Network Formation with Sequential Demands*

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Abstract

This paper introduces a non-cooperative game-theoretic model of sequential network formation, in which players propose links and demand payoffs. Payoff division is therefore endogenous. We show that if the value of networks satisfies size-monotonicity, then each and every equilibrium network is efficient. Formation of networks through bilateral negotiations (link-specific demands) and through absolute participation demands turn out to have the same efficiency properties. The results do not extend to the case in which players can only demand relative shares.

Keywords: Link Formation, Efficient Networks, Graphs, Payoff Division

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1 Introduction

As in the seminal work of Aumann and Myerson (1988), we analyze the formation process of a cooperation structure (or network) as a non-cooperative game, where players move sequentially. The main difference between this paper and Aumann and Myerson (1988) is that we are interested in situations in which it is impossible to pre-assign a fixed imputation to each cooperation structure, i.e., situations in which the distribution of payoffs is endogenous.\(^1\)

Indeed, the formation of international cooperation networks, and, more generally, of any market network, occurs through a bargaining process, in which the demand of a payoff for participation is a crucial variable.

We study an extensive form game in which players sequentially propose links to other players and formulate payoff demands. We consider three variants of the game, corresponding to different ways of formulating payoff demands. In the first case, we assume that players formulate a single absolute demand, representing their final payoff demand. This is representative of situations such as the formation of economic unions, in which negotiations are multilateral in nature, and each player (country) makes an absolute claim on the total surplus from cooperation. In the second variant of the game, players attach to each proposed link a payoff demand, intended as a bilateral proposal to the specific player involved. This case represents situations such as the formation of market networks, where the network forms as a by-product of bilateral negotiations, and the payoffs are the outcomes of bilateral bargaining. Finally, we consider the case in which players demand shares of the final payoff. When the final value of any network is random, then this third version of the game becomes relevant, since absolute demands may induce feasibility problems ex post.

This paper focuses on the efficiency properties of the set of Subgame Perfect Equilibria of these three variants of the link formation game. For the case of absolute participation demands and multiple demands, we show that if efficiency requires that all players are somehow connected (directly or indirectly), then every subgame perfect equilibrium induces an

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\(^1\)Slikker and Van Den Nouweland (1998) studied a link formation game with endogenous payoff division but with a simultaneous-move framework.
efficient network. In contrast, in the share-demands case we show that efficiency may not be achieved in equilibrium.

Our results can be interpreted as contributions to the theoretical debate, stemming from Aumann and Myerson (1988), on the possibility of reconciling efficiency and stability in network formation situations. Jackson and Wolinsky (1996) have demonstrated that efficiency and stability are incompatible under fairly reasonable assumptions on fixed imputation rules. Dutta and Mutuswami (1997) have shown, on the other hand, that a mechanism design approach (where the imputation rules themselves are the mechanisms to play with) can help reconcile efficiency and stability. Our approach reproposes the problem considered by Jackson and Wolinsky (1996) for the situations where imputation rules cannot be chosen by a mechanism designer,\(^2\) and where payoff division is endogenous. We show that efficiency and stability are reconciled if efficiency requires that all players be connected. It is worthwhile to note that the assumption of endogenous payoff division, implicitly introducing equilibrium side payments in the stable network, seems appropriate to deal with the concept of efficiency considered in Jackson and Wolinsky (1996) (and in all the subsequent literature\(^3\)), based as it is on the aggregate value of graphs.

The next section describes the model and presents the link formation game. Section 3 contains the analysis of the set of Subgame Perfect Equilibria. Theorem 1 establishes existence, while theorems 2 and 3 state the efficiency results for the cases of participation demands and bilateral negotiations, respectively. Section 4 concludes the paper.

\(^2\)For example, in all international cooperation problems among countries, it is obvious that no mechanism designer can impose imputations.

2 The Model

2.1 Graphs and Values

Let $N = \{1, \ldots, n\}$ be a finite set of players. A pre-graph on $N$ is a set $A$ of directed arcs (directed segments joining two players in $N$). The arc from player $i$ to player $j$ is denoted by $a^j_i$. A graph $g$ is a set $L$ of links (non-directed segments) joining pairs of players in $N$ (nodes). Denoting by $g^N$ the complete graph, i.e. the set of all possible pairs of players, and by $ij$ the link that joins players $i$ and $j$, every graph $g$ on $N$ can be thought of as a subset of $g^N$. The set of arcs $A$ uniquely induces the graph $g(A) = \{ij : a^j_i \in A \text{ and } a^i_j \in A\}$.

We say that $h \subseteq g$ is a component of $g$ iff:

1. For all $i, j \in N(h)$ there exists a set $\{i_1, \ldots, i_m\} \subseteq N(h)$ such that $i_1 = i$ and $i_m = j$ and such that $i_k i_{k+1} \in h$ for all $k = 1, \ldots, m - 1$;

2. $i \in N(h)$ and $j \not\in N(h)$ implies that $ij \not\in g$.

Let $C(g)$ be the set of components of $g$. $N(h) \equiv \{i \in N : ij \in h \text{ for some } j \in N\}$ will denote the set of players in component $h$. Finally, $L(h)$ will denote the set of links in $h$. We will say that a graph $g$ is connected if it consists of a unique component.

To each graph $g \subseteq g^N$ we associate a value by means of the function $v : \mathcal{G} \rightarrow \mathbb{R}_+$, where $\mathcal{G}$ is the set of all subsets of $g^N$. The real number $v(g)$ represents the aggregate utility produced by the set of agents $N$ organized according to the graph (or network) $g$. We say that a graph $g^*$ is efficient with respect to $v$ if $v(g^*) \geq v(g) \forall g \subseteq g^N$. $\mathcal{G}^*$ will denote the set of efficient networks.

Note that if there are no spillovers the same value function can be applied to each component, and the value of a network equals the sum of the values of its components. On the other hand, if there are spillovers, then the value of a component depends on how the rest of the players are connected. In this paper we consider the standard case of no spillovers.

We make the standard normalization assumption that $v(i) = 0 \forall i$. We also restrict the analysis to anonymous value functions, i.e., such that $v(h)$ does not depend on the identity
of the players in $N(h)$.

### 2.2 The Link Formation Game

We will consider three sequential link formation games with endogenous payoff sharing. These games are all variants of a game $\Gamma(v)$ in which players move sequentially, proposing links and demanding payoffs. The variants, denoted by $\Gamma_1(v)$, $\Gamma_2(v)$, and $\Gamma_3(v)$, differ with respect to the way in which payoffs are demanded. We first introduce the general form of $\Gamma(v)$.

The set of players $N = \{1, \ldots, i, \ldots, n\}$ is exogenously ordered by the function $\rho : N \rightarrow N$. We use the notation $i \leq j$ as equivalent to $\rho(i) \leq \rho(j)$. Players sequentially choose actions according to the order $\rho$. An action $x_i$ for player $i$ is a pair $(a_i, d_i)$, where $a_i$ is a vector of arcs sent by $i$ to players in $N \setminus i$ and $d_i$ is $i$'s payoff demand.

A history $x = (x_1, \ldots, x_n)$ is a vector of actions for each player in $N$. We denote by $X$ the set of possible histories. We will use the notation (borrowed from Harris, 1985)

$$\lambda_i x \equiv (x_1, \ldots, x_{i-1})$$

to identify a subgame.

The game $\Gamma_1$ is obtained by assuming that $d_i \in R_+$ is a real positive number, representing the payoff demand of player $i$ in the game. The game $\Gamma_2$ is obtained by assuming that $d_i$ is a vector of real positive numbers, one for each player to which arcs are sent in the vector $a_i$. The game $\Gamma_3$ is obtained by assuming that $d_i$ is a number between zero and one, representing the demanded share of the payoff originated by $i$'s component. We will describe $\Gamma_1$ in detail; the other two games can be directly obtained by $\Gamma_1$.

#### 2.2.1 Link Formation with Participation Demands

Players' actions induce graphs on the set $N$ as follows. Firstly, we assume that at the beginning no links are formed, i.e., the game starts from the empty graph $g = \{\emptyset\}$. The history $x$ generates the graph $g(x)$ according to the following rule:
• If \( h \in g(a_1, \ldots, a_n) \) and \( h \) is feasible given \( x \), i.e., if

\[
\sum_{i \in N(h)} d_i \leq v(h),
\]

then \( h \in C(g(x)) \);

• If \( h \not\in g(a_1, \ldots, a_n) \) and (1) is violated, then \( h \notin C(g(x)) \) and \( i \in C(g(x)) \) for all \( i \in N(h) \);

• If \( h \not\in g(a_1, \ldots, a_n) \), then \( h \notin C(g(x)) \).

In words, the component \( h \) forms as the outcome of the history \( x \) if and only if the arcs sent in \( x \) generate \( h \) and the demands of payoff of the players in \( N(h) \) are compatible in the sense that they do not exceed the value produced by the component \( h \).

The payoff of player \( i \) is defined as a function of the history \( x \). Letting \( h_i(x) \in C(g(x)) \) denote the component of \( g(x) \) including \( i \), player \( i \) gets

\[
P_i(x) = \begin{cases} 
  d_i & \text{if } L(h_i(x)) \neq \emptyset \\
  0 & \text{otherwise.}
\end{cases}
\]

Note that \( L(h_i(x)) = \emptyset \) and \( N(h_i(x)) = \{i\} \) are two equivalent ways of saying that given the history \( x \) player \( i \) ends up by herself at the end of the game, either because no arcs were sent and reciprocated or because feasibility was violated.

A strategy for player \( i \) is a function \( \sigma_i : \lambda_i x \rightarrow x_i \). A strategy profile for \( \Gamma_1(v) \) is a vector of functions \( \sigma = (\sigma_1, \ldots, \sigma_n) \).

A Subgame Perfect Equilibrium (henceforth SPE) for \( \Gamma_1(v) \) is defined as follows. For any subgame \( \lambda_i x \), let \( \sigma |\lambda_i x \) denote the restriction of the strategy profile \( \sigma \) to the subgame. A strategy profile \( \sigma^* \) is an SPE of \( \Gamma_1(v) \) if for every subgame \( \lambda_i x \) the profile \( \sigma^* |\lambda_i x \) represents a Nash Equilibrium. We will denote by \( f(\lambda_i x) \) the SPE path of the subgame \( \lambda_i x \), i.e., the equilibrium continuation histories after \( \lambda_i x \).
2.2.2 Link Formation with Link-Specific Demands

The game \( \Gamma_2(v) \) (in which players make link-specific demands, one for every arc sent), is almost identical to \( \Gamma_1(v) \), as described in the previous section. The only two differences are the following:

1. The feasibility condition given in (1) is replaced by:

\[
\sum_{i \in N(h)} \sum_{j : j \in h} d^j_i \leq v(h); \tag{3}
\]

2. The payoff for player \( i \) in the component \( h \in C(g(x)) \) is given by

\[
P_i(x) = \begin{cases} 
\sum_{j \neq i : j \in h} d^j_i & \text{if } L(h(x)) \neq \emptyset \\
0 & \text{otherwise} 
\end{cases}
\tag{4}
\]

(instead of (2)). In words, the payoff for player \( i \) from history \( x \) would be equal to the sum of the link-specific demands made by \( i \) to the members of her component whom she is directly linked to.

2.2.3 Link Formation with Share-Demands

The distinguishing feature of game \( \Gamma_3(v) \) is that players demand shares \( (d_i \in [0,1]) \), and such shares refer to the surplus of cooperation given by the component to which \( i \) belongs \((v(h_i(x)))\). The description of the game would again be almost identical to that of \( \Gamma_1(v) \), with only two differences:

1. The feasibility condition given in (1) is replaced by:

\[
\sum_{i \in N(h)} d_i \leq 1; \tag{5}
\]

2. The payoff for player \( i \) in the component \( h \in C(g(x)) \) is given by

\[
P_i(x) = d_i v(h_i(x)) \tag{6}
\]

(instead of (2)).
3 Equilibrium

In this section we analyze the set of SPE of the game, once again giving all the details only for game $\Gamma_1(v)$, and then identifying the differences when moving to the other variants. We first show that SPE always exist. We then study the efficiency properties of SPE in the subsequent sections.

3.1 Existence of Equilibrium

Theorem 1 The game $\Gamma(v)$ always admits Subgame Perfect Equilibria.

Proof. We apply the existence theorem in Harris (1985) for extensive form games with infinite action spaces. That theorem relies on five assumptions, that are all satisfied by $\Gamma(v)$, in all its variants described above. We now list these five assumptions in the terms of $\Gamma(v)$.
1) Each agent’s action space is compact;
2) Each agent’s action space is Hausdorff;
3) The set of possible histories $X$ is closed;
4) The correspondence mapping each history $\lambda_i x$ into the actions’ set for player $i$ is lower hemicontinuous for all $i \in N$;
5) The payoff function $P_i$ is continuous for all $i \in N$.

Assumptions 1 and 3 are satisfied once some finite maximal demand $D$ has been specified for each player. If this maximal level is taken greater than the value of the efficient graph, imposing such constraint on the game does not affect its equilibria. Assumption 4 is satisfied since the set of possible actions for player $i$ does not depend on the previous history of the game. We are going to deal in more detail with assumptions 2 and 5.

Since every metric space is Hausdorff, it is sufficient to show that some metric can be imposed on the set of actions $X_i$ while still maintaining continuity of the payoff function $P_i$. 

9
We thus redefine the action space $X_i$ of player $i$ as follows:

$$X_i = \prod_n ((0, D) \times \{0, 1\})$$

where $(0, D) \subset R$ is the interval in which each player chooses his demands, and $\{0, 1\}$ is interpreted as follows: if player $i$ attaches a 0 to the $j$th player, it means that he does not send to $j$ any arc, and the opposite if he attaches a 1. We define a metric on $X_i$ by choosing some positive big enough distance $Q$ between any two actions that do not include the same set of arcs. We take the euclidean distance between any two actions with the same set of arcs. We now need to make sure that the payoff function $P_i$ is continuous in the topology we have defined. We first note that all converging sequences in $X_i$ are characterized by the same set of arcs along the sequence. Thus, every converging sequence in $X = \prod_n X_i$ has the same sets of arcs along the sequence. Thus, the payoff function is continuous if it is continuous in the demands. It can be checked that $P_i$ is continuous in the demands in the euclidean metric.

QED.

3.2 Efficiency Properties of Equilibria

As discussed in the introduction, one of the main theoretical findings in the theory of networks has to do with the impossibility to reconcile efficiency and stability of networks when payoffs are distributed according to a fixed imputation rule. In the present setting, any conflict between efficiency and stability should translate into the possibility of attaining an inefficient graph as outcome of a SPE history. In this section we prove that no conflict between efficiency and stability arises, as long as merging components always increases the value of the network.

This condition, which we call size monotonicity, is shown to be sufficient to establish efficiency of all SPE of $\Gamma_1(v)$ and $\Gamma_2(v)$. Moreover, Example 1 shows that once size monotonicity is

\footnote{Strictly speaking, this is the rewriting of the action space of player $i$ in $\Gamma_2(v)$. The one for $\Gamma_1(v)$ and $\Gamma_3(v)$ is a special case, since there is only one demand (or share) to make (so the $\prod_n$ sign applies only to $\{0, 1\}$.)}

\footnote{See Jackson and Wolinsky for such an impossibility result. When the fixed imputation rule can be selected by a mechanism designer, then Dutta and Mutuswami (1997) show that efficiency and stability can be reconciled. Recall that in this paper we are concerned with market networks, where a mechanism designer choosing imputation rules can hardly be realistic.}
relaxed, inefficient graphs can arise as SPE of both games. Finally, we will show that even when size monotonicity is satisfied, efficiency cannot be guaranteed in $\Gamma_3(v)$. Here is the definition of the unique axiom needed for our efficiency results.

**Definition 1** The link $ij$ is critical for the graph $g$ if $ij \in g$ and $\#C(g) > \#C(g \setminus ij)$.

In words, a link is critical for a graph if by removing it we increase the number of components. Intuitively, a critical link is essential for the component it belongs to in the sense that without it that component would split in two different components.

**Definition 2** The value function $v$ satisfies Size Monotonicity if and only if for all graphs $g$ and critical link $ij \in g$ it holds that $v(g) > v(g \setminus ij)$.

### 3.2.1 Participation Demands

We first analyze the case of participation demands, represented by the game $\Gamma_1(v)$. The first lemma characterizes the structure of efficient graphs under size monotonicity.

**Lemma 1** Let $v$ satisfy size monotonicity. All efficient graphs are connected, i.e., if $g$ is efficient then $C(g) = \{g\}$.

**Proof.** Consider a graph $g$ such $C(g) = \{h_1, \ldots, h_p\}$ with $p > 1$. Then let $i \in h_1$ and $j \in h_2$, with, therefore, $ij \notin g$. The link $ij$ is a critical link according to definition 1, so that, by size monotonicity of $v$ we have that $v(g) < v(g \cup ij)$, implying that $g$ is not efficient.

**QED.**

The following lemmas prepare the efficiency theorem by proving a few properties of equilibrium graphs under size monotonicity of $v$.

**Lemma 2** Let $v$ satisfy size monotonicity. Let $\lambda_m x$ be an arbitrary history of the game $\Gamma_1(v)$. Then $P_i(f(\lambda_m x)) > 0$ for all $i = 1, \ldots, n - 1$.

**Proof.** Let $n$ be the last player in the ordering $\rho$ and let $m < n$. Consider an arbitrary history $\lambda_m x$. We show that there exists $\varepsilon > 0$ such that if player $m$ plays the action $x_m =$
then it is a dominating strategy for player \( n \) to reciprocate \( m \)'s arc and form some feasible component \( h \) with \( mn \in h \). We first show it is true for \( \varepsilon = 0 \). Consider the continuation history \( \hat{x} = f(\lambda_n x, x_m) \). Let \( x_n = (a_n, d_n) \) be a strategy for player \( n \) such that \( a^m_n \notin a_n \), and let \( h(n) \) be the component including \( n \) if \( x_n \) is played at the history \( \lambda_n \hat{x} \). If we denote by \( h'(n) \) the component obtained by adding the link \( mn \) to \( h(n) \), we conclude that by size monotonicity:

\[
v(h(n)) < v(h'(n)).
\]

This implies that if the component \( h(n) \) is feasible given \( x_n \), the component \( h'(n) \) remains feasible for some demand \( d_n + \delta > d_n \) of player \( n \). If \( h(n) \) is not feasible, then either there exists some positive demand \( d_n \) for player \( n \) such that

\[
\sum_{i \in N(h') \setminus n} d_i + d_n = v(h')
\]

or player \( n \) could just reciprocate player \( m \)'s arc and demand \( d_n = v(mn) > 0 \) (this last inequality follows again from size monotonicity). It follows that it is dominant for \( n \) to reciprocate \( m \)'s arc and get a strictly positive payoff.

We now show that this remains true for some \( \varepsilon > 0 \). For a given \( \varepsilon_m > 0 \), let \( x_m (\varepsilon_m) = (a^m_m, \varepsilon_m) \), and consider again the continuation history \( \hat{x} (\varepsilon_m) = f(\lambda_m x, x_m (\varepsilon_m)) \). Let also \( x_n = (a_n, d_n) \) be a strategy for player \( n \) such that \( a^m_n \notin a_n \), and let \( h(n, \varepsilon_m) \) be the component that includes \( n \) if \( x_n \) is played at the history \( \lambda_n \hat{x} (\varepsilon_m) \) and \( h'(n, \varepsilon_m) \) be the component obtained by adding the link \( mn \) to \( h(n, \varepsilon_m) \). Define

\[
\delta_{\min} \equiv \min_{\varepsilon > 0} v(h'(n, \varepsilon)) - h(n, \varepsilon) > 0,
\]

where the last inequality comes from size monotonicity. Let now \( 0 < \varepsilon_m < \delta_{\min} \). Note first that if \( h(n, \varepsilon_m) \) is feasible, then \( h'(n, \varepsilon_m) \) is feasible for some positive additional demand of player \( n \). Thus, it is possible for player \( n \) to demand a strictly higher payoff than under \( x_n \) (this because \( \varepsilon_m < \delta_{\min} \)). If instead \( h(n, \varepsilon_m) \) is not feasible, then either there exists some positive demand \( d_n \) for player \( n \) such that

\[
\sum_{i \in N(h'(n, \varepsilon_m)) \setminus n} d_i + d_n = v(h'(n, \varepsilon_m))
\]
or player $n$ could just reciprocate player $m$'s arc and demand $d_n = v(mn) - \varepsilon_m > 0$ (this last inequality again follows from size monotonicity). It follows that it is dominant for $n$ to reciprocate $m$'s arc and get a strictly positive payoff.  

**QED.**

**Lemma 3** Let $v$ satisfy size monotonicity. Let $x$ be a SPE history of the game $\Gamma_1(v)$. In the induced graph $g(x)$ all players are connected, i.e., $C(g(x)) = \{g(x)\}$.

**Proof.** Suppose that $C(g(x)) = \{h_1, \ldots, h_k\}$ with $k > 1$. Let again $n$ be the last player in the ordering $\rho$. Note first that there must be some component $h_p$ such that $n \notin h_p$, since otherwise the assumption that $k > 1$ would be contradicted. Also, note that by Lemma 2, $x$ being an equilibrium implies that

$$\sum_{i \in N(h_p)} d_i = v(h_p) \quad \forall p \in \{1, \ldots, k\}.$$

Let us then consider $h_p$ and the last player $m$ in $N(h_p)$ according to the ordering $\rho$. Let $\hat{x}_m(\varepsilon) = (a_m \cup a_n, d_m + \varepsilon)$, with continuation history $\hat{x}(\varepsilon) = f(\lambda_m x, \hat{x}_m(\varepsilon))$. Let $h(n, \varepsilon)$ be the component including $n$ in $g(\hat{x}(\varepsilon))$. Suppose first that $mn \notin h(n, \varepsilon)$ and $m \in h(n, \varepsilon)$ for some $i \in N(h_p)$. Note first that if some player $j > m$ is in $h(n, \varepsilon)$, then by Lemma 2 $h(n, \varepsilon)$ is feasible given $x_n$, and since player $m$ is getting a higher payoff than under $x$, the action $\hat{x}_m(\varepsilon)$ is a profitable deviation for him. We therefore consider the case in which no player $j > m$ is in $h(n, \varepsilon)$, and $h(n, \varepsilon)$ is not feasible. In this case, it is a feasible strategy for player $n$, who is getting a zero payoff under $x_n$, to reciprocate only player $m$'s arc and form the component $h'(n, \varepsilon)$ such that, by size monotonicity,

$$v(h'(n, \varepsilon)) > v(h_p).$$

If $\varepsilon$ is small enough we get

$$v(h'(n, \varepsilon)) - v(h_p) > \varepsilon$$

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Note that there cannot be any equilibrium where the last player demands something unfeasible: since in every equilibrium the last player obtains a zero payoff, one could think that she could then demand anything, making the complete graph unfeasible, but this would entail a deviation by one of the previous players, who would demand $\varepsilon$ less, in order to make $n$ join in the continuation equilibrium. Thus, the unique equilibrium demand of player $n$ is 0.
which implies that reciprocating only player $m$'s arc and demanding $d_n = v(h'(n,\varepsilon)) - v(h_p) - \varepsilon > 0$ is a profitable deviation for player $n$.

We thus can restrict ourselves to the case in which $in \notin h(n,\varepsilon)$ for all $i \in h_p$. Let $h'(n,\varepsilon)$ be obtained by adding the link $mn$ to $h(n,\varepsilon)$. By size monotonicity

$$v(h'(n,\varepsilon)) - v(h(n,\varepsilon)) > 0.$$ 

Let also

$$\delta_{min} \equiv \min_{\varepsilon > 0} [v(h'(n,\varepsilon)) - v(h(n,\varepsilon))] > 0.$$ 

Consider a demand $\varepsilon_m$ such that $0 < \varepsilon_m < \delta_{min}$. As in the proof of Lemma 2, we claim that if player $m$ demands $\varepsilon_m$, then it is dominant for player $n$ to reciprocate player $m$'s link and form the component $h'(n,\varepsilon_m)$. Note first that, given that $0 < \varepsilon_m < \delta_{min}$, if $h(n,\varepsilon_m)$ is feasible, then $h(n,\varepsilon_m)$ is feasible for some positive additional demand (w.r.t. $d_n$) of player $n$. If instead $h(n,\varepsilon_m)$ was not feasible, then player $n$ would be getting a zero payoff, and this would be strictly dominated by reciprocating $m$'s arc and getting a payoff of

$$[v(h'(n,\varepsilon_m)) - v(h(n,\varepsilon_m))] - \varepsilon_m$$

which, again by the fact that $\varepsilon_m < \delta_{min}$, is strictly positive. QED.

**Theorem 2** Let $v$ satisfy size monotonicity. Every SPE of $\Gamma_1(v)$ leads to an efficient network.

**Proof.** The proof is by induction. We first prove, in step 1, that if a given history is not efficient and satisfies a certain condition on payoff demands, then some player has a profitable deviation. Then, in step 2, we show that if some history $x$ such that $g(x) \notin G^*$ is a SPE, then the condition on payoff demands introduced in step 1 would be satisfied, which implies that there exists a profitable deviation from any history that leads to an inefficient network.

**Step 1. Induction argument.**

*Induction Hypothesis (H):* Let $x$ be an arbitrary history such that $g(x) \notin G^*$. Let $m$ be the first player in the ordering $\rho$ such that there is no $x^*$ such that (1) $\lambda_{m+1}x^* = \lambda_{m+1}x$ and
(2) \( g(x^*) \in G^* \). Let \( x \) be such that

\[
\sum_{i=1}^{m} d_i \leq v(g(x)) - \sum_{i=m+1}^{n} d_i.
\]

Then there exists some \( \varepsilon_m > 0 \) such that the action \( x_m^* = (a_m^*, d_m + \varepsilon_m) \) induces a continuation history \( \hat{x} = f(\lambda_m x, x_n^*) \) such that \( g(\hat{x}) \in G^* \) and \( \sum_{i=1}^{n} \hat{d}_i = v(g(\hat{x})) \).

(H) true for player \( n \): Let \( x_n = (a_n, d_n) \). Let player \( m \), as defined in (H), be \( n \). In words, this means that \( n \) could still induce the efficient graph by deviating to some other action. Formally, there exist some arcs \( a_n^* \) and a demand \( d_n' \) such that \( g(x_1, \ldots, x_{n-1}, a_n^*, d_n') \in G^* \) and, therefore, such that \( v(g(x_1, \ldots, x_{n-1}, a_n^*, d_n')) > v(g(x)) \). By (H)

\[
\sum_{i=1}^{n} d_i \leq v(g(x))
\]

and by size monotonicity all players are connected in \( g(x_1, \ldots, x_{n-1}, a_n^*, d_n') \). These two facts imply that player \( n \) can induce the efficient graph and demand \( d_n' = d_n + \varepsilon_n \) with

\[
\varepsilon_n = [v(g^*) - v(g(x))] > 0.
\]

(H) true for player \( m+1 \) implies (H) true for player \( m \): Suppose again that \( x \) is an inefficient history and that \( m \) is the first player in \( x \) such that the action \( a_m \) is not compatible with efficiency in the sense of assumption (H). Let \( a_m^* \) be some action compatible with efficiency and let \( x_m^*(\varepsilon) = (a_m^*, d_m + \varepsilon) \). Let also \( x^*(\varepsilon) = f(\lambda_m x, x_m^*(\varepsilon)) \) represent the corresponding continuation history. We need to show that there exists \( \varepsilon > 0 \) such that \( g(x^*(\varepsilon)) \in G^* \). Note first that in the history \( x^*(\varepsilon) \), the first player \( k \) such that \( a_k \) is not compatible with efficiency must be such that \( k > m \). Since by (H)

\[
\sum_{i=1}^{m} d_i \leq v(g(x)) - \sum_{i=m+1}^{n} d_i
\]

there exists an \( \varepsilon_m > 0 \) such that

\[
\sum_{i=1}^{m-1} d_i + d_m + \varepsilon_m < v(g^*) - \sum_{i=m+1}^{n} d_i.
\]

Thus, if player \( m \) plays \( x_m^*(\varepsilon_m) \), player \( m+1 \) faces a history \( (\lambda_m x, x_m^*(\varepsilon_m)) \) that satisfies the inductive assumption (H). Suppose now that player \( m+1 \) optimally plays some action \( x_{m+1} \) such that no efficient graph is compatible (in the sense of assumption (H)) with
the history \((\lambda_m x, x_m(\epsilon_m), x_{m+1})\). Then, by (H) we know there would be a deviation for player \((m + 1)\), contradicting the assumption that \(x_{m+1}\) is part of the continuation history at \((\lambda_m x, x_m(\epsilon_m))\). Thus, we know that player \((m + 1)\) will optimally play some strategy \(x^*_{m+1}\) such that the continuation history \(f((\lambda_m x, x_m(\epsilon_m), x^*_{m+1}))\) induces a feasible efficient graph.

Step 2. We now show that the induction argument can be applied to each candidate SPE history \(x\) of \(\Gamma_1(v)\) such that \(v(g(x)) < v(g^*)\) (which we want to rule out). This is shown to imply that the first player \(m\) (such that there does not exist \(x^*\) such that \(\lambda_{m+1}x^* = \lambda_{m+1}x\) and \(v(g(x^*)) = v(g^*)\)) has a profitable deviation.

Note first that by Lemma 3 if \(x\) is a SPE history then all players are connected. This, together with Lemma 2, directly implies that

\[
\sum_{i=1}^{n} d_i = v(g(x))
\]

or, equivalently, that

\[
\sum_{i=1}^{m} d_i = v(g(x)) - \sum_{i=m+1}^{n} d_i
\]

for all \(m = 1, \ldots, n\). It follows that the induction argument can be applied to all inefficient SPE histories to conclude that the first player whose action is not compatible with efficiency in the sense of assumption (H) has some action \(x^*_m(\epsilon_m) = (a^*_m, d_m + \epsilon_m)\) such that \(\epsilon_m > 0\) and such that the induced graph \(g(f(\lambda_m x, x^*_m)) \in G^*\) is feasible. Since \(g(f(\lambda_m x, x^*_m))\) is feasible, then the action \(x^*_m(\epsilon)\) represents a deviation for player \(m\), proving the theorem.

QED.

Remark 1 The efficiency theorem extends to the case in which the order of play is random, i.e., in which each mover only knows a probability distribution over the identity of the subsequent mover. Without anonymity, on the other hand, the order of play may become crucial.
3.2.2 Link-Specific Demands

We now consider the case of multiple demands, represented by the game $\Gamma_2(v)$. We show that the result of the previous section extends to this game. Since the proofs are slightly longer but conceptually analogous, we relegate them to the appendix.

Lemma 4 Let $v$ satisfy size monotonicity. Let $\lambda_m x$ be an arbitrary history of the game $\Gamma_2(v)$. Then $P_i(f(\lambda_m x)) > 0$ for all $i = 1, \ldots, n - 1$.

Lemma 5 Let $v$ satisfy size monotonicity. Let $x$ be a SPE history of the game $\Gamma_2(v)$. In the induced graph $g(x)$ all players are connected, i.e., $C(g(x)) = \{g(x)\}$.

Theorem 3 Let $v$ satisfy size monotonicity. Every SPE of $\Gamma_2(v)$ leads to an efficient network.

Remark 1 also extends to $\Gamma_2(v)$.

3.2.3 Eliminating Size Monotonicity

We now show that size monotonicity is a minimal requirement for our efficiency result. The next example shows that if a value function $v$ does not satisfy size monotonicity, then the SPE of $\Gamma_1(v)$ and $\Gamma_2(v)$ may induce an inefficient network.

Example 1 Consider a four-player game with the following value function:

\[
v(h) = \begin{cases} 
9 & \text{if } N(h) = N \\
8 & \text{if } \#N(h) = 3 \text{ and } \#L(h) = 2; \\
5 & \text{if } \#N(h) = 2; \\
0 & \text{otherwise.}
\end{cases}
\]

We show that the history

\[x_1 = \left( (a_1^2, 3), (a_1^3, 3), (a_1^4, 3) \right)\]
\[ x_2 = (a_2^1, 0, (a_2^3, 3), (a_2^1, 3)) \]
\[ x_3 = (a_3^2, 0, (a_3^4, 3)) \]
\[ x_4 = (a_4^3, 0) \]

is a SPE of the game \( \Gamma_2(v) \), leading to the inefficient graph \((12, 23, 34)\).\(^7\)

1. **Player 4:** by the form of \( v \), the last mover has no profitable alternative, given the history \( \lambda_4 x \).

2. **Player 3:** forming the links \((12, 13, 23)\) would let player 3 demand at most 2; forming a link just with player 4 would give player 3 at most 3, since player 4 would have at that node the outside option of going with the first two movers.

3. **Player 2:** If the second mover demands more than 3 to player 3, the latter has the outside option of ignoring the arc of player 2 and reciprocating the arc of player 1, which would then let him demand 3 to player 4 and obtain 3 anyway. Thus, no demand deviation is profitable for player 2. In terms of arcs, it is first of all clear that sending just the backward arc would be worse, since the payoff demanded cannot be greater than 2, given the demand of player 1. The last thing to check for player 2 is that she cannot benefit from sending arcs only to 1 and 4, demanding more than 3 to player 4. Suppose she did that, demanding \( 3 + \epsilon \) to 4. Then player 3 would react by sending an arc just to player 4, demanding \( 3 + \epsilon - \delta \) (\( \epsilon > \delta > 0 \)), which 4 would optimally reciprocate. So, no deviation for player 2.

\(^7\)The same inefficiency result for Example 1 can be proved for the game \( \Gamma_1(v) \). In that case, the corresponding equilibrium history is:

\[ x_1 = (a_1^3, a_1^3, a_1^4), 3) \]
\[ x_2 = (a_2^2, a_2^3, a_2^4), 3) \]
\[ x_3 = (a_3^2, a_3^3), 3) \]
\[ x_4 = (a_4^3, 0) \]
4. **Player 1**: Could she obtain $3 + \epsilon$ in any way? The answer is no, since the second mover can "underbid" by a small $\delta$, as in the argument above, so that player 3 and/or 4 would always prefer to go with player 2.

This example has shown that when size monotonicity is violated then inefficient equilibria may exist.\(^8\)

### 3.2.4 Surplus Shares

Having established that similar efficiency properties hold for the cases of link-specific absolute demands and absolute participation demands, it is reasonable to ask what happens if players can only demand shares, rather than final payoffs.\(^9\) This case is represented by the game $\Gamma_3(v)$.

**Theorem 4** There exist anonymous value functions $v$ such that no SPE of the link formation game $\Gamma_3(v)$ induces an efficient network.

The proof is by example.

**Example 2** Consider a game with four players and the following value function:

$$
v(g^N) = 2 + \epsilon \\
v(ij) = 1 \\
v(ij, jk) = v(ij, jk, ik) = 1 + \delta$

In this example there is no way to endogenously obtain the complete graph (which is the efficient one). To see this, assume that the first two players have sent all possible arcs (thereby completing one link) and call their share demands $\alpha$ and $\beta$. If player 3 sends an arc to player 4 he can get at most a payoff of $[1 - (1 - \alpha + \beta)(1 + \delta)]$. For $\delta \to 0$ this amount tends to $(\alpha + \beta)$. So, for player 3 not to do this, $(\alpha + \beta)$ must satisfy

$$
2 + \epsilon(1 - \alpha - \beta) = \alpha + \beta
$$

---

\(^8\)The proof of this for $\Gamma_1(v)$ is much easier, and is therefore omitted.

\(^9\)As mentioned in the introduction, this case would be very relevant if the value of networks was uncertain.
which implies

$$\alpha + \beta = \frac{2 + \epsilon}{3 + \epsilon}.$$  

Player 2 obtains, on the proposed path, \((\frac{3}{3+\epsilon} - \alpha)2 + \epsilon\). If \(\alpha < \frac{1}{2}\), the best alternative for player 2 is to reciprocate just the proposed link with player 1, obtaining \(1 - \alpha\). Thus, for 2 to follow the efficient path, \(\alpha\) must satisfy

$$(\frac{2 + \epsilon}{3 + \epsilon} - \alpha)2 + \epsilon = (1 - \alpha).$$

For \(\epsilon \to 0\) the solution of this equation (in terms of \(\alpha\)) tends (from the right) to \(\frac{1}{3}\). Hence, for \(\epsilon\) small enough, player 1 has a profitable deviation from the efficient path, demanding 1 and sending all arcs. It can be easily shown that the continuation equilibrium of this action yields an inefficient network with two pairs, rather than the grandcoalition. QED.

4 Conclusions

This paper has shown that when no designer exists and the payoff sharing is endogenous, as in many market networks, the subgame perfect equilibrium of a sequential link formation game, in which the relevant players demand absolute payoffs, leads to an efficient network, as long as the aggregate value of networks changes monotonically with the size of the network. On the other hand, endogenous payoff division is not sufficient to obtain optimality when the optimal network has more than one component. The situations in which players form a network by subsequent bilateral negotiations or absolute payoff demands have been fully characterized, and then compared with the situations where players make demands in terms of shares of the final value of their component. In the latter case, efficiency may not be achieved in equilibrium.

We plan to extend the model of this paper to a dynamic setting, where more than one round of sequential demands can be played. Our conjecture is that with multiple rounds efficiency and stability can be reconciled also for situations where players demand shares.
Appendix

Proof of Lemma 4.

Let $n$ be the last player in the ordering $\rho$ and let $m < n$. Consider an arbitrary history $\lambda_m x$. We show that there exists a demand $d^n_m > 0$ such that if player $m$ plays the action $x_m = (a^n_m, d^n_m)$ then it is a dominating strategy for player $n$ to reciprocate $m$'s arc and form some feasible component $h$ with $mn \in h$.

For a given $d^n_m > 0$, let $x_m (d^n_m) = (a^n_m, d^n_m)$, and consider again the continuation history $\dot{x} (d^n_m) = f (\lambda_m x, x_m (d^n_m))$. Let also $x_n = (a_n, d_n)^10$ be a strategy for player $n$ such that $a^n_m \not\in a_n$. Let $h(n, d^n_m)$ be the component that includes $n$ if $x_n$ is played at the history $\lambda_m \dot{x} (d^n_m)$ and $h'(n, d^n_m)$ be the component obtained by adding the link $mn$ to $h(n, d^n_m)$. Define

$$\delta_{\text{min}} \equiv \min_{d^n_m > 0} v (h'(n, d^n_m)) - v (h(n, d^n_m)) > 0,$$

where the last inequality comes from size monotonicity. Let now $0 < d^n_m < \delta_{\text{min}}$. Note first that if $h(n, d^n_m)$ is feasible, then $h'(n, d^n_m)$ is feasible for some positive demand $d^n_n$ of player $n$. Thus, player $n$ can get a strictly higher payoff than under $x_n$ (this because $\epsilon < \delta_{\text{min}}$). If instead $h(n, d^n_m)$ is not feasible, then either there exists some positive demand $d^n_n$ for player $n$ such that

$$\sum_{i \in N(h'(n,d^n_m))} \sum_{j: ij \in h'(n,d^n_m)} d^n_i + d^n_n = v (h'(n, d^n_m))$$

or player $n$ could just reciprocate player $m$'s arc and demand her $d^n_n = v(mn) - d^n_m > 0$ (this last inequality again follows from size monotonicity). It follows that it is dominant for $n$ to reciprocate $m$'s arc and get a strictly positive payoff. \textbf{QED.}

Proof of Lemma 5.

Suppose that $C(g(x)) = \{h_1, \ldots, h_k\}$ with $k > 1$. Let again $n$ be the last player in the ordering $\rho$. Note first that there must be some component $h_p$ such that $n \not\in h_p$, since otherwise the assumption that $k > 1$ would be contradicted. Also, note that by Lemma 4, $x$

\footnote{Recall that in game $\Gamma_2(v)$ $d_n$ is a vector, with as many dimensions as the number of arcs sent by $n$.}
being an equilibrium implies that for all $p = 1, \ldots, k$

$$\sum_{i \in N(h_p)} \sum_{j : j \in h_p} d_i^j = v(h_p).$$

Let us then consider $h_p$ and the last player $m$ in $N(h_p)$ according to the ordering $p$. Let $\hat{x}_m(d_m^n) = (a_m \cup a_m^n, d_m \cup d_m^n)$, with continuation history $\hat{x}(d_m^n) = f(\lambda_m x, \hat{x}_m(d_m^n))$. Let $h(n, d_m^n)$ be the component including $n$ in $g(\hat{x}(d_m^n))$. Suppose first that $mn \notin h(n, d_m^n)$ and $in \in h(n, d_m^n)$ for some $i \in N(h_p)$. Conside then the demand

$$\hat{d} _m^n < \min_{j \in N(h_p)} \{d_j^n\}.$$ 

Let now player $m$ play $\hat{d}_m^n$. Suppose that still $in \in N(h(n, d_m^n))$ for some $i \in N(h_p)$. Then it would be a profitable deviation for player $n$ to reciprocate the arc sent by $m$ instead of the arc sent by some other player $i \in N(h_p)$, to which a demand $d_i^n > d_m^n$ is attached.

Suppose now that $in \notin N(h(n, d_m^n))$ for all $i \in N(h_p)$. Let $h'(n, d_m^n)$ be obtained by adding the link $mn$ to $h(n, d_m^n)$. By size monotonicity

$$v(h'(n, d_m^n)) - v(h(n, d_m^n)) > 0.$$

Now let

$$\delta_{\text{min}} \equiv \min_{d_m^n > 0} [v(h'(n, d_m^n)) - v(h(n, d_m^n))] > 0.$$

Consider now a demand $0 < d_m^n < \delta_{\text{min}}$. As in the proof of Lemma 4, we claim that it is dominant for player $n$ to reciprocate player $m$'s link and form a feasible component. Note first that, given that $0 < d_m^n < \delta_{\text{min}}$, if $h(n, d_m^n)$ is feasible, then $h(n, d_m^n)$ is feasible for some positive demand $d_m^n$ of player $n$. If instead $h(n, d_m^n)$ was not feasible, then player $n$ would be getting a zero payoff, and this would be strictly dominated by reciprocating $m$'s arc and getting a payoff of $[v(h'(n, d_m^n)) - v(h(n, d_m^n))] - d_m^n$, which, again by the fact that $d_m^n < \delta_{\text{min}}$, is strictly positive.

**Proof of Theorem 3.**

We proceed by first showing by induction, in step 1, that if a given history is not efficient and satisfies a certain condition on payoff demands, then some player has a profitable deviation. In step 2 we establish that if a history $x$, leading to an inefficient graph, was SPE, then
it would have to satisfy the condition on payoff demands described in step 1, which implies that there exists a profitable deviation from any such history $x$ leading to an inefficient graph.

**Step 1. Induction Argument.**

**Induction Hypothesis (H):** Let $x$ be an arbitrary history such that $g(x) \notin G^*$. Let $m$ be the first player in the ordering $\rho$ such that there is no $x^*$ such that (1) $\lambda_{m+1} x^* = \lambda_{m+1} x$ and (2) $g(x^*) \in G^*$. Let $x$ be such that

$$\sum_{i=1}^{m} \sum_{j:ij \in N(h(i))} d_i^j \leq v(g(x)) - \sum_{i=m+1}^{n} \sum_{j:ij \in N(h(i))} d_i^j.$$ 

Then there exists some $\varepsilon_m > 0$ such that the action $x_m^* = (a_m^*, d_m + \varepsilon_m)$ induces a history $\hat{x} = f(\lambda_m x, x_m^*)$ such that $g(\hat{x}) \in G^*$ and $\sum_{i=1}^{n} \sum_{j:ij \in N(h(i))} d_i^j = v(g(\hat{x})).$

(H) **true for player $n$:** Let $x_n = (a_n, d_n)$. By assumption (H), there exists some arcs $a_n^*$ such that $g(\lambda_n a, a_n^*) \in G^*$ and, therefore, such that $v(g(\lambda_n a, a_n^*)) > v(g(x))$. By (H)

$$\sum_{i=1}^{n} \sum_{j:ij \in N(h(i))} d_i^j \leq v(g(x))$$

Moreover, by size monotonicity all players are connected in $g(\lambda_n a, a_n^*)$.\(^{11}\) These two facts imply that player $n$ can induce the efficient graph and demand the vector $d_n + \varepsilon_n$, where

$$\varepsilon_n = [v(g(\lambda_n a, a_n^*)) - v(g(x))] > 0.$$ 

(H) **true for player $m + 1$ implies (H) true for player $m$:** Suppose again that $x$ is an inefficient history and that $m$ is the first player in $x$ such that the action $a_m$ is not compatible with efficiency (in the sense of assumption (H)). Let $a_m^*$ be some vector of arcs compatible with efficiency and let $x_m^* (\varepsilon) = (a_m^*, d_m + \varepsilon)$. Let $x^* (\varepsilon) \equiv f(\lambda_m x, x_m^* (\varepsilon))$ represent the relative continuation history. We need to show that there exists $\varepsilon > 0$ such that $g(x^* (\varepsilon)) \in G^*$. Note first that in the history $x^* (\varepsilon)$ the first player $k$ such that $a_k$ is not compatible with efficiency must be such that $k > m$. Also, since by (H)

$$\sum_{i=1}^{m} \sum_{j:ij \in N(h(i))} d_i^j \leq v(g(x)) - \sum_{i=m+1}^{n} \sum_{j:ij \in N(h(i))} d_i^j$$

\(^{11}\) $\lambda_a a$ constitutes a slight abuse of notation, describing the history of arcs sent before the turn of player $i$. 

23
there exists an $\varepsilon_m > 0$ such that

$$
\sum_{i=1}^{m-1} \sum_{j: i \in N(h(i))} d_i^j + \sum_{j: m \in N(h(m))} (d_m^j + \varepsilon_m) < v(g^*) - \sum_{i=m+1}^{n} \sum_{j: i \in N(h(i))} d_i^j.
$$

Thus, if player $m$ plays $x_m^* (\varepsilon_m)$, player $m + 1$ faces a history $(\lambda_m x, x_m^* (\varepsilon_m))$ that satisfies the inductive assumption (H). Suppose now that player $m + 1$ optimally plays some action $x_{m+1}$ such that no efficient graph is compatible (in the sense of assumption (H)) with the history $(\lambda_m x, x_m^* (\varepsilon_m), x_{m+1})$. Then, by (H) we know there would be a deviation for player $m + 1$, contradicting the assumption that $x_{m+1}$ is part of the continuation history at $(\lambda_m x, x_m^* (\varepsilon_m))$. Thus, we know that player $m + 1$ will optimally play some strategy $x_{m+1}^*$ such that the continuation history $f ((\lambda_m x, x_m^* (\varepsilon_m), x_{m+1}^*))$ induces a feasible efficient graph.

**Step 2.** We now show that the induction argument can be applied to each SPE history $x$ of $\Gamma_2 (v)$ such that $g(x) \notin \mathcal{G}^*$. This is shown to imply that the first player $m$ such that there is no $x^*$ such that $\lambda_{m+1} x^* = \lambda_{m+1} x$ and $g(x^*) \in \mathcal{G}^*$ has a profitable deviation.

Note first that by Lemma 5 if $x$ is a SPE history then all players are connected. This, together with Lemma 4, directly implies that

$$
\sum_{i=1}^{n} \sum_{j: i \in N(h(i))} d_i^j = v(g(x))
$$

or, equivalently, that

$$
\sum_{i=1}^{m} \sum_{j: i \in N(h(i))} d_i^j = v(g(x)) - \sum_{i=m+1}^{n} \sum_{j: i \in N(h(i))} d_i^j
$$

for all $m = 1, \ldots, n$. It follows that the induction argument can be applied to all inefficient SPE histories, to conclude that the first player whose action is not compatible with efficiency in the sense of (H), has some action $x_m^* (\varepsilon_m) = (a_m^*, d_m + \varepsilon_m)$ such that $\varepsilon_m > 0$ and such that the induced graph $g(f(\lambda_m x, x_m^* (\varepsilon_m))) \in \mathcal{G}^*$ is feasible. Since $g(f(\lambda_m x, x_m^*))$ is feasible, then the action $x_m^* (\varepsilon_m)$ represents a deviation for player $m$, proving the theorem. **QED.**
References


