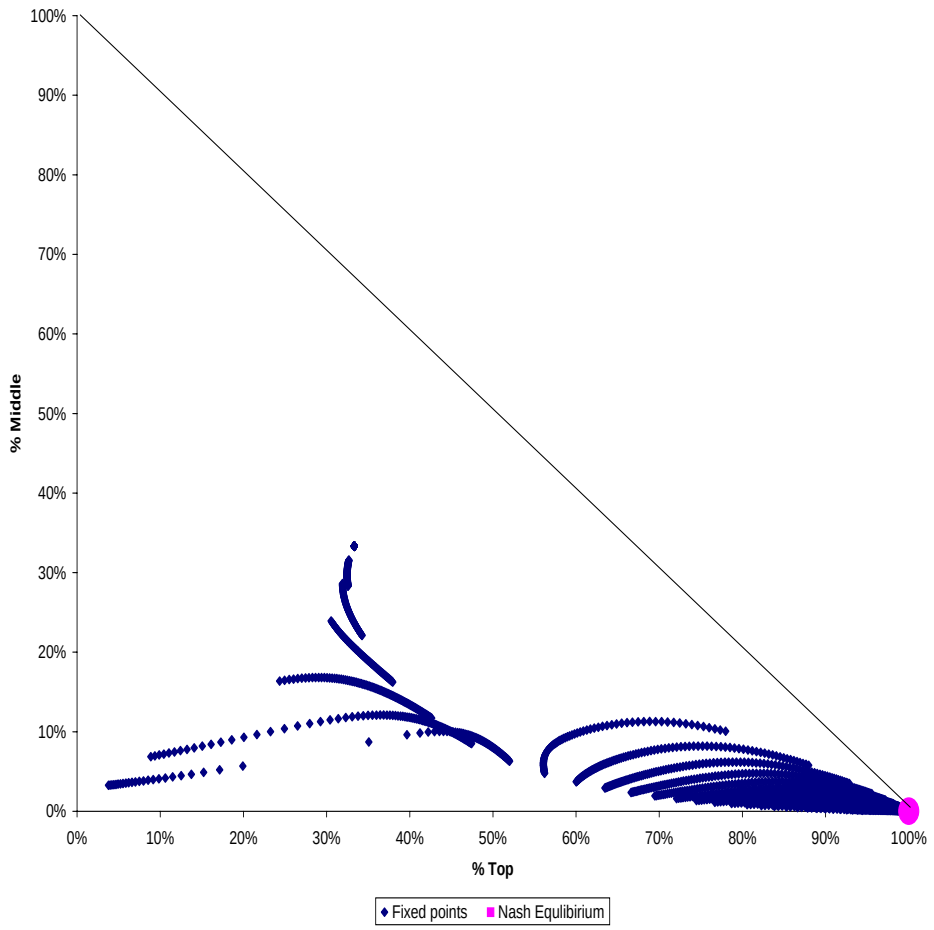


Figure 2: Matrix 4- Distribution of fixed points using different parameters ($\delta \in [0, 1]$ by 0.01 and $\lambda \in [0, 2]$ by 0.1)

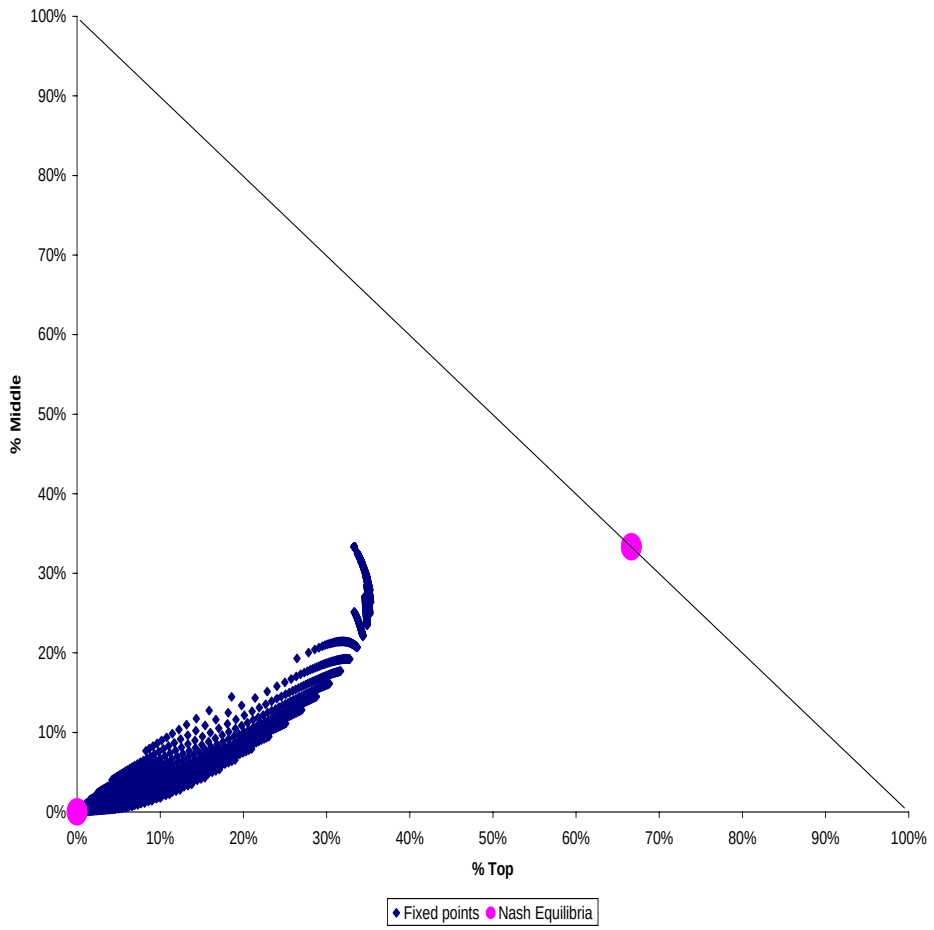


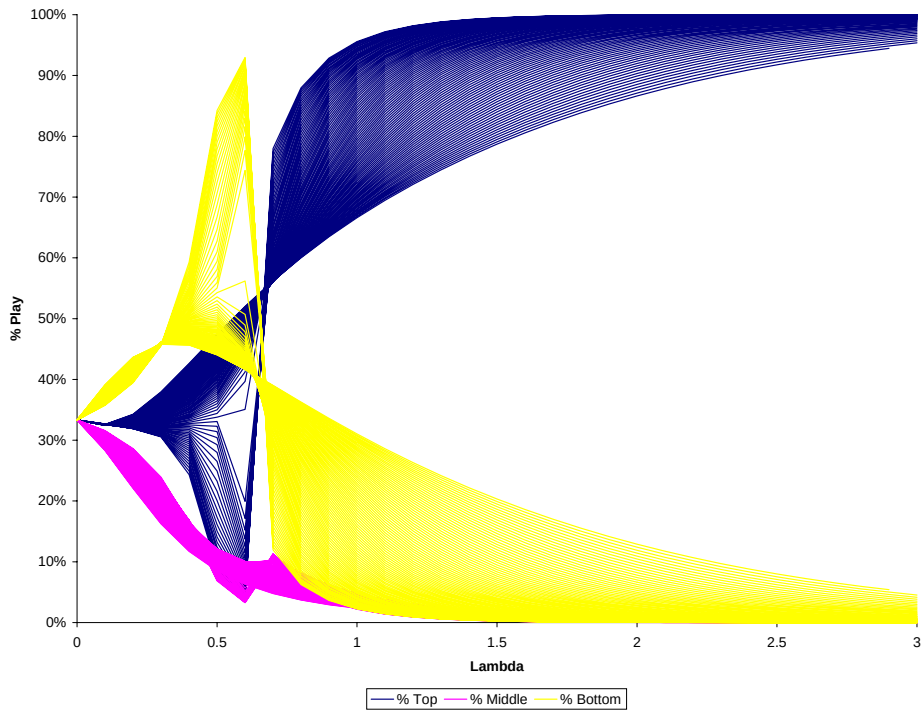
Table 1: Matrix 2 Fixed points and stability simulations

δ	λ	ϕ	%Top	%Middle	%Bottom	Eigen-value1	Eigen-value2	Stable
0	0	0.1	0.33	0.33	0.33	-6.91	-2.30	1
		0.5	0.33	0.33	0.33	-2.08	-0.69	1
		1	0.33	0.33	0.33	0.00	0.00	0
	0.5	0.1	0.37	0.19	0.44	-1.10	-1.10	1
		0.5	0.37	0.19	0.44	-0.33	-0.33	1
		1	0.37	0.19	0.44	0.00	0.00	0
	1	0.1	0.96	0.02	0.02	0.54	0.54	0
		0.5	0.96	0.02	0.02	0.16	0.16	0
		1	0.96	0.02	0.02	0.00	0.00	0
	1.5	0.1	0.99	0.00	0.00	1.68	0.23	0
		0.5	0.99	0.00	0.00	0.50	0.07	0
		1	0.99	0.00	0.00	0.00	0.00	0
	2	0.1	1.00	0.00	0.00	2.07	-1.09	0
		0.5	1.00	0.00	0.00	0.62	-0.33	0
		1	1.00	0.00	0.00	0.00	0.00	0
0.5	0	0.1	0.33	0.33	0.33	-6.91	-2.30	1
		0.5	0.33	0.33	0.33	-2.08	-0.69	1
		1	0.33	0.33	0.33	0.00	0.00	0
	0.5	0.1	0.46	0.15	0.39	-4.25	-1.49	1
		0.5	0.46	0.15	0.39	-1.28	-0.45	1
		1	0.46	0.15	0.39	0.00	0.00	0
	1	0.1	0.82	0.05	0.13	-1.50	-1.50	1
		0.5	0.82	0.05	0.13	-0.45	-0.45	1
		1	0.82	0.05	0.13	0.00	0.00	0
	1.5	0.1	0.96	0.01	0.03	-1.11	-1.11	1
		0.5	0.96	0.01	0.03	-0.34	-0.34	1
		1	0.96	0.01	0.03	0.00	0.00	0
	2	0.1	0.99	0.00	0.01	-1.33	-1.33	1
		0.5	0.99	0.00	0.01	-0.40	-0.40	1
		1	0.99	0.00	0.01	0.00	0.00	0
1	0	0.1	0.33	0.33	0.33	-6.91	-2.30	1
		0.5	0.33	0.33	0.33	-2.08	-0.69	1
		1	0.33	0.33	0.33	0.00	0.00	0
	0.5	0.1	0.50	0.12	0.38	-7.39	-1.79	1
		0.5	0.50	0.12	0.38	-2.22	-0.54	1
		1	0.50	0.12	0.38	0.00	0.00	0
	1	0.1	0.71	0.04	0.24	-4.25	-4.25	1
		0.5	0.71	0.04	0.24	-1.28	-1.28	1
		1	0.71	0.04	0.24	0.00	0.00	0
	1.5	0.1	0.83	0.02	0.15	-3.70	-3.70	1
		0.5	0.83	0.02	0.15	-1.11	-1.11	1
		1	0.83	0.02	0.15	0.00	0.00	0
	2	0.1	0.90	0.01	0.10	-3.32	-3.32	1
		0.5	0.90	0.01	0.10	-1.00	-1.00	1
		1	0.90	0.01	0.10	0.00	0.00	0

Table 2: Matrix 4 Fixed points and stability simulations

δ	λ	ϕ	%Top	%Middle	%Bottom	Eigen-value1	Eigen-value2	Stable
0	0	0.1	0.33	0.33	0.33	-6.91	-2.30	1
		0.5	0.33	0.33	0.33	-2.08	-0.69	1
		1	0.33	0.33	0.33	0.00	0.00	0
	0.5	0.1	0.35	0.29	0.37	-1.73	-3.29	1
		0.5	0.35	0.29	0.37	-0.52	-0.99	1
		1	0.35	0.29	0.37	0.00	0.00	0
	1	0.1	0.02	0.02	0.95	2.49	-1.53	0
		0.5	0.02	0.02	0.95	0.75	-0.46	0
		1	0.02	0.02	0.95	0.00	0.00	0
	1.5	0.1	0.00	0.00	0.99	6.18	-1.38	0
		0.5	0.00	0.00	0.99	1.86	-0.42	0
		1	0.00	0.00	0.99	0.00	0.00	0
	2	0.1	0.00	0.00	1.00	7.43	-1.40	0
		0.5	0.00	0.00	1.00	2.24	-0.42	0
		1	0.00	0.00	1.00	0.00	0.00	0
0.5	0	0.1	0.33	0.33	0.33	-6.91	-2.30	1
		0.5	0.33	0.33	0.33	-2.08	-0.69	1
		1	0.33	0.33	0.33	0.00	0.00	0
	0.5	0.1	0.35	0.26	0.39	-2.60	-4.50	1
		0.5	0.35	0.26	0.39	-0.78	-1.35	1
		1	0.35	0.26	0.39	0.00	0.00	0
	1	0.1	0.25	0.15	0.61	-1.40	-2.87	1
		0.5	0.25	0.15	0.61	-0.42	-0.86	1
		1	0.25	0.15	0.61	0.00	0.00	0
	1.5	0.1	0.03	0.01	0.95	1.50	-2.67	0
		0.5	0.03	0.01	0.95	0.45	-0.80	0
		1	0.03	0.01	0.95	0.00	0.00	0
	2	0.1	0.01	0.00	0.99	2.13	-2.59	0
		0.5	0.01	0.00	0.99	0.64	-0.78	0
		1	0.01	0.00	0.99	0.00	0.00	0
1	0	0.1	0.33	0.33	0.33	-6.91	-2.30	1
		0.5	0.33	0.33	0.33	-2.08	-0.69	1
		1	0.33	0.33	0.33	0.00	0.00	0
	0.5	0.1	0.35	0.25	0.40	-5.74	-3.43	1
		0.5	0.35	0.25	0.40	-1.73	-1.03	1
		1	0.35	0.25	0.40	0.00	0.00	0
	1	0.1	0.32	0.18	0.51	-4.42	-4.42	1
		0.5	0.32	0.18	0.51	-1.33	-1.33	1
		1	0.32	0.18	0.51	0.00	0.00	0
	1.5	0.1	0.23	0.09	0.67	-4.76	-3.06	1
		0.5	0.23	0.09	0.67	-1.43	-0.92	1
		1	0.23	0.09	0.67	0.00	0.00	0
	2	0.1	0.14	0.03	0.83	-4.43	-2.48	1
		0.5	0.14	0.03	0.83	-1.33	-0.75	1
		1	0.14	0.03	0.83	0.00	0.00	0

Figure 3: Matrix 2- How the fixed points depend on λ allowing δ to vary.



Matrix2 (from Stahl and Wilson,1994)

$$\text{Matrix}\,2 = \begin{bmatrix} 4 & 1 & 7 \\ 2 & 8 & 0 \\ 3 & 10 & 6 \end{bmatrix} \tag{16}$$

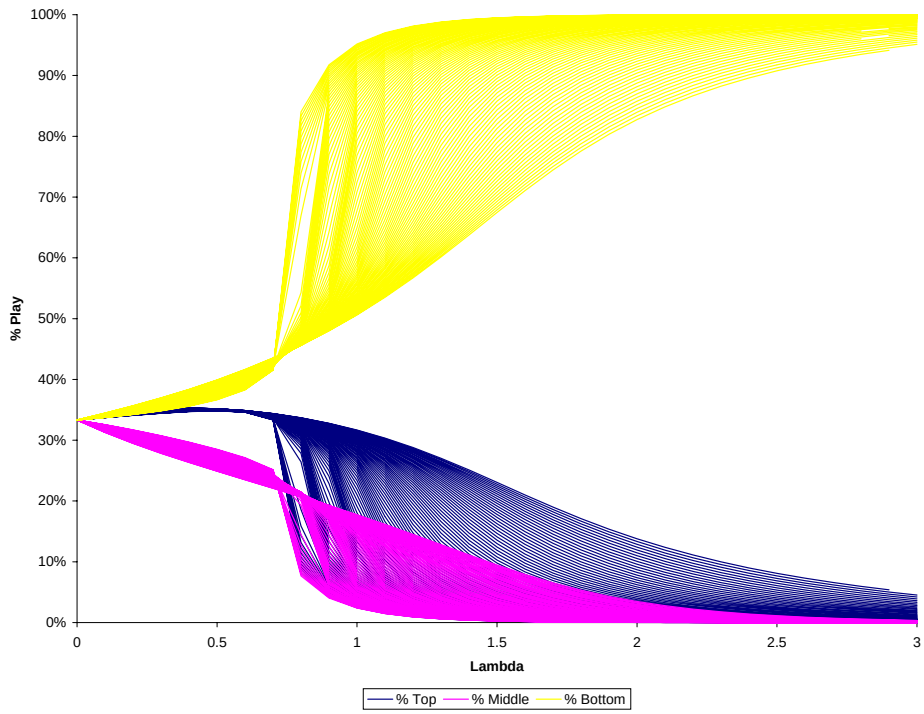
has a single Nash equilibrium at (1,0,0), Matrix 4 is a HDB matrix (from Cheung and Friedman 1997) where:

$$\text{Matrix}\,4 = \begin{bmatrix} -2 & 8 & 3 \\ 0 & 4 & 2 \\ -1 & 6 & 4 \end{bmatrix} \tag{17}$$

with a mixed Nash equilibrium at $(2/3, 1/3, 0)$ and a pure Nash equilibrium at $(0, 0, 1)$.

The location of the fixed points depends only on λ and δ and not ϕ. Figures 3 and 4 show how the fixed points change as lambda changes (allowing

Figure 4: Matrix 4- How the fixed points depend on λ allowing δ to vary



for different values of ϕ and δ). As expected, in both matrices, small values of lambda do "pull" the fixed point to the center of the simplex. As the multiplier becomes large, the fixed points move towards a pure strategy NE action. The importance of the underlying matrix can be seen by comparing the behavior in the two matrices. Matrix 4 has all of the fixed points in a narrow band between the center of the simplex and the Nash. Matrix 2 however exhibits a more complex arrangement, where the fixed points depend heavily on δ. When $\delta < 0.1$ the fixed points are between the center of the simplex and (0,0,1). This is highly inconsistent with the Nash equilibrium of the payoff matrix. Further work is needed to determine why this is the case. Figures 5 and 6 show how the fixed points depend on δ by identifying the points were $\delta <= 0.5$ and $\delta > 0.5$. Note that the in Matrix 2, effect of small δ values ($\delta < 0.1$) can be clearly seen and in Matrix 4, the mixed NE is *not* the limit of the fixed points.

Figure 5: Matrix 2: How fixed points depend on δ

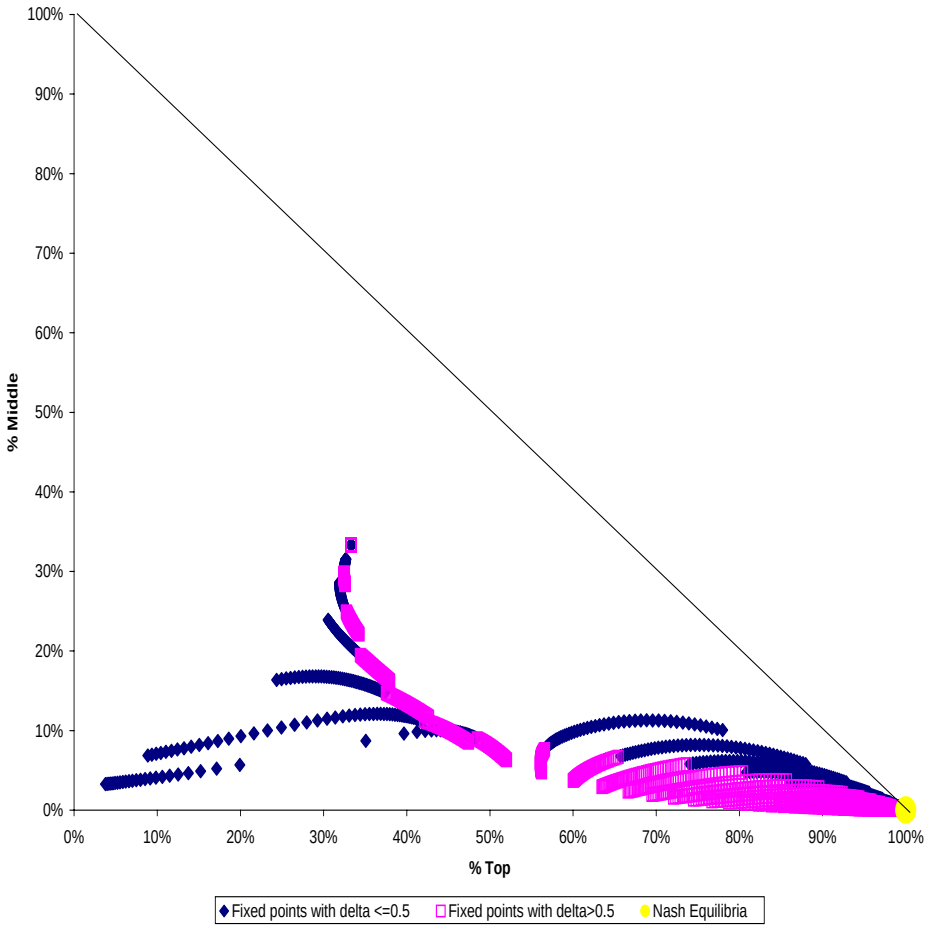
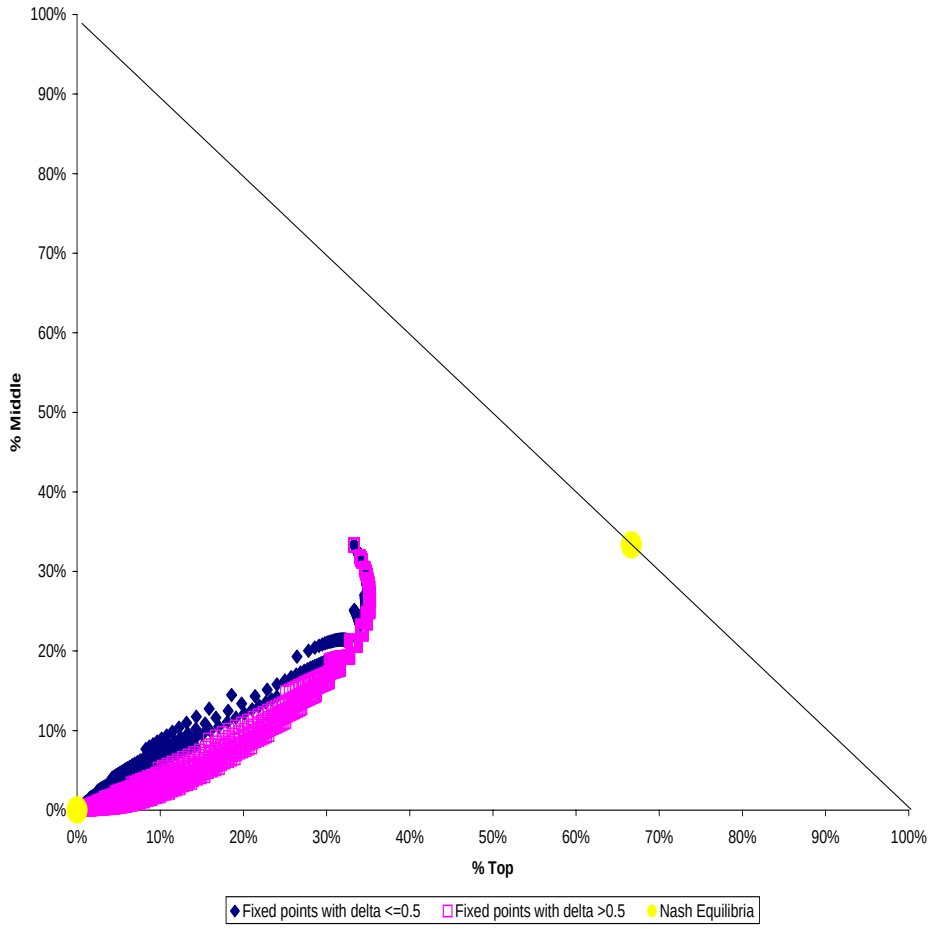


Figure 6: Matrix 4: How fixed points depend on δ



4 Stability

4.1 Single population games

To look at the stability of the fixed points of a model we need to look at the behavior predicted by the model in the region around the fixed point. Using the time limit of the model to find the asymptotic equivalent of the model,

$$
\begin{aligned}
P &= \lim_{t\to\infty}\left(\frac{\int_0^t \phi^{t-\tau}\pi_j(\delta, c_\tau, s_\tau)\,\mathrm{d}\tau}{\int_0^t \phi^{\tau}\,\mathrm{d}\tau}\right) && (18)\\
&= \lim_{t\to\infty}\left(\frac{\int_0^t \phi^{t-\tau}\pi_j(\delta, c_\tau, s_\tau)\,\mathrm{d}\tau}{\left(\frac{-1+\phi^t}{\ln(\phi)}\right)}\right) && (19)\\
&\approx -\ln(\phi)\int_0^t \phi^{t-\tau}\pi_j(\delta, c_\tau, s_\tau)\,\mathrm{d}\tau\ . && (20)
\end{aligned}
$$

Looking at how the model changes over time,

$$
\begin{aligned}
\dot{P} &\approx \frac{\partial}{\partial t}\left(-\ln(\phi)\int_0^t \phi^{t-\tau}\pi_j(\delta, c_\tau, s_\tau)\,\mathrm{d}\tau\right) && (21)\\
&\approx -\ln\phi\,\frac{\partial}{\partial t}\left(\int_0^t \phi^{t-\tau}\pi_j(\delta, c_\tau, s_\tau)\,\mathrm{d}\tau\right) && (22)\\
&\approx -\ln\phi\left(\ln\phi\int_0^t \phi^{t-\tau}\pi_{\tau,j}(\delta, c_\tau, s_\tau)\,\mathrm{d}\tau + \pi_{t,j}(\delta, c_t, s_t)\right) && (23)\\
&\approx \ln\phi\cdot P - \ln\phi\cdot\pi_t. && (24)\\
&\approx \ln(\phi)\begin{bmatrix} p_1-(s_1+\delta s_2+\delta s_3)(m_{1,1}s_1+m_{1,2}s_2+m_{1,3}s_3)\\ p_2-(\delta s_1+s_2+\delta s_3)(m_{2,1}s_1+m_{2,2}s_2+m_{2,3}s_3)\\ p_3-(\delta s_1+\delta s_2+s_3)(m_{3,1}s_1+m_{3,2}s_2+m_{3,3}s_3)\end{bmatrix}. && (25)
\end{aligned}
$$

Substituting,

$$
\begin{aligned}
s_1 &= \tfrac{e^{\lambda p_1}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}\\
s_2 &= \tfrac{e^{\lambda p_2}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}\\
s_3 &= \tfrac{e^{\lambda p_3}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}
\end{aligned}
\qquad (26)
$$

gives,

$$
\dot{P}_t \approx ln(\phi)\begin{bmatrix} p_1-\left(\frac{e^{\lambda p_1}+\delta e^{\lambda p_2}+\delta e^{\lambda p_3}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}\right)\cdot\left(\frac{m_{1,1}e^{\lambda p_1}+m_{1,2}e^{\lambda p_2}+m_{1,3}e^{\lambda p_3}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}\right)\\ p_2-\left(\frac{\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}\right)\cdot\left(\frac{m_{2,1}e^{\lambda p_1}+m_{2,2}e^{\lambda p_2}+m_{2,3}e^{\lambda p_3}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}\right)\\ p_3-\left(\frac{\delta e^{\lambda p_1}+\delta e^{\lambda p_2}+e^{\lambda p_3}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}\right)\cdot\left(\frac{m_{3,1}e^{\lambda p_1}+m_{3,2}e^{\lambda p_2}+m_{3,3}e^{\lambda p_3}}{e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}}\right)\end{bmatrix}. \qquad (27)
$$

Using a first order Taylor expansion to linearize $\dot{P}_t$ around the interior fixed point $Q^* = [q_1^* \; q_2^* \; q_3^*]'$, and using the Jacobian $J_{q^*} = \dot{P}'_{q^*}$,

$$\dot{P}_t \approx \dot{P}_{q^*} + J_{q^*} \cdot (P - Q^*) \tag{28}$$

$$\dot{P}_t - \dot{P}_{q^*} \approx J_{q^*} \cdot (P - Q^*) \tag{29}$$

redefining $W = P - Q^*$,

$$\dot{W}_t \approx J_{q^*} \cdot W \tag{30}$$

where,

$$J_{q^*} = \left. \begin{bmatrix} \frac{\partial}{\partial p_1}\dot{P}_1 & \frac{\partial}{\partial p_2}\dot{P}_1 & \frac{\partial}{\partial p_3}\dot{P}_1 \\ \frac{\partial}{\partial p_1}\dot{P}_2 & \frac{\partial}{\partial p_2}\dot{P}_2 & \frac{\partial}{\partial p_3}\dot{P}_2 \\ \frac{\partial}{\partial p_1}\dot{P}_3 & \frac{\partial}{\partial p_2}\dot{P}_3 & \frac{\partial}{\partial p_3}\dot{P}_3 \end{bmatrix} \right|_{p_1=q_1^*, p_2=q_2^*, p_3=q_3^*} . \tag{31}$$

Appendix A explicitly lists the elements of the Jacobian.

Although this is a three choice game, the system (assuming a single population game) is actually two dimensional since probabilities sum to one. Using Haigh's (1975) theorem to reduce the matrix, the entries of the 2 by 2 transformation matrix Z are $z_{l,k} = j_{l,k} - j_{l,k+1} - j_{l+1,k} + j_{l+1,k+1}$. So that,

$$Z = \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix} . \tag{32}$$

see Appendix A for the elements of the Z.

For the model to be asymptotically stable, the eigenvalues of the linearized equation around the fixed point have to have negative real parts (Hirsch & Smale 1974). The eigenvalues of the Z matrix (Equation 32) are:

$$\begin{array}{l} \frac{1}{2}\, z_{1,1} + \frac{1}{2}\, z_{2,2} + \frac{1}{2}\sqrt{{z_{1,1}}^2 - 2\, z_{1,1} z_{2,2} + {z_{2,2}}^2 + 4\, z_{1,2} z_{2,1}} \\ \frac{1}{2}\, z_{1,1} + \frac{1}{2}\, z_{2,2} - \frac{1}{2}\sqrt{{z_{1,1}}^2 - 2\, z_{1,1} z_{2,2} + {z_{2,2}}^2 + 4\, z_{1,2} z_{2,1}}. \end{array} \tag{33}$$

Proposition 1 *Let M be a 3x3 single population game, and $\dot{P}$ be the equation of movement over time. Let Z be the matrix defined above with elements $\{z_{i,j}\}$. Then, an interior fixed point Q^* is asymptotically stable iff*

$z_{1,1} + z_{2,2} \geq 0$ *and either* $z_{1,1} \cdot z_{2,2} > z_{1,2} \cdot z_{2,1}$ *or* ${z_{1,1}}^2 - 2\, z_{1,1} z_{2,2} + {z_{2,2}}^2 + 4\, z_{1,2} z_{2,1} \leq 0$

Proof: For a interior fixed point to be asymptotic stable, both eigenvalues of the Z matrix must have negative real parts. If $\frac{1}{2}\, z_{1,1} + \frac{1}{2}\, z_{2,2} < 0$, the second eigenvalue will have negative real parts. The asymptotic stability then depends on the first Eigenvalue and its square root term. If there are only imaginary roots (${z_{1,1}}^2 - 2\, z_{1,1} z_{2,2} + {z_{2,2}}^2 + 4\, z_{1,2} z_{2,1} \leq 0$) there is stability. If there are real roots, to have stability,

$$\left| \frac{1}{2}(z_{1,1} + z_{1,2}) \right| > \frac{1}{2}\sqrt{{z_{1,1}}^2 - 2\, z_{1,1} z_{2,2} + {z_{2,2}}^2 + 4\, z_{1,2} z_{2,1}} \tag{34}$$

and since $\frac{1}{2}\, z_{1,1} + \frac{1}{2}\, z_{2,2} < 0$,

$$\begin{aligned}
\frac{1}{2}(z_{1,1} + z_{1,2}) &< \frac{1}{2}\sqrt{{z_{1,1}}^2 - 2\, z_{1,1} z_{2,2} + {z_{2,2}}^2 + 4\, z_{1,2} z_{2,1}} && (35)\\
(z_{1,1} + z_{1,2})^2 &< {z_{1,1}}^2 - 2\, z_{1,1} z_{2,2} + {z_{2,2}}^2 + 4\, z_{1,2} z_{2,1} && (36)\\
z_{1,1}^2 + 2 z_{1,1} z_{2,2} + z_{2,2}^2 &< {z_{1,1}}^2 - 2\, z_{1,1} z_{2,2} + {z_{2,2}}^2 + 4\, z_{1,2} z_{2,1} && (37)\\
2 z_{1,1} z_{2,2} &< -2\, z_{1,1} z_{2,2} + 4\, z_{1,2} z_{2,1} && (38)\\
z_{1,1} z_{2,2} &< z_{1,2} z_{2,1} && (39)
\end{aligned}$$

Conversely, if $\frac{1}{2}\, z_{1,1} + \frac{1}{2}\, z_{2,2} > 0$, the first eigenvalue can not be negative and thus, there is no asymptotic stability.

The role of $\ln \phi$ in determining asymptotic stability is important since all of the elements of Z are factors of $\ln \phi$ (and thus both eigenvalues are factors of $\ln \phi$ as well). When $\phi = (0, 1)$, $\ln \phi < 0$. Values of $\phi \geq 1$ lead to $\ln \phi \geq 0$ and thus change the signs of the eigenvalues. Matrices and parameter values that are asymptotically stable with $\phi = (0, 1)$ will not be asymptotically stable with $\phi \geq 1$ (though the converse is not necessarily true). Values of $\phi > 1$ correspond to a discount rate consistent with "imprinting": where initial periods have more weight than subsequent periods. The special case $\phi = 1$

corresponds to Fictitious play and leads to asymptotically neutral stability[4]. It is also important to note that this is the only place the parameter ϕ enters in the asymptotic stability of a fixed point Q^*. Assuming $\phi = (0, 1)$, it is the underlying matrix and λ and δ determine stability. Of course, ϕ affects the convergence *rate* for stable fixed points.

4.2 Stability in two population games

In the case of two (or more) population games, the stability requirement changes. The P_t vectors for each of the populations are stacked. For two populations, $P_t = [p_1^1\ p_2^1\ , p_3^1\ p_1^2\ p_2^2, p_3^2\ ]$, and Z includes all of the cross derivatives so that,

$$\dot{P}_t \approx \begin{bmatrix} ln(\phi^1)p_1^1 - ln(\phi^1)(s_1^1 + \delta^1 s_2^1 + \delta^1 s_3^1)(m_{1,1}^2 s_1^2 + m_{1,2}^2 s_2^2 + m_{1,3}^2 s_3^2) \\ ln(\phi^1)p_2^1 - ln(\phi^1)(\delta^1 s_1^1 + s_2^1 + \delta^1 s_3^1)(m_{2,1}^2 s_1^2 + m_{2,2}^2 s_2^2 + m_{2,3}^2 s_3^2) \\ ln(\phi^1)p_3^1 - ln(\phi^1)(\delta^1 s_1^1 + \delta^1 s_2^1 + s_3^1)(m_{3,1}^2 s_1^2 + m_{3,2}^2 s_2^2 + m_{3,3}^2 s_3^2) \\ ln(\phi^2)p_1^2 - ln(\phi^2)(s_1^2 + \delta^2 s_2^2 + \delta^2 s_3^2)(m_{1,1}^1 s_1^1 + m_{1,2}^1 s_2^1 + m_{1,3}^1 s_3^1) \\ ln(\phi^2)p_2^2 - ln(\phi^2)(\delta^2 s_1^2 + s_2^2 + \delta^2 s_3^2)(m_{2,1}^1 s_1^1 + m_{2,2}^1 s_2^1 + m_{2,3}^1 s_3^1) \\ ln(\phi^2)p_3^2 - ln(\phi^2)(\delta^2 s_1^2 + \delta^2 s_2^2 + s_3^2)(m_{3,1}^1 s_1^1 + m_{3,2}^1 s_2^1 + m_{3,3}^1 s_3^1) \end{bmatrix} \tag{40}$$

Substituting,

$$\begin{aligned} s_1^1 &= \frac{e^{\lambda^1 p_1^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}, & s_1^2 &= \frac{e^{\lambda^2 p_1^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}} \\ s_2^1 &= \frac{e^{\lambda^1 p_2^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}, & s_2^2 &= \frac{e^{\lambda^2 p_2^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}} \\ s_3^1 &= \frac{e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}, & s_3^2 &= \frac{e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}} \end{aligned} \tag{41}$$

[4]The intuition is that with Fictitious play the process stagnates: people don't respond appreciably to the last payoff since it averaged with an infinite set of previous payoffs.

gives,

$$\dot{P}_t \approx \begin{bmatrix} \ln(\phi^1)p_1^1 - \ln(\phi^1)\left(\frac{e^{\lambda^1 p_1^1}+\delta^1 e^{\lambda^1 p_2^1}+\delta^1 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1}+e^{\lambda^1 p_2^1}+e^{\lambda^1 p_3^1}}\right)\cdot\left(\frac{m_{1,1}^1 e^{\lambda^2 p_1^2}+m_{1,2}^1 e^{\lambda^2 p_2^2}+m_{1,3}^1 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2}+e^{\lambda^2 p_2^2}+e^{\lambda^2 p_3^2}}\right) \\ \ln(\phi^1)p_2^1 - \ln(\phi^1)\left(\frac{\delta^1 e^{\lambda^1 p_1^1}+e^{\lambda^1 p_2^1}+\delta^1 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1}+e^{\lambda^1 p_2^1}+e^{\lambda^1 p_3^1}}\right)\cdot\left(\frac{m_{2,1}^1 e^{\lambda^2 p_1^2}+m_{2,2}^1 e^{\lambda^2 p_2^2}+m_{2,3}^1 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2}+e^{\lambda^2 p_2^2}+e^{\lambda^2 p_3^2}}\right) \\ \ln(\phi^1)p_3^1 - \ln(\phi^1)\left(\frac{\delta^1 e^{\lambda^1 p_1^1}+\delta^1 e^{\lambda^1 p_2^1}+e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1}+e^{\lambda^1 p_2^1}+e^{\lambda^1 p_3^1}}\right)\cdot\left(\frac{m_{3,1}^1 e^{\lambda^2 p_1^2}+m_{3,2}^1 e^{\lambda^2 p_2^2}+m_{3,3}^1 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2}+e^{\lambda^2 p_2^2}+e^{\lambda^2 p_3^2}}\right) \\ \ln(\phi^2)p_1^2 - \ln(\phi^2)\left(\frac{e^{\lambda^2 p_1^2}+\delta^2 e^{\lambda^2 p_2^2}+\delta^2 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2}+e^{\lambda^2 p_2^2}+e^{\lambda^2 p_3^2}}\right)\cdot\left(\frac{m_{1,1}^2 e^{\lambda^1 p_1^1}+m_{1,2}^2 e^{\lambda^1 p_2^1}+m_{1,3}^2 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1}+e^{\lambda^1 p_2^1}+e^{\lambda^1 p_3^1}}\right) \\ \ln(\phi^2)p_2^2 - \ln(\phi^2)\left(\frac{\delta^2 e^{\lambda^2 p_1^2}+e^{\lambda^2 p_2^2}+\delta^2 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2}+e^{\lambda^2 p_2^2}+e^{\lambda^2 p_3^2}}\right)\cdot\left(\frac{m_{2,1}^2 e^{\lambda^1 p_1^1}+m_{2,2}^2 e^{\lambda^1 p_2^1}+m_{2,3}^2 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1}+e^{\lambda^1 p_2^1}+e^{\lambda^1 p_3^1}}\right) \\ \ln(\phi^2)p_3^2 - \ln(\phi^2)\left(\frac{\delta^2 e^{\lambda^2 p_1^2}+\delta^2 e^{\lambda^2 p_2^2}+e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2}+e^{\lambda^2 p_2^2}+e^{\lambda^2 p_3^2}}\right)\cdot\left(\frac{m_{3,1}^2 e^{\lambda^1 p_1^1}+m_{3,2}^2 e^{\lambda^1 p_2^1}+m_{3,3}^2 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1}+e^{\lambda^1 p_2^1}+e^{\lambda^1 p_3^1}}\right) \end{bmatrix} \tag{42}$$

and using the same Taylor expansion and re-definition of P as with a single population (Equations 28-30), $J = \dot{P}'_{q^*}$ becomes,

$$J = \dot{P}'_{q^{1*},q^{2*}} = \left.\begin{bmatrix} \frac{\partial}{\partial p_1^1}\dot{P}_1 & \frac{\partial}{\partial p_2^1}\dot{P}_1 & \frac{\partial}{\partial p_3^1}\dot{P}_1 & \frac{\partial}{\partial p_1^2}\dot{P}_1 & \frac{\partial}{\partial p_2^2}\dot{P}_1 & \frac{\partial}{\partial p_3^2}\dot{P}_1 \\ \frac{\partial}{\partial p_1^1}\dot{P}_2 & \frac{\partial}{\partial p_2^1}\dot{P}_2 & \frac{\partial}{\partial p_3^1}\dot{P}_2 & \frac{\partial}{\partial p_1^2}\dot{P}_2 & \frac{\partial}{\partial p_2^2}\dot{P}_2 & \frac{\partial}{\partial p_3^2}\dot{P}_2 \\ \frac{\partial}{\partial p_1^1}\dot{P}_3 & \frac{\partial}{\partial p_2^1}\dot{P}_3 & \frac{\partial}{\partial p_3^1}\dot{P}_3 & \frac{\partial}{\partial p_1^2}\dot{P}_3 & \frac{\partial}{\partial p_2^2}\dot{P}_3 & \frac{\partial}{\partial p_3^2}\dot{P}_3 \\ \frac{\partial}{\partial p_1^1}\dot{P}_4 & \frac{\partial}{\partial p_2^1}\dot{P}_4 & \frac{\partial}{\partial p_3^1}\dot{P}_4 & \frac{\partial}{\partial p_1^2}\dot{P}_4 & \frac{\partial}{\partial p_2^2}\dot{P}_4 & \frac{\partial}{\partial p_3^2}\dot{P}_4 \\ \frac{\partial}{\partial p_1^1}\dot{P}_5 & \frac{\partial}{\partial p_2^1}\dot{P}_5 & \frac{\partial}{\partial p_3^1}\dot{P}_5 & \frac{\partial}{\partial p_1^2}\dot{P}_5 & \frac{\partial}{\partial p_2^2}\dot{P}_5 & \frac{\partial}{\partial p_3^2}\dot{P}_5 \\ \frac{\partial}{\partial p_1^1}\dot{P}_6 & \frac{\partial}{\partial p_2^1}\dot{P}_6 & \frac{\partial}{\partial p_3^1}\dot{P}_6 & \frac{\partial}{\partial p_1^2}\dot{P}_6 & \frac{\partial}{\partial p_2^2}\dot{P}_6 & \frac{\partial}{\partial p_3^2}\dot{P}_6 \end{bmatrix}\right|_{p^1=q^{1*},p^2=q^{2*}} \tag{43}$$

see Appendix B for all of the elements of this matrix.

The dimension of this matrix must then be reduced (although there are three choices available to both groups, the probabilities of these three choices must sum to one, thus there are only two dimensions for each group). As is the case with the single population, for there to be asymptotical stability, the eigenvalues of the reduced matrix must have negative real parts.

Figure 7: Matrix 2: Stable and non-stable fixed points assuming $\phi = 0.5$ and allowing δ to increment between 0 and 1 by 0.01 and λ to increment between 0 and 3 by 0.1

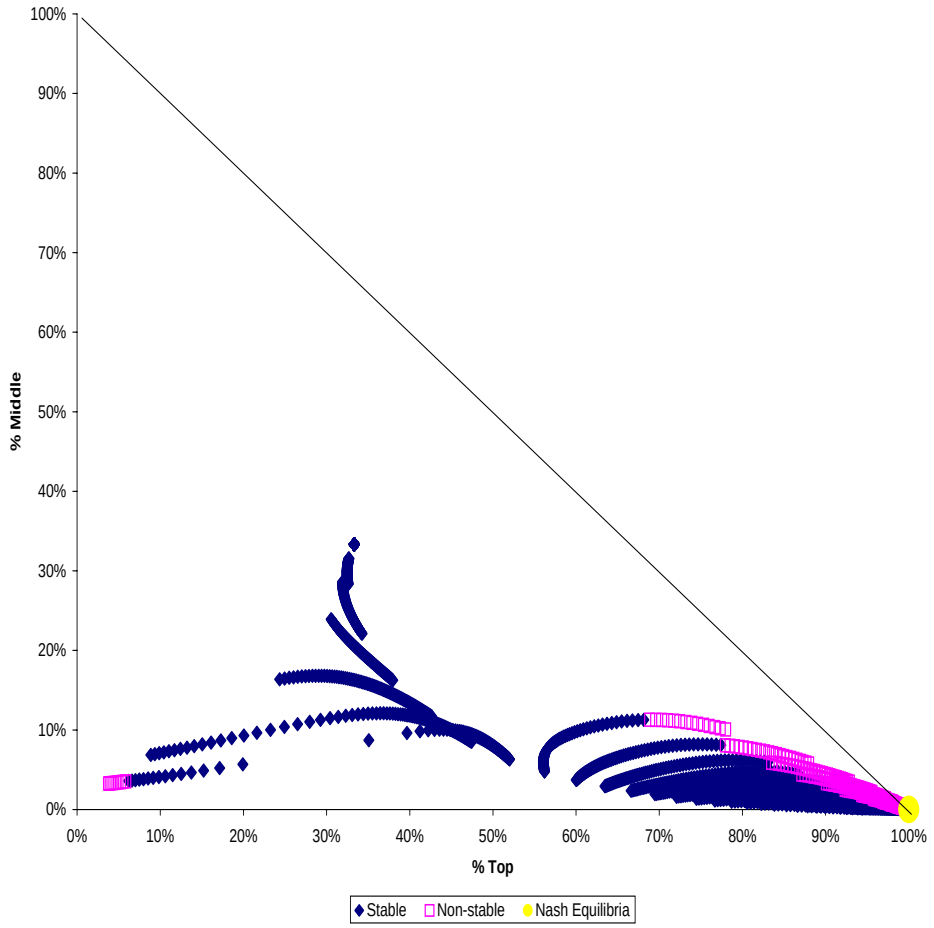
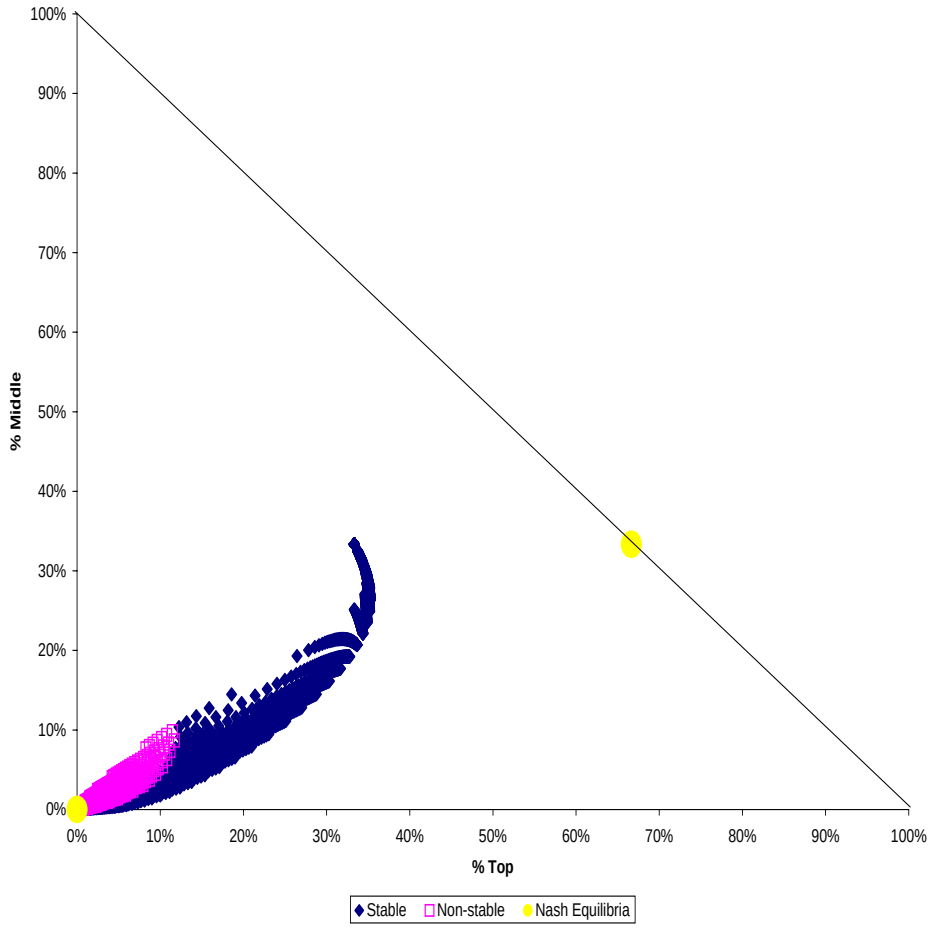


Figure 8: Matrix 4: Stable and non-stable fixed points assuming $\phi = 0.5$ and allowing δ to increment between 0 and 1 by 0.01 and λ to increment between 0 and 3 by 0.1



4.3 Stability in specific matrices –Single population games

Tables 1 and 2 show the eigenvalues and stability characteristics for Matrices 2 and 4 for a variety of fixed points. In both cases, when δ is very small it is very difficult to achieve stability[5]. Since rote learning models require $\delta = 0$, rote learning models would generally not be stable in these cases. As predicted, for both matrices, $\phi = 1$ leads to non-stability. In Matrix 4, this leads to the points around the pure strategy NE not being stable while the fixed points nearer to the center of the simplex are stable. The larger λ, the more likely the fixed point is to be stable. Figures 7 and 8 show the same information graphically. As can be seen from the figures, there are definite regions of stability and non-stability.

Figures 9 and 10 show the regions of stability and non-stability as a function of the parameter space. As was shown in the previous section, the discount parameter (ϕ) does not affect the region of stability.

It is now possible, in conjunction with actual estimates of parameters from experimental data to have specific predictions of long run behaviour for a matrix **??**.

5 Conclusion

This paper has set out to fill gaps in the theory of learning in games. A continuous time model is proposed and the conditions for its fixed points and their stability properties are derived. This, together with the payoff matrix, provides a prediction about expected behaviour. As expected, the Nash equilibrium of the payoff matrix is important. However, the parameters of the model also play a large role in both the location of the fixed points and in their stability. One of the features of this model is that with slight modifications –adding initial propensities, allowing the discounts in the nu-

[5]Unless λ is also small.

Figure 9: Matrix 2 stable fixed points by parameter values when $\phi = 0.1$ and 0.9

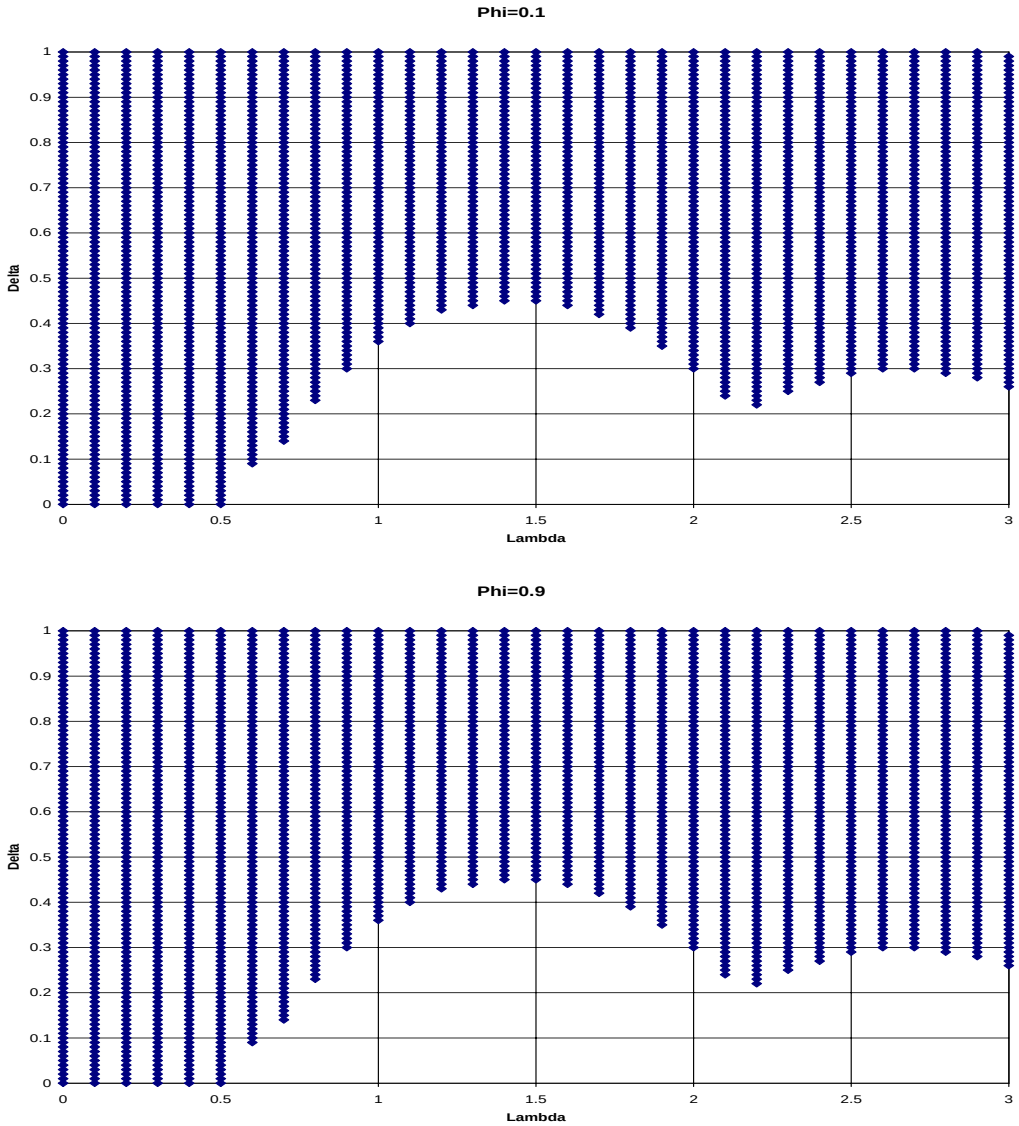
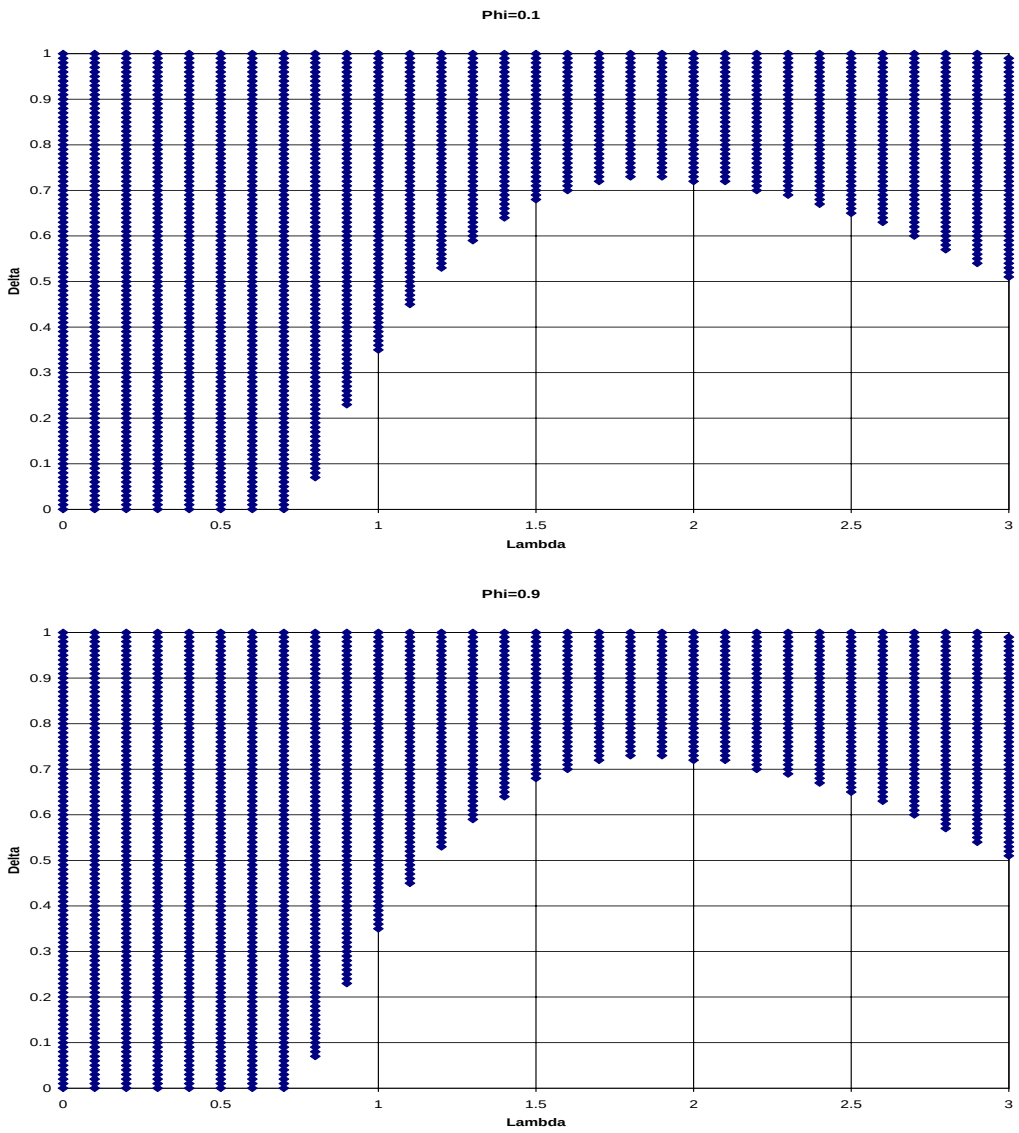


Figure 10: Matrix 4 stable fixed points by parameter values when $\phi = 0.1$ and 0.9



merator and denominator to be independent of each other and adding an initial experience term– the Camerer & Ho (1998c) EWA model can be recovered and hence the Cheung & Friedman (1997) and Roth & Erev (1995) models as well.

The location of the fixed points depends only on the underlying matrix, λ, and δ. The λ parameter pulls the fixed points towards the center of the simplex as λ becomes small. The role of δ is harder to quantify. For Matrix 2, very small δ values ($0 < \delta < 0.1$) lead to fixed points in the region of (0,0,1) which is very far form the Nash equilibria of the payoff matrix. In both matrices however, small δ values "pull" the fixed points away from (0,0,1), these points are only stable for very small λ values ($\lambda \leq 0.5$ for Matrix 2 and ($\lambda \leq 0.7$ for Matrix 4).

The stability of fixed points depends both on the underlying payoff matrix and on the parameter values ϕ and δ. In the single population case, as long as $\phi = (0, 1)$, ϕ has no effect on stability. If $\phi = 1$ (Fictitious play) there is neutral stability.

Looking at two matrices, the "pure" rote learning version of this model (constraining $\delta = 0$) does not lead to stable fixed points (unless λ is very small which would indicate a poorly specified model).

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A Some more on single population stability

Using a first order Taylor expansion to linearize $\dot{P}_t$ (Equation 25) around the fixed point $Q^* = [q_1^* \; q_2^* \; q_3^*]'$, and using the Jacobian $J_{q^*} = \dot{P}'_{q^*}$,

$$\dot{P}_t \approx \dot{P}_{q^*} + J_{q^*} \cdot (P - Q^*) \tag{44}$$

$$\dot{P}_t - \dot{P}_{q^*} \approx J_{q^*} \cdot (P - Q^*) \tag{45}$$

redefining $W = P - Q^*$,

$$\dot{W}_t \approx J_{q^*} \cdot W \tag{46}$$

where,

$$J_{q^*} = \dot{P}'_{q^*} = \left. \begin{bmatrix} \frac{\partial}{\partial p_1}\dot{P}_1 & \frac{\partial}{\partial p_2}\dot{P}_1 & \frac{\partial}{\partial p_3}\dot{P}_1 \\ \frac{\partial}{\partial p_1}\dot{P}_2 & \frac{\partial}{\partial p_2}\dot{P}_2 & \frac{\partial}{\partial p_3}\dot{P}_2 \\ \frac{\partial}{\partial p_1}\dot{P}_3 & \frac{\partial}{\partial p_2}\dot{P}_3 & \frac{\partial}{\partial p_3}\dot{P}_3 \end{bmatrix} \right|_{p_1=q_1^*, p_2=q_2^*, p_3=q_3^*} \tag{47}$$

and,

$$\frac{\partial}{\partial p_1}\dot{P}_1 = \frac{\partial}{\partial p_1}\ln(\phi)\left(p_1 - \left(\frac{e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{48}$$

$$= \ln(\phi)\left(1 - \frac{\lambda e^{\lambda p_1}(m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^2} - \frac{\lambda m_{1,1}e^{\lambda p_1}(e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_1}(e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3})(m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^3}\right) \tag{49}$$

$$\frac{\partial}{\partial p_2}\dot{P}_1 = \frac{\partial}{\partial p_2}\ln(\phi)\left(p_1 - \left(\frac{e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{50}$$

$$= \ln(\phi)\left(-\frac{\delta\lambda e^{\lambda p_2}(m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^2} - \frac{\lambda m_{1,2}e^{\lambda p_2}(e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_2}(e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3})(m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^3}\right) \tag{51}$$

$$\frac{\partial}{\partial p_3}\dot{P}_1 = \frac{\partial}{\partial p_3}\ln(\phi)\left(p_1 - \left(\frac{e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{52}$$

$$= \ln(\phi)\left(-\frac{\delta\lambda e^{\lambda p_3}(m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^2} - \frac{\lambda m_{1,3}e^{\lambda p_3}(e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_3}(e^{\lambda p_1} + \delta e^{\lambda p_2} + \delta e^{\lambda p_3})(m_{1,1}e^{\lambda p_1} + m_{1,2}e^{\lambda p_2} + m_{1,3}e^{\lambda p_3})}{\left(e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}\right)^3}\right) \tag{53}$$

$$\frac{\partial}{\partial p_1}\dot{P}_2 = \frac{\partial}{\partial p_1}\ln(\phi)\left(p_2 - \left(\frac{\delta e^{\lambda p_1} + e^{\lambda p_2} + \delta e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{2,1}e^{\lambda p_1} + m_{2,2}e^{\lambda p_2} + m_{2,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{54}$$

$$= \ln(\phi)\left(-\frac{\delta\lambda e^{\lambda p_1}(m_{2,1}e^{\lambda p_1}+m_{2,2}e^{\lambda p_2}+m_{2,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} - \frac{\lambda m_{2,1}e^{\lambda p_1}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_1}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})(m_{2,1}e^{\lambda p_1}+m_{2,2}e^{\lambda p_2}+m_{2,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^3}\right) \tag{55}$$

$$\frac{\partial}{\partial p_2}\dot{P}_2 = \frac{\partial}{\partial p_1}\ln(\phi)\left(p_2 - \left(\frac{\delta e^{\lambda p_1} + e^{\lambda p_2} + \delta e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{2,1}e^{\lambda p_1} + m_{2,2}e^{\lambda p_2} + m_{2,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{56}$$

$$= \ln(\phi)\left(1 - \frac{\lambda e^{\lambda p_2}(m_{2,1}e^{\lambda p_1}+m_{2,2}e^{\lambda p_2}+m_{2,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} - \frac{\lambda m_{2,2}e^{\lambda p_2}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_2}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})(m_{2,1}e^{\lambda p_1}+m_{2,2}e^{\lambda p_2}+m_{2,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^3}\right) \tag{57}$$

$$\frac{\partial}{\partial p_3}\dot{P}_2 = \frac{\partial}{\partial p_1}\ln(\phi)\left(p_2 - \left(\frac{\delta e^{\lambda p_1} + e^{\lambda p_2} + \delta e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{2,1}e^{\lambda p_1} + m_{2,2}e^{\lambda p_2} + m_{2,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{58}$$

$$= \ln(\phi)\left(-\frac{\delta\lambda e^{\lambda p_3}(m_{2,1}e^{\lambda p_1}+m_{2,2}e^{\lambda p_2}+m_{2,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} - \frac{\lambda m_{2,3}e^{\lambda p_3}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_3}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})(m_{2,1}e^{\lambda p_1}+m_{2,2}e^{\lambda p_2}+m_{2,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^3}\right) \tag{59}$$

$$\frac{\partial}{\partial p_1}\dot{P}_3 = \frac{\partial}{\partial p_1}\ln(\phi)\left(p_3 - \left(\frac{\delta e^{\lambda p_1} + \delta e^{\lambda p_2} + e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{3,1}e^{\lambda p_1} + m_{3,2}e^{\lambda p_2} + m_{3,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{60}$$

$$= \ln(\phi)\left(-\frac{\delta\lambda e^{\lambda p_1}(m_{3,1}e^{\lambda p_1}+m_{3,2}e^{\lambda p_2}+m_{3,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} - \frac{\lambda m_{3,1}e^{\lambda p_1}(\delta e^{\lambda p_1}+\delta e^{\lambda p_2}+e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_1}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})(m_{3,1}e^{\lambda p_1}+m_{3,2}e^{\lambda p_2}+m_{3,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^3}\right) \tag{61}$$

$$\frac{\partial}{\partial p_2}\dot{P}_3 = \frac{\partial}{\partial p_2}\ln(\phi)\left(p_3 - \left(\frac{\delta e^{\lambda p_1} + \delta e^{\lambda p_2} + e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{3,1}e^{\lambda p_1} + m_{3,2}e^{\lambda p_2} + m_{3,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{62}$$

$$= \ln(\phi)\left(-\frac{\delta\lambda e^{\lambda p_2}(m_{3,1}e^{\lambda p_1}+m_{3,2}e^{\lambda p_2}+m_{3,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} - \frac{\lambda m_{3,2}e^{\lambda p_2}(\delta e^{\lambda p_1}+\delta e^{\lambda p_2}+e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_2}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})(m_{3,1}e^{\lambda p_1}+m_{3,2}e^{\lambda p_2}+m_{3,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^3}\right) \tag{63}$$

$$\frac{\partial}{\partial p_3}\dot{P}_3 = \frac{\partial}{\partial p_3}\ln(\phi)\left(p_3 - \left(\frac{\delta e^{\lambda p_1} + \delta e^{\lambda p_2} + e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\left(\frac{m_{3,1}e^{\lambda p_1} + m_{3,2}e^{\lambda p_2} + m_{3,3}e^{\lambda p_3}}{e^{\lambda p_1} + e^{\lambda p_2} + e^{\lambda p_3}}\right)\right) \tag{64}$$

$$= \ln(\phi)\left(1 - \frac{\lambda e^{\lambda p_3}(m_{3,1}e^{\lambda p_1}+m_{3,2}e^{\lambda p_2}+m_{3,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} - \frac{\lambda m_{3,3}e^{\lambda p_3}(\delta e^{\lambda p_1}+\delta e^{\lambda p_2}+e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^2} + \frac{2\lambda e^{\lambda p_3}(\delta e^{\lambda p_1}+e^{\lambda p_2}+\delta e^{\lambda p_3})(m_{3,1}e^{\lambda p_1}+m_{3,2}e^{\lambda p_2}+m_{3,3}e^{\lambda p_3})}{\left(e^{\lambda p_1}+e^{\lambda p_2}+e^{\lambda p_3}\right)^3}\right) \tag{65}$$

Using Haigh's (1975) theorem to reduce the matrix, the entries of the n by n transformation matrix Z are $z_{l,k} = j_{l,k} - j_{l,k+1} - j_{l+1,k} + j_{l+1,k+1}$. So

that,

$$Z = \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix}. \tag{66}$$

Letting,

$$\begin{aligned}
X_1 &= m_{1,1}e^{\lambda q_1^*} + m_{1,2}e^{\lambda q_2^*} + m_{1,3}e^{\lambda q_3^*} && (67)\\
X_2 &= m_{2,1}e^{\lambda q_1^*} + m_{2,2}e^{\lambda q_2^*} + m_{2,3}e^{\lambda q_3^*} && (68)\\
X_3 &= m_{3,1}e^{\lambda q_1^*} + m_{3,2}e^{\lambda q_2^*} + m_{3,3}e^{\lambda q_3^*} && (69)\\
E &= e^{\lambda q_1^*} + e^{\lambda q_2^*} + e^{\lambda q_3^*} && (70)\\
E_1 &= e^{\lambda q_1^*} + \delta e^{\lambda q_2^*} + \delta e^{\lambda q_3^*} && (71)\\
E_2 &= \delta e^{\lambda q_1^*} + e^{\lambda q_2^*} + \delta e^{\lambda q_3^*} && (72)\\
E_3 &= \delta e^{\lambda q_1^*} + \delta e^{\lambda q_2^*} + e^{\lambda q_3^*} && (73)
\end{aligned}$$

$$\begin{aligned}
z_{1,1} &= \ln(\phi)\left(2 + 2\lambda(e^{\lambda q_1^*} - e^{\lambda q_3^*})\left(\frac{E_1X_1 - E_3X_3}{E^3}\right) + \right.\\
&\quad \left.\lambda\left(\frac{(\delta e^{\lambda q_3^*} - e^{\lambda q_1^*})X_1 + (\delta e^{\lambda q_1^*} - e^{\lambda q_3^*})X_3 + (m_{1,3}e^{\lambda q_3^*} - m_{1,1}e^{\lambda q_1^*})E_1 + (m_{1,3}e^{\lambda q_1^*} + m_{3,3}e^{\lambda q_3^*})E_3}{E^2}\right)\right) && (74)\\
z_{1,2} &= \ln\phi\left(1 + 2\lambda(e^{\lambda q_2^*} - e^{\lambda q_3^*})\left(\frac{E_1X_1 + E_3X_3}{E^3}\right) + \right.\\
&\quad \left.\lambda\left(\frac{\delta(e^{\lambda q_3^*} - e^{\lambda q_2^*})X_1 + (\delta e^{\lambda q_2^*} - e^{\lambda q_3^*})X_3 + (m_{1,3}e^{\lambda q_3^*} - m_{1,2}e^{\lambda q_2^*})E_1 + (m_{3,2}e^{\lambda q_2^*} + m_{3,3}e^{\lambda q_3^*})E_3}{E^2}\right)\right) && (75)\\
z_{2,1} &= \ln\phi\left(1 + 2\lambda(e^{\lambda q_1^*} - e^{\lambda q_2^*})\left(\frac{E_2X_2 - E_3X_3}{E^3}\right) + \right.\\
&\quad \left.\lambda\left(\frac{\delta(e^{\lambda q_3^*} - e^{\lambda q_1^*})X_2 + (\delta e^{\lambda q_1^*} - e^{\lambda q_3^*})X_3 + (m_{2,3}e^{\lambda q_3^*} - m_{2,1}e^{\lambda q_1^*})E_2 + (m_{3,1}e^{\lambda q_1^*} - m_{3,3}e^{\lambda q_3^*})E_3}{E^2}\right)\right) && (76)\\
z_{2,2} &= \ln\phi\left(2 + 2\lambda(e^{\lambda q_2^*} - e^{\lambda q_3^*})\left(\frac{E_2X_2 - E_3X_3}{E^3}\right) + \right.\\
&\quad \left.\lambda\left(\frac{(\delta e^{\lambda q_3^*} - e^{\lambda q_2^*})X_2 + (\delta e^{\lambda q_2^*} - e^{\lambda q_3^*})X_3 + (m_{2,3}e^{\lambda q_3^*} - m_{2,2}e^{\lambda q_2^*})E_2 + (m_{3,2}e^{\lambda q_2^*} + m_{3,3}e^{\lambda q_3^*})E_3}{E^2}\right)\right) && (77)
\end{aligned}$$

For the model to be asymptotically stable, the eigenvalues of the linearized equation around the fixed point have to have negative real parts (Hirsch & Smale 1974). The eigenvalues of the $\tilde{Z}$ matrix (Equation 32) are:

$$\begin{aligned}
&\tfrac{1}{2}\,z_{1,1} + \tfrac{1}{2}\,z_{2,2} + \tfrac{1}{2}\sqrt{{z_{1,1}}^2 - 2\,z_{1,1}z_{2,2} + {z_{2,2}}^2 + 4\,z_{1,2}z_{2,1}}\\
&\tfrac{1}{2}\,z_{1,1} + \tfrac{1}{2}\,z_{2,2} - \tfrac{1}{2}\sqrt{{z_{1,1}}^2 - 2\,z_{1,1}z_{2,2} + {z_{2,2}}^2 + 4\,z_{1,2}z_{2,1}}.
\end{aligned} \tag{78}$$

B Two population stability

The P_t vectors for each of the populations are stacked. So that $P_t = [p_1^1\ p_2^1\ p_3^1\ p_1^2\ p_2^2\ p_3^2]$ and,

$$\dot{P}_t \approx \begin{bmatrix} ln(\phi^1)p_1^1 - ln(\phi^1)(s_1^1 + \delta^1 s_2^1 + \delta^1 s_3^1)(m_{1,1}^2 s_1^2 + m_{1,2}^2 s_2^2 + m_{1,3}^2 s_3^2) \\ ln(\phi^1)p_2^1 - ln(\phi^1)(\delta^1 s_1^1 + s_2^1 + \delta^1 s_3^1)(m_{2,1}^2 s_1^2 + m_{2,2}^2 s_2^2 + m_{2,3}^2 s_3^2) \\ ln(\phi^1)p_3^1 - ln(\phi^1)(\delta^1 s_1^1 + \delta^1 s_2^1 + s_3^1)(m_{3,1}^2 s_1^2 + m_{3,2}^2 s_2^2 + m_{3,3}^2 s_3^2) \\ ln(\phi^2)p_1^2 - ln(\phi^2)(s_1^2 + \delta^2 s_2^2 + \delta^2 s_3^2)(m_{1,1}^1 s_1^1 + m_{1,2}^1 s_2^1 + m_{1,3}^1 s_3^1) \\ ln(\phi^2)p_2^2 - ln(\phi^2)(\delta^2 s_1^2 + s_2^2 + \delta^2 s_3^2)(m_{2,1}^1 s_1^1 + m_{2,2}^1 s_2^1 + m_{2,3}^1 s_3^1) \\ ln(\phi^2)p_3^2 - ln(\phi^2)(\delta^2 s_1^2 + \delta^2 s_2^2 + s_3^2)(m_{3,1}^1 s_1^1 + m_{3,2}^1 s_2^1 + m_{3,3}^1 s_3^1) \end{bmatrix} \tag{79}$$

Substituting,

$$\begin{aligned} s_1^1 &= \frac{e^{\lambda^1 p_1^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}, \quad & s_1^2 &= \frac{e^{\lambda^2 p_1^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}} \\ s_2^1 &= \frac{e^{\lambda^1 p_2^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}, \quad & s_2^2 &= \frac{e^{\lambda^2 p_2^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}} \\ s_3^1 &= \frac{e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}, \quad & s_3^2 &= \frac{e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}} \end{aligned} \tag{80}$$

gives,

$$\dot{P}_t \approx \begin{bmatrix} \ln(\phi^1)p_1^1 - \ln(\phi^1)\left(\frac{e^{\lambda^1 p_1^1} + \delta^1 e^{\lambda^1 p_2^1} + \delta^1 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}\right) \cdot \left(\frac{m_{1,1}^1 e^{\lambda^2 p_1^2} + m_{1,2}^1 e^{\lambda^2 p_2^2} + m_{1,3}^1 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}}\right) \\ \ln(\phi^1)p_2^1 - \ln(\phi^1)\left(\frac{\delta^1 e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + \delta^1 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}\right) \cdot \left(\frac{m_{2,1}^1 e^{\lambda^2 p_1^2} + m_{2,2}^1 e^{\lambda^2 p_2^2} + m_{2,3}^1 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}}\right) \\ \ln(\phi^1)p_3^1 - \ln(\phi^1)\left(\frac{\delta^1 e^{\lambda^1 p_1^1} + \delta^1 e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}\right) \cdot \left(\frac{m_{3,1}^1 e^{\lambda^2 p_1^2} + m_{3,2}^1 e^{\lambda^2 p_2^2} + m_{3,3}^1 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}}\right) \\ \ln(\phi^2)p_1^2 - \ln(\phi^2)\left(\frac{e^{\lambda^2 p_1^2} + \delta^2 e^{\lambda^2 p_2^2} + \delta^2 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}}\right) \cdot \left(\frac{m_{1,1}^2 e^{\lambda^1 p_1^1} + m_{1,2}^2 e^{\lambda^1 p_2^1} + m_{1,3}^2 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}\right) \\ \ln(\phi^2)p_2^2 - \ln(\phi^2)\left(\frac{\delta^2 e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + \delta^2 e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}}\right) \cdot \left(\frac{m_{2,1}^2 e^{\lambda^1 p_1^1} + m_{2,2}^2 e^{\lambda^1 p_2^1} + m_{2,3}^2 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}\right) \\ \ln(\phi^2)p_3^2 - \ln(\phi^2)\left(\frac{\delta^2 e^{\lambda^2 p_1^2} + \delta^2 e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}}{e^{\lambda^2 p_1^2} + e^{\lambda^2 p_2^2} + e^{\lambda^2 p_3^2}}\right) \cdot \left(\frac{m_{3,1}^2 e^{\lambda^1 p_1^1} + m_{3,2}^2 e^{\lambda^1 p_2^1} + m_{3,3}^2 e^{\lambda^1 p_3^1}}{e^{\lambda^1 p_1^1} + e^{\lambda^1 p_2^1} + e^{\lambda^1 p_3^1}}\right) \end{bmatrix} \tag{81}$$

Letting,

$$X_1^1 = m_{1,1}^1 e^{\lambda^2 q_1^{2*}} + m_{1,2}^1 e^{\lambda^2 q_2^{2*}} + m_{1,3}^1 e^{\lambda^2 q_3^{2*}} \tag{82}$$

$$X_1^2 = m_{1,1}^2 e^{\lambda^1 q_1^{1*}} + m_{1,2}^2 e^{\lambda^1 q_2^{1*}} + m_{1,3}^2 e^{\lambda^1 q_3^{1*}} \tag{83}$$

$$X_2^1 = m_{2,1}^1 e^{\lambda^2 q_1^{2*}} + m_{2,2}^1 e^{\lambda^2 q_2^{2*}} + m_{2,3}^1 e^{\lambda^2 q_3^{2*}} \tag{84}$$

$$X_2^2 = m_{2,1}^2 e^{\lambda^1 q_1^{1*}} + m_{2,2}^2 e^{\lambda^1 q_2^{1*}} + m_{2,3}^2 e^{\lambda^1 q_3^{1*}} \tag{85}$$

$$X_3^1 = m_{3,1}^1 e^{\lambda^2 q_1^{2*}} + m_{3,2}^1 e^{\lambda^2 q_2^{2*}} + m_{3,3}^1 e^{\lambda^2 q_3^{2*}} \tag{86}$$

$$
\begin{aligned}
X_3^2 &= m_{3,1}^2 e^{\lambda^1 q_1^{1*}} + m_{3,2}^2 e^{\lambda^1 q_2^{1*}} + m_{3,3}^2 e^{\lambda^1 q_3^{1*}} && (87)\\
E^1 &= e^{\lambda^1 q_1^{1*}} + e^{\lambda^1 q_2^{1*}} + e^{\lambda^1 q_3^{1*}} && (88)\\
E &= e^{\lambda q_1^*} + e^{\lambda q_2^*} + e^{\lambda q_3^*} && (89)\\
E_1^1 &= e^{\lambda^1 q_1^{1*}} + \delta^1 e^{\lambda^1 q_2^{1*}} + \delta^1 e^{\lambda^1 q_3^{1*}} && (90)\\
E_1^2 &= e^{\lambda^2 q_1^{2*}} + \delta^2 e^{\lambda^2 q_2^{2*}} + \delta^2 e^{\lambda^2 q_3^{2*}} && (91)\\
E_2^1 &= \delta^1 e^{\lambda^1 q_1^{1*}} + e^{\lambda^1 q_2^{1*}} + \delta^1 e^{\lambda^1 q_3^{1*}} && (92)\\
E_2^2 &= \delta^2 e^{\lambda^2 q_1^{2*}} + e^{\lambda^2 q_2^{2*}} + \delta^2 e^{\lambda^2 q_3^{2*}} && (93)\\
E_3^1 &= \delta^1 e^{\lambda^1 q_1^{1*}} + \delta^1 e^{\lambda^1 q_2^{1*}} + e^{\lambda^1 q_3^{1*}} && (94)\\
E_3^2 &= \delta^2 e^{\lambda^2 q_1^{2*}} + \delta^2 e^{\lambda^2 q_2^{2*}} + e^{\lambda^2 q_3^{2*}} && (95)
\end{aligned}
$$

and using the same Taylor expansion and re-definition of P as with a single population (Equations 28-30), $J = \dot{P}'_{q^*}$ becomes,

$$
J = \dot{P}'_{q^{1*},q^{2*}} = \left.\begin{bmatrix}
\frac{\partial}{\partial p_1^1}\dot{P}_1 & \frac{\partial}{\partial p_2^1}\dot{P}_1 & \frac{\partial}{\partial p_3^1}\dot{P}_1 & \frac{\partial}{\partial p_1^2}\dot{P}_1 & \frac{\partial}{\partial p_2^2}\dot{P}_1 & \frac{\partial}{\partial p_3^2}\dot{P}_1 \\
\frac{\partial}{\partial p_1^1}\dot{P}_2 & \frac{\partial}{\partial p_2^1}\dot{P}_2 & \frac{\partial}{\partial p_3^1}\dot{P}_2 & \frac{\partial}{\partial p_1^2}\dot{P}_2 & \frac{\partial}{\partial p_2^2}\dot{P}_2 & \frac{\partial}{\partial p_3^2}\dot{P}_2 \\
\frac{\partial}{\partial p_1^1}\dot{P}_3 & \frac{\partial}{\partial p_2^1}\dot{P}_3 & \frac{\partial}{\partial p_3^1}\dot{P}_3 & \frac{\partial}{\partial p_1^2}\dot{P}_3 & \frac{\partial}{\partial p_2^2}\dot{P}_3 & \frac{\partial}{\partial p_3^2}\dot{P}_3 \\
\frac{\partial}{\partial p_1^1}\dot{P}_4 & \frac{\partial}{\partial p_2^1}\dot{P}_4 & \frac{\partial}{\partial p_3^1}\dot{P}_4 & \frac{\partial}{\partial p_1^2}\dot{P}_4 & \frac{\partial}{\partial p_2^2}\dot{P}_4 & \frac{\partial}{\partial p_3^2}\dot{P}_4 \\
\frac{\partial}{\partial p_1^1}\dot{P}_5 & \frac{\partial}{\partial p_2^1}\dot{P}_5 & \frac{\partial}{\partial p_3^1}\dot{P}_5 & \frac{\partial}{\partial p_1^2}\dot{P}_5 & \frac{\partial}{\partial p_2^2}\dot{P}_5 & \frac{\partial}{\partial p_3^2}\dot{P}_5 \\
\frac{\partial}{\partial p_1^1}\dot{P}_6 & \frac{\partial}{\partial p_2^1}\dot{P}_6 & \frac{\partial}{\partial p_3^1}\dot{P}_6 & \frac{\partial}{\partial p_1^2}\dot{P}_6 & \frac{\partial}{\partial p_2^2}\dot{P}_6 & \frac{\partial}{\partial p_3^2}\dot{P}_6
\end{bmatrix}\right|_{p^1=q^{1*},p^2=q^{2*}} \qquad (96)
$$

With,

$$
\begin{aligned}
\frac{\partial}{\partial p_1^1}\dot{P}_1 &= \ln(\phi^1) + \frac{\ln(\phi^1)\lambda^1 e^{\lambda^1 p_1^1} X_1^1}{E^1 E^2}\left(\frac{E_1^1 - E^1}{E^1}\right) && (97)\\
\frac{\partial}{\partial p_2^1}\dot{P}_1 &= \frac{\ln(\phi^1)\lambda^1 e^{\lambda^1 p_2^1} X_1^1}{E^1 E^2}\left(\frac{E_1^1 - \delta^1 E^1}{E^1}\right) && (98)\\
\frac{\partial}{\partial p_3^1}\dot{P}_1 &= \frac{\ln(\phi^1)\lambda^1 e^{\lambda^1 p_3^1} X_1^1}{E^1 E^2}\left(\frac{E_1^1 - \delta^1 E^1}{E^1}\right) && (99)\\
\frac{\partial}{\partial p_1^2}\dot{P}_1 &= \frac{\ln(\phi^1)\lambda^2 e^{\lambda^2 p_1^2} E_1^1}{E^1 E^2}\left(\frac{X_1^1 - m_{1,1}^1 E^2}{E^2}\right) && (100)\\
\frac{\partial}{\partial p_2^2}\dot{P}_1 &= \frac{\ln(\phi^1)\lambda^2 e^{\lambda^2 p_2^2} E_1^1}{E^1 E^2}\left(\frac{X_1^1 - m_{1,2}^1 E^2}{E^2}\right) && (101)\\
\frac{\partial}{\partial p_3^2}\dot{P}_1 &= \frac{\ln(\phi^1)\lambda^2 e^{\lambda^2 p_3^2} E_1^1}{E^1 E^2}\left(\frac{X_1^1 - m_{1,3}^1 E^2}{E^2}\right) && (102)\\
\frac{\partial}{\partial p_1^1}\dot{P}_2 &= \frac{\ln(\phi^1)\lambda^1 e^{\lambda^1 p_1^1} X_2^1}{E^1 E^2}\left(\frac{E_2^1 - E^1}{E^1}\right) && (103)
\end{aligned}
$$

$$\frac{\partial}{\partial p_2^1}\dot{P}_2 = \ln(\phi^1) + \frac{\ln(\phi^1)\lambda^1\, e^{\lambda^1\, p_2^1} X_2^1}{E^1 E^2}\left(\frac{E_2^1 - E^1}{E^1}\right) \tag{104}$$

$$\frac{\partial}{\partial p_3^1}\dot{P}_2 = \frac{\ln(\phi^1)\lambda^1\, e^{\lambda^1\, p_3^1} X_2^1}{E^1 E^2}\left(\frac{E_2^1 - \delta^1 E^1}{E^1}\right) \tag{105}$$

$$\frac{\partial}{\partial p_1^2}\dot{P}_2 = \frac{\ln(\phi^1)\lambda^2\, e^{\lambda^2\, p_1^2} E_2^1}{E^1 E^2}\left(\frac{X_2^1 - m_{2,1}^1 E^2}{E^2}\right) \tag{106}$$

$$\frac{\partial}{\partial p_2^2}\dot{P}_2 = \frac{\ln(\phi^1)\lambda^2\, e^{\lambda^2\, p_2^2} E_2^1}{E^1 E^2}\left(\frac{X_2^1 - m_{2,2}^1 E^2}{E^2}\right) \tag{107}$$

$$\frac{\partial}{\partial p_3^2}\dot{P}_2 = \frac{\ln(\phi^1)\lambda^2\, e^{\lambda^2\, p_3^2} E_2^1}{E^1 E^2}\left(\frac{X_2^1 - m_{2,3}^1 E^2}{E^2}\right) \tag{108}$$

$$\frac{\partial}{\partial p_1^1}\dot{P}_3 = \frac{\ln(\phi^1)\lambda^1\, e^{\lambda^1\, p_1^1} X_3^1}{E^1 E^2}\left(\frac{E_3^1 - \delta^1 E^1}{E^1}\right) \tag{109}$$

$$\frac{\partial}{\partial p_2^1}\dot{P}_3 = \frac{\ln(\phi^1)\lambda^1\, e^{\lambda^1\, p_2^1} X_3^1}{E^1 E^2}\left(\frac{E_3^1 - \delta^1 E^1}{E^1}\right) \tag{110}$$

$$\frac{\partial}{\partial p_3^1}\dot{P}_3 = \ln(\phi^1) + \frac{\ln(\phi^1)\lambda^1\, e^{\lambda^1\, p_3^1} X_3^1}{E^1 E^2}\left(\frac{E_3^1 - E^1}{E^1}\right) \tag{111}$$

$$\frac{\partial}{\partial p_1^2}\dot{P}_3 = \frac{\ln(\phi^1)\lambda^2\, e^{\lambda^2\, p_1^2} E_3^1}{E^1 E^2}\left(\frac{X_3^1 - m_{3,1}^1 E^2}{E^2}\right) \tag{112}$$

$$\frac{\partial}{\partial p_2^2}\dot{P}_3 = \frac{\ln(\phi^1)\lambda^2\, e^{\lambda^2\, p_2^2} E_3^1}{E^1 E^2}\left(\frac{X_3^1 - m_{3,2}^1 E^2}{E^2}\right) \tag{113}$$

$$\frac{\partial}{\partial p_3^2}\dot{P}_3 = \frac{\ln(\phi^1)\lambda^2\, e^{\lambda^2\, p_3^2} E_3^1}{E^1 E^2}\left(\frac{X_3^1 - m_{3,3}^1 E^2}{E^2}\right) \tag{114}$$

$$\frac{\partial}{\partial p_1^1}\dot{P}_4 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_1^1} E_1^2}{E^1 E^2}\left(\frac{X_1^2 - m_{1,1}^2 E^2}{E^2}\right) \tag{115}$$

$$\frac{\partial}{\partial p_2^1}\dot{P}_4 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_2^1} E_1^2}{E^1 E^2}\left(\frac{X_2^2 - m_{1,1}^2 E^2}{E^2}\right) \tag{116}$$

$$\frac{\partial}{\partial p_3^1}\dot{P}_4 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_3^1} E_1^2}{E^1 E^2}\left(\frac{X_2^2 - m_{1,1}^2 E^2}{E^2}\right) \tag{117}$$

$$\frac{\partial}{\partial p_1^2}\dot{P}_4 = \ln(\phi^2) + \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_1^2} E_1^2}{E^1 E^2}\left(\frac{E_1^2 - E^2}{E^2}\right) \tag{118}$$

$$\frac{\partial}{\partial p_2^2}\dot{P}_4 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_2^2} E_1^2}{E^1 E^2}\left(\frac{E_1^2 - E^2}{E^2}\right) \tag{119}$$

$$\frac{\partial}{\partial p_3^2}\dot{P}_4 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_3^2} E_1^2}{E^1 E^2}\left(\frac{E_1^2 - E^2}{E^2}\right) \tag{120}$$

$$\frac{\partial}{\partial p_1^1}\dot{P}_5 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_1^1} E_2^2}{E^1 E^2}\left(\frac{X_2^2 - m_{2,1}^2 E^2}{E^2}\right) \tag{121}$$

$$\frac{\partial}{\partial p_2^1}\dot{P}_5 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_2^1} E_2^2}{E^1 E^2}\left(\frac{X_2^2 - m_{2,2}^2 E^2}{E^2}\right) \tag{122}$$

$$\frac{\partial}{\partial p_3^1}\dot{P}_5 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_2^1} E_2^2}{E^1 E^2}\left(\frac{X_2^2 - m_{2,3}^2 E^2}{E^2}\right) \tag{123}$$

$$\frac{\partial}{\partial p_1^2}\dot{P}_5 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_1^2} E_2^2}{E^1 E^2}\left(\frac{E_2^2 - E^2}{E^2}\right) \tag{124}$$

$$\frac{\partial}{\partial p_2^2}\dot{P}_5 = \ln(\phi^2) + \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_2^2} E_2^2}{E^1 E^2}\left(\frac{E_2^2 - E^2}{E^2}\right) \tag{125}$$

$$\frac{\partial}{\partial p_3^2}\dot{P}_5 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_2^2} E_2^2}{E^1 E^2}\left(\frac{E_2^2 - E^2}{E^2}\right) \tag{126}$$

$$\frac{\partial}{\partial p_1^1}\dot{P}_6 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_1^1} E_3^2}{E^1 E^2}\left(\frac{X_3^2 - m_{3,1}^2 E^2}{E^2}\right) \tag{127}$$

$$\frac{\partial}{\partial p_2^1}\dot{P}_6 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_2^1} E_3^2}{E^1 E^2}\left(\frac{X_3^2 - m_{3,2}^2 E^2}{E^2}\right) \tag{128}$$

$$\frac{\partial}{\partial p_3^1}\dot{P}_6 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_3^1} E_3^2}{E^1 E^2}\left(\frac{X_3^2 - m_{3,2}^2 E^2}{E^2}\right) \tag{129}$$

$$\frac{\partial}{\partial p_1^2}\dot{P}_6 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_1^2} E_3^2}{E^1 E^2}\left(\frac{E_3^2 - E^2}{E^2}\right) \tag{130}$$

$$\frac{\partial}{\partial p_2^2}\dot{P}_6 = \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_1^2} E_3^2}{E^1 E^2}\left(\frac{E_3^2 - E^2}{E^2}\right) \tag{131}$$

$$\frac{\partial}{\partial p_3^2}\dot{P}_6 = \ln(\phi^2) + \frac{\ln(\phi^2)\lambda^1 e^{\lambda^1 p_1^2} E_3^2}{E^1 E^2}\left(\frac{E_3^2 - E^2}{E^2}\right) \tag{132}$$