Fair Bargains: Distributive Justice and Nash Bargaining Theory

by Marco Mariotti

Department of Economics, Royal Holloway College, University of London, Egham, Surrey TW20 0EX, UK.

Abstract: the Suppes-Sen dominance relation is a weak and widely accepted criterion of distributive justice. I propose its application to Nash bargaining theory. The Nash Bargaining Solution (NBS) is characterised by replacing the controversial Independence of Irrelevant Alternatives axiom with an axiom embodying the Suppes-Sen principle. The characterisation is more robust than the standard one with respect to variations in the domain of bargaining problems. It is also shown that a subset of Nash's axioms imply the Suppes-Sen relation.

JEL class. number: C71, C78.

I am deeply indebted, with the usual disclaimers, to Paul Madden for comments, bibliographical suggestions, encouragement, and not least for originating my interest in the Suppes-Sen relation. Useful suggestions have also come from Vincenzo Denicolò, Joseph Greenberg, Paola Manzini, Herve Moulin, Martin Osborne, Fioravante Patrone, Hyun Shin, Stefano Zamagni, two anonymous referees and from the seminar audiences at the 1997 Summer Meeting of the Econometric Society in Pasadena and at University College London.
1. Introduction

In the axiomatic theory of bargaining initiated by Nash (1950) one defines a set of admissible bargaining problems (e.g., convex problems) and imposes some desirable requirements (axioms) on the solution function, which picks an element from each problem; the aim is to characterise uniquely such a function. One interpretation of the axioms is as properties that should be satisfied by the choices of a fair arbitrator (e.g., Myerson (1990, p. 372-3), Young (1994, ch. 7), Mas Colell et al. (1993, ch. 22.E) and, implicitly, Hammond (1991), p. 203). The axioms that characterise the Nash Bargaining Solution (NBS) include the powerful Independence of Irrelevant Alternative (IIA), which has been extensively discussed and criticised. Indeed, although IIA and its variations may be viewed as a relevant criterion of rationality for individual choice, it is difficult to see it as a compelling requirement of a fair arbitration. As Binmore (1992, p.196) puts it:

"Some authors misunderstand Nash's motives in formulating his bargaining solution and imagine that his axioms can be sensibly interpreted as criteria for a 'fair arbitration scheme' - other axiom systems have been introduced to characterise other so-called 'bargaining solutions' that do make sense as fair arbitration schemes'.

I agree with Binmore's statement concerning Nash's axioms. In this paper I aim to show that - whatever Nash's motives were - there are nonetheless strong reasons to interpret Nash's solution as the expression of a fair arbitrator's decisions under certain informational conditions. I will illustrate a striking property of the NBS, which supports this interpretation and allows a characterisation that dispenses with IIA altogether. It turns out that the NBS is the only scale covariant bargaining solution satisfying the criterion of distributive justice known as Suppes-Sen dominance (Suppes (1960), Sen (1970), chs. 9 and 10)."1

Given utility vectors $x$ and $y$, $x$ is said to Suppes-Sen dominate (SS-dominate) $y$ if and only if there exists a permutation of $x$ that Pareto dominates $y$. I refer to the application of this dominance criterion as the Suppes-Sen principle. There are two main ideas behind the Suppes-Sen principle. First, 'fair' decisions should be, in some sense, impartial: if $x$ is considered 'more just' than $y$, this judgement should not depend on the position of any particular pre-specified individual. Secondly, interpersonal comparisons of utility levels are meaningful (recall how, in social choice theory, Arrow's (1951) impossibility theorem and in particular Sen's (1970, p. 123-30) analogous result when cardinal intensities of preferences are available imply that utilities must satisfy some form of interpersonal comparability if reasonable and defined social choices are to be made). As far as I am aware, the relevance of the Suppes-Sen principle to Nash bargaining theory viewed as a theory of fair arbitration has not been studied. I propose to do so, as follows.

In Nash's bargaining theory a disagreement point is given, which is relevant for the outcome of the problem. On my interpretation, a bargaining problem is a special kind of collective decision problem. A fair arbitrator who accepts ordinal interpersonal utility comparisons will want to apply the Suppes-Sen principle to the players' utilities net of the disagreement utility. Many scale-dependent bargaining solutions yield SS-undominated outcomes for each problem - consider, for example, the Utilitarian (Myerson, 1981) and the Equitarian (Kalai, 1977, Roth, 1979) solutions. At first blush, it might appear that also all scale covariant solutions which satisfy the axioms of Pareto optimality and Anonymity (or Symmetry) will be bound to be compatible with the Suppes-Sen principle, the most obvious candidate being perhaps the Kalai and Smorodinsky solution (KSS), identified with relative egalitarianism. Surprisingly, this is not the case. There exists only one scale-covariant solution which yields SS-undominated outcomes in each bargaining problem, and this is the NBS.

The validity of the last assertion as a more mathematical fact is not in doubt, but one important point of interpretation should be discussed. Scale covariant

---

1 The most notable modification of Nash's axioms concerning IIA is by Leinberg (1988). He replaces IIA with Stability, which applies when the number of players is a variable.

2 Keln (1971) uses the term 'fundamental dominance', Blackaby and Donohue (1977) simply refer to 'dominance'. The term 'Suppes-Sen dominance' seems to be due to Suppes (1983).
bargaining solutions can determine solution outcomes without the use of interpersonal utility comparisons of any sort. What is the meaning, then, of adopting a criterion that hinges on such interpersonal comparisons, albeit of an ordinal nature? My methodological premise is that it is logically meaningful to compare utilities across individuals, but that it is practically difficult, if not impossible, to obtain empirically the necessary scheme of interpersonal scaling. 1 I quote from Elster and Roemer (1991, p. 10-11):

"Let us assume that there is a fact of the matter in an interpersonal comparison of well-being... It does not follow that we could ever discover it. Statements about the past pose similar problems. We tend to assume that there is a fact of the matter by virtue of which statements about the past are true if true, false if false. We may never be able to establish what the fact of the matter is - for example, whether it was raining when Caesar crossed the Rubicon. But that does not affect the existence of a fact of the matter. In one sense, other minds are just as inaccessible to us as the past. We need not entertain doubts about their existence and their essential similarity to our own, but we may despair at ever getting the details right."

Along these lines, consider an arbitrator who does not receive sufficiently detailed information to make interpersonal utility comparisons in practice, but who nonetheless believes that the Suppes-Sen principle is meaningful and is a necessary condition of impartial decision making in the (possibly only theoretical) circumstances in which it can be used. The property of the NBS proved in this paper means that by implementing the NBS, the arbitrator can ensure that, even if given the necessary additional information, he would not be found in violation of the Suppes-Sen principle.

Besides yielding a novel characterisation of the NBS which dispenses with one of the controversial axioms in Nash’s system, the present approach has the added benefit of being robust to the choice of domain of the solution function Nash (1950)

confined himself to the class of convex problems; his axioms are not consistent on a wider domain. My characterisation, on the contrary, is robust to different and, one could argue, more realistic specifications of the domain. Thus, I also argue, makes bargaining problems more directly comparable to standard social decision problems.

The rest of the paper is organised as follows. The next section contains the main definitions and result. In section 3 I discuss extensively the the Suppes-Sen principle and the informational context I consider appropriate for the proposed interpretation of the NBS. Section 4 studies the extension of the characterisation to non-standard domains. A strengthening of the Suppes-Sen dominance criterion is briefly discussed in section 5, which concludes.

2. Main Characterisation

In Nash’s (1950) theory, an n-person bargaining problem is a pair (S,d), where \( S \subseteq N^n \) and \( d \in S \). The interpretation is that S is the set of feasible utilities attainable by the players and d is the disagreement point which results if no agreement is attained. In order to enhance expository clarity, in the main text I make two simplifications:

\[
\begin{align*}
(1) & \quad n = 2, \\
(2) & \quad d = 0 = (0,0)
\end{align*}
\]

None of the results depends on (1). In the appendix I show in full how some definitions and the main result generalise. Nothing conceptual is lost by assuming (1) but much is gained in readability. As for (2), this is a much used convention which saves on notation and is also immaterial, provided that the assumption of Scale Covariance made below is transformed to include the weak requirement of translation consistency. 2

---

1 This does not mean, of course, that my characterisation is logically stronger just because it applies to a larger domain (although it is logically stronger on the same domain as Nash’s).

2 This requirement is satisfied by all the main solutions.
A bargaining problem can thus be simply described as a set \( S \subseteq \mathbb{R}^I \). Let \( \Pi \) be a collection of bargaining problems. A solution on \( \Pi \) is a function \( \varphi: \Pi \rightarrow \mathbb{R}^I \) such that \( \varphi(S) \in S \) for all \( S \in \Pi \). The following restrictions on \( S \) are standard:

A1) \( S \) is compact;
A2) \( S \) is convex;
A3) there exists \( s \in S \) such that \( s > 0 \).

Restriction (A2) in particular is not trivial; we shall see later how non-convex sets of feasible alternatives can be included in the domain. Let \( \Gamma \) denote the collection of all bargaining problems satisfying (A1) through (A3). Some well-known properties that can be imposed on a solution are the following:

**Weak Pareto Optimality (WPO):** \( s > \varphi(S) \Rightarrow s \in S \).

**Strong Individual Rationality (SIR):** \( \varphi(S) > 0 \).

**Covariance with Positive Scale Transformations (Cov):** let \( t: \mathbb{R}^I \rightarrow \mathbb{R}^I \) be a positive, linear, component by component transformation given by \( t(x) = (\lambda x_1, \lambda x_2) \), with \( \lambda, \lambda > 0 \), for all \( x \in \mathbb{R}^I \), and for any \( X \in \mathbb{R}^I \) let \( t(X) = \{ y \in \mathbb{R}^I \mid y = t(x) \text{ for some } x \in X \} \). Then, \( \varphi(t(S)) = t(\varphi(S)) \).

**Symmetry (Sym):** suppose that \( s \in S \Rightarrow (s,s) \in S \). Then, \( \varphi(S) = (s,s) \).

**Anonymity (AN):** let \( \pi: \mathbb{R}^I \rightarrow \mathbb{R}^I \) be a map such that \( \pi(x) = (x_i, x_j) \) for all \( x \in \mathbb{R}^I \). Then, \( \varphi(\pi(S)) = \pi(\varphi(S)) \).

**Independence of Irrelevant Alternatives (IIA):** \( S \subseteq T \subseteq S \Rightarrow \varphi(T) = \varphi(S) \).

In this paper I introduce the following axiom, discussed extensively in the next section:

**Supples-Sen Proportionality (SSP):** \((s,s) > \varphi(S) \Rightarrow s \in S\).

The Nash Bargaining Solution (NBS) \( \varphi: \Gamma \rightarrow \mathbb{R}^I \) is defined by \( \varphi(S) = \arg \max_{0 < x \in \mathbb{R}^I} x \).

Nash (1950) proved that the NBS is the only solution on \( \Gamma \) that satisfies WPO, IIA, COV and SYM (or AN). I will refer to these four axioms as Nash’s axioms. The main result is the following:

**Theorem 1:** a solution \( \varphi: \Gamma \rightarrow \mathbb{R}^I \) satisfies COV and SSP if and only if \( \varphi \sim \nu \).

**Proof:** given \( S \in \Gamma \), suppose that there existed \( s \in S \) with \( t(x) > \varphi(S) \). Then also \( t(x) > \varphi(S) \in \varphi(S) \), a contradiction. Thus, together with the well-known facts that the NBS satisfies COV and WPO, proves the "if" part of the statement.

For the "only if" part, let \( S \in \Gamma \) and suppose by contradiction that \( s = \varphi(S) \cdot \nu \). I will show that there then exists \( T \in \Gamma \) such that \( \varphi(T) \) is SS-dominated. If there exists \( t \in T \) with \( t > s \) we are done, so assume that \( s \) is weakly Pareto optimal. Distinguish three cases.

**Case 1:** \( s > 0 \). Given any point \( x \in \mathbb{R}^I \), let \( H(x) \) denote the branch of the symmetric hyperbola going through \( x \), that is, \( H(x) = (y \in \mathbb{R}^I \mid y_1 = -x_2) \). Clearly, there exists \( t \in H(x) \) such that \( \varphi(S) > t \). Consider now the positive linear transformation \( \tau \) defined by \( \tau(x) = \tau(1) \) and \( \tau(1) = t(1) \). Such a transformation is defined (not uniquely) by \( \tau(x) = (\lambda_1 x_1, \lambda_2 x_2) \) for all \( x \in \mathbb{R}^I \) where \( \lambda_1, \lambda_2 > 0 \), \( \lambda_1 \lambda_2 < 1 \), \( \lambda_1 = s_1 \), and \( \lambda_2 = s_2 \). Since \( s \in H(x) \), these equations have a solution.

Let \( \tau(S) = T \). We have \( \tau(\varphi(S)) = \tau(t) = (\tau(1), \tau(1)) \). Therefore \( \tau(S) \) it is SS-dominated by \( \tau(\varphi(S)) \) in \( T \), and by SSP it must be \( \tau(S) = \varphi(T) \). However, by CDV it must be \( \varphi(T) = \varphi(\varphi(S)) = \tau(\varphi(S))) = \tau(\varphi(S)) = \tau(T) \), a contradiction.

**Case 2:** \( s = 0 \) (the case \( s = 0 \) is treated analogously, note that it cannot be \( s = 0 \) if \( s \) is weakly Pareto optimal). Since \( \varphi(S) > 0 \), there exists \( s > 0 \) such that \( t(x) > \varphi(S) \). Let \( \lambda_2 > 0 \) be such that \( \lambda_2 = s \). Define the transformation \( \tau \) by \( \tau(x) = (x, \lambda x) \). Now the argument of the previous case applies to \( \tau(S) \).
Case 3: \( s_1 > 0, s_2 < 0 \). Let \( t \in \mathbb{R}^2 \), with \( t_1 < 0 \) and \( t_2 > 0 \), be such that \( s_1 t_1 = s_2 t_2 \). In addition, it is clearly possible to choose such a large negative value for \( t_1 \) that \( t < v(S) \). Again define \( \lambda_1, \lambda_2 > 0 \) and \( t \) as in case 1 and argue analogously. 0

3. The Interpretation of Suppes-Sen Practicess

In this section I discuss the interpretation of SSP in a context in which interpersonal comparisons of utility cannot be used.

In general, given utility vectors \( s_i \in \mathbb{R}^2 \), \( s \) is said to SS-dominate \( t \) if \( s > t \) or \( (s_1, s_2) > (t_1, t_2) \). Consider first a situation in which utility levels are interpersonally comparable. Then requiring that a choice should not be SS-dominated merely combines an 'anonymity' principle with the Pareto optimality principle. In other words, it is a Pareto principle for utility vectors where all information regarding personal identity has been erased. As such, this requirement expresses impartiality in the use of Pareto optimality, and seems hardly objectionable. Mathematically, it amounts to preventing the choice of utility vector which is first-order stochastically dominated, that is:

Observation 3.1: Given \( s \in \mathbb{R}^2 \), \( s \) SS-dominate \( t \) if and only if \( s^* > t^* \), where, for any \( u \in \mathbb{R}^2 \), \( u^* \) denotes the vector \( u \) with its components listed in ascending order.

This result is well-known to hold for general vectors in \( \mathbb{R}^n \) (see e.g. Maddon 1970 and the references therein) and is immediately checked for the two-person case.

First-order stochastic dominance is a universally accepted criterion to rank distributions of magnitudes regarded as interpersonally comparable (such as incomes).

Observation 3.1 then shows how weak a requirement SS-dominance is in a context of interpersonally comparable utility. This is confirmed by the fact that, when applied to social welfare functionals, SS-dominance does not discriminate almost at all. All main social welfare functionals, from maximin to utilitarian, will pick SS-undominated alternatives.

Similar remarks apply to the SSP axiom applied to bargaining solutions which do not satisfy COV. For these solutions, SSP is an appealing requirement, but not particularly useful in characterizations. Consider now a collective decision-making context in which:

- There is cardinal utility information, that is, each person’s utility is represented by a cardinal equivalence class of utility functions.
- Interpersonal comparisons of utility cannot be made, that is, the collective decision should be invariant with respect to independent cardinal rescalings of people’s utilities;

- Interpersonal comparisons of utility are meaningful, that is, there exists an (unknown) rescaling of each person’s utility which makes utilities interpersonally comparable.

In this informational context, a utility vector which is SS-undominated when expressed using some particular representative utility functions from the equivalence classes representing preferences may become SS-dominated if the rescaling which makes utilities interpersonally comparable were to be applied. Only those utility vectors that are SS-undominated for all possible scalings of utility will guarantee that the Suppes-Sen principle is not violated. This is exactly the type of requirement embodied by SSP in the presence of COV: requiring a chosen utility vector not to be SS-dominated while simultaneously identifying all bargaining solutions which are cardinal transformations of each other (as COV does) amounts to requiring that a utility vector should not be dominated for any scaling of utility. In other words, imposing SSP in the presence of COV amounts to assuming that the arbitrator will want to ensure that he would not violate the Suppes-Sen principle whatever utility units turned out to be those that make utility comparable across persons.

This is a stronger requirement than the Suppes-Sen principle itself. I do not claim it is a compelling requirement for fair arbitration. However, it is certainly a reasonable principle which allows, in view of Theorem 2.1, a transparent ethical interpretation for the NBS: it is the only solution that (i) does make use of interpersonal utility comparisons; (ii) and guarantees that the solution outcome will not be SS-dominated in the unknown 'correct' scaling of utilities. In addition, SSP is also logically implied by a subset of Nash’s axioms on comprehensive problems. Let \( A \) be the set of all problems satisfying (A1) through (A3) and
4. Other Domains and Multisolusions

The assumption that $S$ is convex is usually justified by the fact that alternatives are expressed in von Neumann-Morgenstern utilities and that lotteries are available. These two requirements, although fairly standard, are not always palpable. If players are not expected utility maximisers, or if in some underlying game in strategic form no correlating device is available, or simply if players are not willing or able to randomise at all, the feasible set will not be convex or even a continuum\(^3\). In addition, as Moulin (1996, p. 126) observes, "It is hard to believe that our search for operational criteria of fairness should be confined to a convex world\(^4\)."

Fortunately, the characterisation in terms of SSP given above is quite robust to variations of the basic setting. Since the set of maximisers of the Nash product is not necessarily a singleton when the domain is not convex, in this section I turn to multisolusions. Let $\Pi$ be a collection of bargaining problems. Then a multisolusion on $\Pi$ is a correspondence $\varphi : \Pi \rightarrow \mathbb{R}^I$ such that $\varphi(S) \subseteq S$ for all $S \in \Pi$. The NBS viewed as a multisolusion is defined analogously to the solution, the KSS and the ES are always single-valued when they are well-defined. Some axioms are redefined accordingly; a star indicates that they refer to multisolusions:

- **Weak Pareto Optimality (WPO\(^*\)):** $s > t \in \varphi(S) \Rightarrow t \in S$.
- **Strong Individual Rationality (SIR\(^*\)):** $s \in \varphi(S) \Rightarrow t > 0$.
- **Symmetry (SYM\(^*\)):** Suppose that $s \in S \Rightarrow (s, s) \in S$. Then, $s \in \varphi(S) \Rightarrow (s, s) \in \varphi(S)$.
- **Independence of Irrelevant Alternatives (IIA\(^*\)):** $S \subseteq T$ and $\varphi(T) \cap S = \emptyset$\(^*\) $\Rightarrow \varphi(S) = \varphi(T) \cap S$.
- **Suppes-Sen Proofness (SSP\(^*\)):** $(s, s) > t \in \varphi(S) \Rightarrow s > t \in \varphi(S) \Rightarrow t \not\in S$.

---

\(^3\) After two crucial contributions by Kangoo (1980) and Herrero (1989) the convexity assumption has received much attention recently; see, e.g., Cosley and Wilkey (1996), Markov (1997a,b) and Zhou (1997).
COV and AN remain unchanged.

I consider two interesting domains. The first is $\Sigma$, the class of problems $S$ that satisfy (A1) and (A3) in section 2 and such that, in addition, $S$ is comprehensive (A4 in the previous section). The second domain is $\Theta$, the class of problems $S$ that satisfy (A3) of section 2 and such that, in addition, $S$ contains a finite number of alternatives. For comparison, I summarise next some results of Mariotti (1997a).

**Theorem 4.1:** There exists no solution $\Psi : \Sigma \rightarrow R^2$ that satisfies Nash's axioms. There exists however a multisoluation $\Psi : \Sigma \otimes \Theta \rightarrow R^2$ that satisfies WPO*, COV, SYM* and IIA*. This multisoluation is unique and it is the NBS. Finally, the NBS is also the only multisoluation $\Psi : \Theta \rightarrow R^2$ that satisfies these axioms.

Single-valuedness is thus incompatible with Nash's axioms. I also note that there exists no characterisation in terms of WPO*, COV, SYM* and IIA* for the NBS multisoluation $\Psi : \Sigma \rightarrow R^1$ (Kaneko 1980) has a characterisation on this domain which involves also an upper-semicontinuity axiom). $\Sigma$ is the natural domain to consider when randomisations are available but the players are not necessarily expected utility maximisers (Rubinstein et al., 1992), or they cannot correlate their strategies in the underlying strategic form description. $\Theta$ is the natural domain to consider when randomisations are not available at all. In the present approach, we have:

**Theorem 4.2:** A multisoluation $\Psi : \Pi \rightarrow R^1$, with $\Pi \in (\Sigma, \Theta, \Sigma \otimes \Theta)$, satisfies COV and SSP if and only if $\Psi \in \Psi$. In particular, there exist solutions $\Psi : \Pi \rightarrow R^1$, with $\Pi \in (\Sigma, \Theta, \Sigma \otimes \Theta)$, which satisfy COV and SSP, and such solutions are all selections from the NBS.

Here, the notation $\Psi \in \Psi$ means $\Psi(S) \subseteq \Psi(T)$ for all $S \in \Pi$. The proof of this theorem uses the same argument used for theorem 2.1, so I will not repeat it here.

These results suggest that, unlike the standard characterisation, the characterisation of the NBS proposed here is relatively independent of the precise structural properties of the feasible set in the domain. One of the main advantages of this feature is that it makes bargaining problems more directly comparable to standard social decision problems, in the following sense: In the latter type problem one typically has a given set $X$ of 'physical' alternatives, and individual preferences on $X$ are then allowed to vary. In traditional bargaining theory, to the contrary, also the set $X$ must be allowed to vary; otherwise, one might not be able to obtain in the feasible domain the problems needed for the proof (e.g., in Nash's 1950 case, one needs a symmetric rectangle to apply IIA). My characterisation overcomes this difficulty, because the only axiom which involves comparisons of different bargaining problems is COV: it is perfectly possible, then, to think of the set of physical alternatives as fixed.

5. **Concluding remarks**

In this paper I have supported and characterised the interpretation of the NBS as an expression of distributive justice. It is the only solution that satisfies a 'conservative' criterion of impartiality in arbitration. In particular, the NBS is the only solution that reconciles two powerful yet conflicting needs on the one hand, interpersonal comparisons of utility should be not used in its calculation; on the other hand, basic principles of fairness relying on such comparisons should not be violated by its outcomes. I note here that the NBS satisfies a much stronger 'fairness' criterion than SSP, related to second-order stochastic dominance (or Generalised Lorenz dominance)\(^g\) Given $x, y \in R^1$, $x$ is said to GL-dominate $y$ if there exists $\alpha \in [0,1]$ such that $(a_1 + (1-\alpha)a_2, (1-\alpha)a_1 + \alpha a_2) > t$. It is immediately verified that $\alpha = 0$ or $\alpha = 1$ in the inequality implies SS-dominance.

---

\(^a\) The only additional structural requirement needed in the bargaining framework is, of course, the presence of a dominated 'reference' alternative.

\(^b\) The mentioned characterisation of the NBS in terms of Covariance by Landberg (1988) also lends support, from a different perspective, to this interpretation. See Young (1994, ch. 7) for a discussion of this interpretation, Kristoff and Pares (1990) for the relation with microeconomic bargaining, and Thoemmes (1990) for a general discussion of the Covariance principle. I should also mention the original approach by Gauthier; in Gauthier (1986) it is claimed that the NSS is somewhere in an expression of rational bargaining and of fairness, but it appears from Gauthier and Sagi (1993) that he now views the NSS as a better candidate to perform that double role.

\(^h\) I think there has been too much attention to this point. The terminology follows Madden (1996) and Shoven (1983).
Generalised Lorenz Proofness (GLP): \( (a_3 + (1-a_2)a_1 + a_1) > q(S), \alpha \in [0,1] \Rightarrow s \in S. \)

It is easy to verify that the NBS cannot yield a GL-dominated outcome. By definition, the feasible set \( S \) is bounded above at \( v(S) \) by the symmetric hyperbola through \( v(S) \). The set of points \( t = (\alpha v(S) + (1-\alpha)v(S), (1-\alpha)v(S) + \alpha v(S)) \) with \( \alpha \in [0,1] \) is the segment joining \( v(S) \) and \( v(S) \). This segment - connecting a point of \( H(v(S)) \) with another point which, being symmetric to the first, is also on \( H(v(S)) \) - lies entirely above \( H(v(S)) \). Therefore for any point \( t \) that \( GL \)-dominates \( v(S) \) it must be \( t \in S \). Thus:

Corollary 5.1: let \( \phi: \Gamma \rightarrow 1^\Phi \) be a solution satisfying COV and SSP. Then it also satisfies GLP.

Clearly, the concept of GL-dominance requires \emph{cardinal} interpersonal comparability of utility, not merely ordinal comparability as required by the Suppes-Sen criterion. However, the interpretation of GPL in the presence of COV is based on observations analogous to those of section 3.

It should be emphasised in conclusion that the interpretation I propose will only be valid in circumstances when the axiomatic solution of a bargaining problem à la Nash can be held to be appropriate for issues of fairness. In particular:

i) there is one distinguished point \( d \) which is allowed to be relevant for solving the distributional problem;

ii) cardinal utility information is available;

iii) the decision can be made on the basis of utility information alone.

Sen (1976), for example, has discussed situations in which the disagreement point should have no bearing on justice issues (note, however, that the interpretation of \( d \) as disagreement point is not necessary; all that matters is that there exists a Pareto dominated point for which Arrow's Independence can be violated). Unfortunately, as shown in section 3, the 'conservative' criterion proposed here will lead to an impossibility result when applied to a general social choice context without a reference point. If (ii) and (iii) are not good assumptions (e.g. Roemer 1986, 1990, 1996), other methods and procedures for deciding fairly will be more useful (see e.g. Young, 1994) and Brams and Taylor, 1996).

Appendix

All the definitions and arguments for the results of the text generalise easily to the \( n \)-person case. In this appendix, by way of illustration, the definitions of SSP and GLP and the proof of Theorem 2.1 are given. The generalisations of domains, standard axioms and solutions are obvious. Given \( s \in 1^\Phi \), \( s \) is said to SS-dominate \( t \) if \( As > t \) for some permutation matrix \( A \); \( s \) is said to GL-dominate \( t \) if \( As > t \) for some bistochastic matrix \( A \).

Suppes-Sen Proofness (SSP): \( As > q(S) \) for some permutation matrix \( A \Rightarrow s \in S. \)

Generalised Lorenz Proofness (GLP): \( As > q(S) \) for some bistochastic matrix \( A \Rightarrow s \in S. \)

Theorem A.2.1: A solution \( \phi: \Gamma \rightarrow 1^\Phi \) satisfies COV and SSP if and only if \( \phi = \nu \).

Proof: 'If': I note a stronger property of \( v \), namely that it satisfies GPL. That this is so follows from the fact that the Nash product is a symmetric increasing concave function and from standard characterisation results available in the literature.\[1\]

'Only if': let \( S \in \Gamma \) and suppose by contradiction that \( s = \phi(s) \neq v(S) \). I will show that then there exists \( 
\text{The relevant result for the derivation is: if } A \text{ GL-dominates } t \text{ if and only if } As \geq t \text{ for all increasing symmetric quasi-concave multi-valued functions } A, \text{ with strict inequality for some such } A. \text{ See for instance Madden (1996, Theorem 2). For general surveys see e.g. Maskin (1986), where the relationship with Shub-Nguyen is noted.}

\[1\]
Case 1: s > 0. Given any point $x \in \mathbb{R}^n$, let $H(x)$ denote the symmetric hyperboloid going through $x$, that is, $H(x) = \{ y \in \mathbb{R}^n : \Pi_1 y = \Pi_2 x \}$. Clearly, there exists $t \in H(x)$ such that $v(x) > t$. In addition it is possible to choose $t$ so that $t = x$ for all $i \in \{1, \ldots, n\}$. Consider now a positive linear transformation $\tau$ defined by $\tau(t) = A\tau(t)$, where $A$ is the non-permutation matrix which moves the $i$th component to the $(i+1)$th place (setting $n+1 = 1$). That is:

$$
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
$$

I show that such a transformation $\tau$ exists (not uniquely). Denote $S$ and $T$ the non-zero diagonal matrices with the components of $s$ and $t$, respectively, on their diagonal, and denote $\lambda$ the non-zero vector of coefficients representing $\tau$ (that is, $\tau(x) = \lambda s x$ for all $x \in \mathbb{R}^n, i \in \{1, \ldots, n\}$). It must be proved that the homogeneous system:

(A1) $S - AT = 0$

has (a class of) strictly positive solutions in $\lambda$. (A1) has nontrivial solutions if and only if:

(A2) $S - AT = 0$.

We have:

$$
K = S - AT = \begin{bmatrix}
s_1 & 0 & \cdots & 0 & -t_1 \\
-t_1 & s_2 & 0 & \cdots & 0 \\
0 & -t_2 & s_3 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -t_{n-1} & s_n \\
0 & 0 & \cdots & 0 & -t_n
\end{bmatrix}
$$

Expanding along the first row:

(A3) $|K| = s_1 |M_{11}| + (-1)^{n+1}(c_{11}|M_{11}|)$. 

where $M_{ij}$ denotes the minor of $K$ obtained by removing the $i$th row and $j$th column. By the properties of triangular matrices (e.g., Birkhoff and Lane (1951), p. 303), $|M_{11}| = \Pi_{1\alpha} \Pi_{11}$ and $|M_{11}| = \Pi_{1\alpha} l_1$. Note that the second term on the RHS of (A3) is negative for all $n$. Therefore (A2) holds if and only if $\Pi_{1\alpha} l_1 \neq 0$, or, equivalently, if and only if $t \in H(x)$. Since $t$ was chosen exactly in this way, (A1) has nontrivial solutions.

Suppose now that $\lambda^*$ is a nontrivial solution of (A1) and that $\lambda^*_i < 0$ (resp. $= 0$) for some $i \in \{1, \ldots, n\}$. This means (by inspection of $K$ and the fact that $s_i > 0$) that $s^*_i < 0$ (resp. $= 0$). Consequently, $\lambda^*_i < 0$ (resp. $= 0$) for all $i \in \{1, \ldots, n\}$. The case $\lambda^*_i = 0$ is excluded by nontriviality. If $\lambda^*_i < 0$, then $\lambda^* > 0$ is also a solution. We conclude that the desired $t$ exists in this case.

Now let $v(S) = T$. We have $v(S) > t$ (by $\lambda^* > 0$). Therefore $A_1 v(S) = t(S)$. Therefore $A_1 v(S) = t(S)$, and by SSP it must be $v(T) = t(S)$. However, by COV it must be $v(T) = v(S)$, $v(t(S)) = v(S)$, a contradiction.

Case 2: There exists $s_i > 0$ for $i \in \{1, \ldots, n\}$. Without loss of generality, write $s_i$ (possibly relabeling the axes) in such a way that the first $k$ components are positive and the other negative; that is, let $k$ be such that $s_i > 0$ for $1 \leq i < k$ and $s_i < 0$ for $k \leq i < n$. Now let $s_i \in \mathbb{R}^n$ have components with signs as follows: $s_i > 0$ for $1 \leq i < k$ and $s_i = 0$ for $k \leq i < n$. In addition, let $t < V(S)$ (this is possible since $v(S) < 0$). Define the system (A1) as in case 1. The matrix $K$ now has one or more rows whose entries are all zero (certainly the last row, since $t_k = 0$), therefore (A1) has nontrivial solutions. If $\lambda^* > 0$ is a nontrivial solution, by the choice of sign of $t$ we now have that $\lambda_1 s_1 = t_k$, whenever $s > 0$ (and hence $s > 0$). Therefore $\lambda^*$ and have the same sign for $1 \leq i \leq k$. Since the other $\lambda^*_i, k \leq i \leq n$, are all free variables, the choice $\lambda^*_i > 0$ is certainly allowed, and the proof for this case concludes as in case 1.
Case 3: There exists $i \in \{1,2,\ldots,n\}$ with $s_i < 0$ for $i \in I$ and $s_i > 0$ for $i \in \{1,2,\ldots,n\} \setminus I$. Without loss of generality, let $k$ be such that $s_i > 0$ for $1 \leq i \leq k$ and $s_i < 0$ for $k < i \leq n$. Define $g = (s_1, s_2, \ldots, s_k, -s_{k+1}, \ldots, -s_n, 0)$. Let $t \in \mathbb{R}^n$ have the following properties:

(a) $\text{sign } t_s = -\text{sign } s_i$;

(b) $\Pi_s t_s = -\Pi_s s_i$;

(c) $t < g(s)$.

Given (b), (c) is possible by making the negative components of $t$ sufficiently large in absolute value. At this point the argument proceeds in a way analogous to case 1 and will not be repeated. 0

Observation: By reduction (using induction) to echelon form of $K$ it is easy to see that in fact the null-space of $K$ in case 1 has dimension 1 for all $n$. The transformation $\tau$ is therefore subject to exactly the same degree of freedom as in the two-person case.

References

Arrow, K.J. (1951) "Social choice and individual values", Wiley, New York.


Hannanwood, P.J. (1991) "Interpersonal comparisons of utility: Why and how they are and should be made", ch. 7 of Ester and Roemer.


