Escalating Games, Coordination and Dominance Solvability*

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Abstract

Inspired by the model of Kalai and Satterthwaite (1994), I define a class of abstract games which are proved to be dominance-solvable. I show moreover that, in the leading subclass of coordination games, they are solvable on the unique Pareto-dominant outcome.

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1 Introduction

A *coordination game* is a game which possesses several Nash equilibria, one of which Pareto dominates the others. The prototype example is displayed in table 1.

	A	B
A	1,1	0,0
B	0,0	1,1

Table 1

In practice, in these games people are often observed to be able to focus on the Pareto dominant equilibrium, especially when there are no complicating issues of risk-dominance. Unfortunately, as in the example just shown, the inefficient equilibria of coordination games are sometimes robust to the standard refinements of the Nash equilibrium concept. In these cases, coordination would have to be explained by different equilibrium theories[1]. Consider, however, the example of table 2.

	A	B	C
A	9,11	5,6	1,0
B	5,6	5,6	5,6
C	1,0	5,6	1,0

Table 2

This game also has multiple equilibria. Of the four equilibria in pure strategies, (A, A) Pareto dominates the other three. The two dominated equilibria (B, C) and (C, B), however, involve the use of weakly dominated strategies. The remaining Pareto dominated equilibrium (B, B) does *not* involve the use of dominated strategies; but it can be discarded after one further round of elimination. In other words, the game is dominance-solvable and the Pareto dominant equilibrium (A, A) is the unique dominance-solvable outcome. This example, therefore, admits a particularly neat solution of the coordination problem

As will be seen later, a situation similar to the one of this game holds in some economic applications. What is the property common to these coordination games that makes them dominance-solvable?

[1] Or by different theories altogether.

In this paper, I introduce and study a class of dominance-solvable n-person games, which include coordination games as a special but leading case, called *escalating games*. They are characterised by the following properties:

(*i*) players choose from the same strategy set S;

(*ii*) it is possible to impose on S a linear order $\succ$ such that the payoff to each player depends only on the $\succ$-maximum component of any strategy profile s.

(*iii*) for any player, there are no ties between the payoffs attained at different values of $\max\{s_1, ..., s_n\}$, $s \in S$.

When an escalating game is a game of coordination, the following additional property holds:

(*iv*) although the payoff functions may depend on $\max\{s_1, ..., s_n\}$, $s \in S$, in different ways, for each player there exists a common level $c \in S$, called the *cooperative* level, at which the payoff attains a (strict) global maximum.

The game of table 2 is, as shown below, an escalating coordination game. An escalating game which does not satisfy property (*iv*) above is displayed in table 3. Note that it is dominance-solvable.

	A	B	C
A	9,0	5,11	1,6
B	5,11	5,11	5,11
C	1,6	5,11	1,6

Table 3

In the game of table 2, consider for example the ordering on $\{A, B, C\}$ given by $A \succ B \succ C$. With this ordering, the payoff that a player gets when, for example, he plays the highest strategy, A, depends on which strategy is chosen by the other player. Therefore this ordering does not satisfy condition (*ii*) above. Similarly, with the ordering $C \succ B \succ A$, the payoff that a player gets when he plays C depends on the strategy chosen by the other player. Things are different, however, when the ordering $B \succ C \succ A$ is considered. With this ordering the payoff accruing to each player from a particular strategy profile s can be described in a simplified manner, as follows:

– if the highest component of s is A (that is, $s = (A, A)$), then the payoffs are $(9, 11)$;

– if the highest component of s is C (that is $s = (C, C)$, or $s = (A, C)$, or $s = (C, A)$), then the payoffs are $(1, 0)$;

– if the highest component of s is B (as is the case in all remaining five strategy profiles), then the payoffs are $(5,6)$.

This reasoning shows that in this case condition (ii) defining an escalating coordination game is satisfied: only the maximum component of a strategy profile determines payoffs. The other two conditions are also obviously satisfied. The cooperative level of this escalating coordination game is A.

The terminology I use is due to the following interpretation, which is appropriate in some applications (in particular, it emerges from the price competition model by Kalai and Satterthwaite ([2]), discussed later, which has inspired the present study). Suppose that the strategies in a game can be ordered in terms of some attribute (for example, 'quality') on which payoffs depend. Consider a game in which chosen levels of the attribute other than the maximum matter. Condition (ii) can then be interpreted as defining an 'escalating' variation of this game, in which higher chosen levels of the attribute are instantaneously 'matched' by all players. In order to qualify as a coordination game, the players must agree on the optimal level of the attribute (condition (iv)).

In the next section, I will show that conditions (i), (ii) and (iii) are sufficient to guarantee that the game is dominant-solvable. If condition (iv) holds as well, then only the cooperative level survives iterated elimination of weakly dominated strategies. In the third section I discuss two economic applications. Section four concludes.

2 Definitions and Results

First, I define escalating games and escalating coordination games more precisely. The set of players is denoted $N = \{1, 2, ..., n\}$, $n \geq 2$. In an *escalating* game, the strategy set common to all players is denoted S, and its n-fold Cartesian product S^n. The payoff functions are denoted u_i. There exists a linear order $\succ$ (a complete transitive asymmetric relation) on S such that:

$$\text{for all } i \in N, u_i(s) = \pi_i(max\{s_1, ..., s_n\}) \text{ for all } s = (s_1, ..., s_n) \in S^n$$

for some functions $\pi_i : S \to \mathcal{R}$.

and

$$\text{for all } i \in N, \pi_i(x) \neq \pi_i(x') \text{ for all } x, x' \in S$$

In an escalating *coordination* game there exists in addition a *cooperative level* $c \in S$ such that:

$$\text{for all } i \in N, \pi_i(c) > \pi_i(x) \text{ for all } x \in S, x \neq c$$

Note that it may be the case that $\pi_i(c) \neq \pi_j(c)$. Note also that the profile $\mathbf{c} = (c, c, ..., c)$ is obviously a strict Nash equilibrium. In addition, we have:

Claim : In an escalating coordination game, the equilibrium $\mathbf{c}$ Pareto-dominates all other Nash equilibria.

Proof: Suppose in contradiction to the statement that $s \in S^n$ is another Nash equilibrium with $u_i(s) > u_i(\mathbf{c})$. In the sequel I will use the notation $\max\{r\}$ for $\max\{r_1, ..., r_k\} \in S^k$, and r_{-i} for the profile r without its i^{th} component. It cannot be $c \succ \max\{s\}$, or every player i could improve by switching from s_i to c. On the other hand, if $\max\{s\} \succ c$, say $\max\{s\} = s_i$ and consider the profile $t = (s_i, \mathbf{c}_{-i})$. By the definition of an escalating coordination game, $u_j(t) = u_j(s) > u_j(\mathbf{c})$ for all j. But then player i could improve at $\mathbf{c}$ by switching from c to s_i, a contradiction with $\mathbf{c}$ being a Nash equilibrium. □

Finally, I assume that S is *finite above* in the sense that, for any $x \in S$, the set $\{s_i S \in | s_i \succ x\}$ is finite.

A game is *dominance-solvable*[2] if all players are indifferent between all outcomes (the dominance-solvable outcomes) that survive the procedure of iterated *simultaneous* elimination of weakly dominated strategies.

Theorem 1 : *An escalating coordination game is dominance-solvable (in pure strategies), and its unique dominance-solvable outcome is the profile of cooperative levels* $\mathbf{c} = (c, c, ..., c)$.

Proof: The argument proceeds in a series of simple steps.

Step 1: for all $i = 1, ..., n$, all strategies s_i with $c \succ s_i$ are eliminated in the first round.

If $c \succ \max\{s_{-i}\}$, then choosing $s_i = c$ yields a strictly higher payoff than choosing s_i with $c \succ s_i$ (remember that c is a strict global maximiser of the

[2] See e.g. Osborne and Rubinstein ([4], p.63). Moulin ([3]) seems to have introduced the term.

π_i). If $\max\{s_{-i}\} \succ c$ or $\max\{s_{-i}\} = c$, then choosing $s_i = c$ yields the same payoff as choosing s_i with $c \succ s_i$.

Step 2: for all $i = 1, ..., n$, $s_i = c$ survives any round of elimination.

By induction on the number of rounds. Since c is a strict better response than any $s_i \neq c$ against all other players choosing c, c survives the first round; and if c has survived the k^{th} round for all players, then it survives the next round as well.

Step 3: in order for $s_i \succ c$ to survive a round of elimination, s_i must yield a strictly better payoff than c against some profile s_{-i}, survived until that round, such that $s_i \succ \max\{s_{-i}\} \succ c$.

That $s_i \succ c$ must yield a strictly better payoff than c against *some* profile in order to survive a round follows from c being a strictly better response than s_i with $c \succ s_i$ against c_{-i}, and from step 2. That $\max\{s_{-i}\} \succ c$ follows from the fact that $s_i \succ c$ yields a strictly worse payoff than c when $c \succ \max\{s_{-i}\}$. That $s_i \succ \max\{s_{-i}\}$ follows from the fact that c would yield the same payoff as s_i (given that $s_i \succ c$) against a profile s_{-i} such that $\max\{s_{-i}\} \succ s_i$ or $\max\{s_{-i}\} = s_i$.

Step4: all strategies $s_i \succ c$ are eventually eliminated.

Suppose not. Let d be the minimum, taken over all S, of all the strategies higher than c which, after all rounds of elimination, yield a strictly better payoff than c, for some players, against some profile which has also survived all rounds. Note that d is well-defined by the fact that S is finite above.

Say that d can be set optimally by player j against some s_{-j} which has survived. By step 3 it must be the case that $d \succ \max\{s_{-j}\} \succ c$. But this means that another strategy lower than d and higher than c has survived all rounds, contradicting the fact that d is a minimum. □

One of the criticisms levelled against procedures of iterated elimination of dominated strategies is that different orders of elimination may yield different results. However, this criticism does not apply here:

Corollary 2 : *In an escalating coordination game any order of iterated elimination of weakly dominated strategies yields the dominance-solvable outcome.*

Proof: The argument of Step 2 of the previous proof shows in fact that c cannot be eliminated in *any* order of elimination. The arguments in the other

steps show that at any round, some surviving strategy $s_i \neq c$ is dominated by c itself. The conclusion in the statement thus follows immediately. □

Remark 1 : *The above theorem and corollary can be strenghtened by dispensing with the assumption that there are no ties in the levels attained by the payoff functions. All that matters is the feature that a strict maximum is attained at c.*

The next result shows that what really drives the dominance-solvability of escalating coordination games is the 'escalation' feature rather than the coordination feature, which is only responsible for the Pareto optimality of the outcome.

Theorem 3 : *Any escalating game is dominance-solvable.*

Proof: In view of theorem 1, we only need to consider the case of an escalating game which is not a coordination game. Denote by $m_i \in S$ the level of $\max\{s\}$, $s \in S$, at which i's payoff attains its maximum. Note first that, for all i, all strategies s_i with $m_i \succ s_i$ are eliminated in the first round. This follows from the same argument used to prove step 1 in the proof of theorem 1.

The rest of the proof is by induction on the cardinality of S. Suppose first that $|S| = 2$. Because the game is not a coordination game, there exist $i, j \in N$ such that $m_i \neq m_j$. So there exists a player i such that $m_i \succ x$ for some $x \in S$. Such a strategy x is eliminated in the first round, so that m_i is the only surviving strategy for player i. Therefore, for any player j, after the first round the only possible payoff is $\pi_j(m_i)$. This shows that the game is dominance-solvable when $|S| = 2$.

Suppose now that a game is dominance-solvable if $|S| \leq k$ and consider a game G such that $|S| = k+1$. Recall again that after the first round of elimination in G, for all i, all strategies s_i with $m_i \succ s_i$ are eliminated in the first round. This means that for any player s_i, after the first round, all strategies lower than $\max\{m_1, ..., m_n\}$ yield exactly the same payoff, against any surviving strategy combination, as $s_i = \max\{m_1, ..., m_n\}$. Therefore, after the first round, any strategy dominated by a strategy s_i with $\max\{m_1, ..., m_n\} \succ s_i$ is also dominated by $s_i = \max\{m_1, ..., m_n\}$ itself. Moreover, any strategy $s_i \succ \max\{m_1, ..., m_n\}$ or $s_i = \max\{m_1, ..., m_n\}$ yields against a strategy profile t with $\max\{m_1, ..., m_n\} \succ \max\{t\}$ the same payoff it would yield against all players playing $\max\{m_1, ..., m_n\}$ itself. As a consequence, after the first round, no strategy is eliminated or kept just because of the presence of strategies lower than $\max\{m_1, ..., m_n\}$. Now let G' be the game obtained by deleting, for all players, all strategies lower than $\max\{m_1, ..., m_n\}$ from G after the first round of elimination has been completed. Because of the previous reasoning, the procedure of simultaneous iterated elimination of dominated strategies leads to the same set of outcomes whether applied to G' or to G after the first round of elimination has been completed. But G' is clearly an escalating game, and by the induction hypothesis it is dominance-solvable. This shows that G itself is dominance-solvable. □

Remark 2 : *The assumption that there are no ties in the levels attained by the payoff functions is necessary for the above theorem to hold, as the example of table 4 shows.*

	A	B	C
A	9,5	9,5	9,5
B	9,5	10,5	10,5
C	9,5	10,5	10,2

Table 4

This example, in which strategies B and C survive for both players, also illustrates a main difference between the present treatment and that of Kalai and Satterthwaite ([2]), further discussed later. Rather than the concept of dominance-solvability, they use the idea of a *maximal sequentially dominant reduction*. This procedure does not require simultaneous elimination of all strategies, and, crucially, it allows at any round the elimination of a strategy which is not dominated, but which does not dominate a strategy which survives the round. This would allow one to eliminate, say, strategy B in the example above. In general, it is this that ensures that any maximal sequentially dominant reduction is a singleton even when there are payoff ties (although there may be more than one such reduction).

3 Two Applications

3.1 Price Competition

Kalai and Satterthwaite ([2]) have introduced a modified Bertrand game in which first firms announce prices (in $\mathcal{R}_+$) and then each firm sells at the lowest announced price. Their formulation captures the idea that attempts to undercut the other firms are immediately matched (as is the case when, for example, there is a 'price-matching clause'), and is proposed as a formalisation of the classic kinked demand model of oligopoly. So, the payoff $u_i(p)$ for firm i, given a price n-tuple $p = (p_1, ..., p_n)$ is defined by

$$u_i(p) = f_i(min\{p_1, ..., p_n\})$$

where f_i is some given function $f_i : \mathcal{R}_+ \to \mathcal{R}$.

[2] show that if f_i satisfies a condition called multipeakedness[3], and if there exists a price q which is a strict global maximiser of each f_i, then any maximal sequentially dominant reduction of the game (there may be more than one) induces a price q.

The result of this paper can be applied to discrete versions of the model of [2], with S any discrete subset of $\mathcal{R}_+$ such that there exists η such that $|x - x'| \geq \eta$ for all $x, x' \in S$ (this ensures that S is finite above with the ordering specified in the sequel). The order $\succ$ on S that makes this an escalating coordination game (if q as defined above exists) is as follows: $x \succ x'$ if and only if $x < x'$.

In this application, the approach of this paper yields a simplified proof of a result analogous to that in [2]. In addition, unlike their version, it predicts uniquely the surviving strategy profile. Finally, it allows one to drop the assumption of multipeakedness. Such an assumption is clearly limiting in an abstract setting (see the example in the conlcuding section), but there are also examples where it is violated even in some simple oligopoly games, as shown in the sequel.

Two identical firms produce a homogeneous product at constant marginal costs normalised to zero. The demand function is given by $h = h(x)$, where h is total output. In this case it is natural to set $f_i(x) = h(x)x/2$. A

[3]The assumption of multipeakedness is as follows. For each firm i, there exists a *finite* sequence $\{r_1, ..., r_k\}$, with $r_j \in \mathcal{R}_+$ for all $j = 1, ..., k$ and $r_j < r_{j+1}$ for $j = 1, ..., k-1$, such that f_i is (weakly) monotonic on each interval $[r_j, r_{j+1}]$ for $j = 1, ..., k-1$, and f_i is nonincreasing on $[r_k, \infty)$.

cooperative price for this model would be the monopoly price m such that $h(m)m > h(x)x$ for all $x \in \mathcal{R}_+, x \neq m$. There are no necessary economic restrictions on the demand function $h(x)$ which imply the multipeakedness of fi. In particular, there is no necessary economic restriction which guarantees the existence of some $y \in \mathcal{R}_+$ such that $h'(x)x + h(x) \leq 0$ for all $x > y$. A good way to visualise a case where this condition is violated is to consider a demand curve which is tangent to successively lower level curves of the rectangular hyperbolae $hx = k$. Each point of tangency corresponds to a local maximum for f_i,which thus exhibits an infinite number of peaks and is thus not multipeaked in the sense of [2].

3.2 Voting over the Production Of a Public Good

A public good can be produced in any of k distinct levels, $1, 2, ..., K$. The utility net of the (predetermined) share of the cost that agent i derives from the production of level k of the public good is $v_i(k)$. Agents simultaneously and independently declare the level of production they would like to see implemented. Suppose that nobody can be forced to consume a level of the good greater than the one he has declared[4]. The amount produced is thus equal to the minimum declared level.

Even when this game is a coordination one, in the sense that there exists k^* such that, for all i, $v_i(k^*) > v_i(k)$ for all $k = 1, ..., K$ with $k \neq k*$, there may be many undominated inefficient equilibria, depending on the v_i. A person may not declare his preferred level of production to 'protect himself' against other people doing the same, a silly outcome but a possible equilibrium nonetheless! However, an immediate application of theorem 1 guarantees that in this case the game is uniquely dominance solvable on $(k^*, k^*, ..., k^*)$ for any number of players, K and set of v_i with the above property.

4 Concluding Remarks

This paper has described a general property of games that makes them dominance-solvable and, in the leading case of coordination games, solvable on the unique Pareto optimal outcome. Applied to the particular model ([2]) that has inspired the present paper, the result I have presented yields a different outlook on the orginal result, by applying a different solution concpet and dispensing with one of the assumptions. But what I hope especially validates the approach proposed here is its applicability to quite different economic situations, as well as to a host of abstract examples of games in strategic form. I conclude with an example of an infinite two-person escalating coordination game which does not satisfy multipeakedness or finiteness above[5], and with an infinite number of Nash equilibria to boot, yet solvable in only two rounds. Players choose from $\{1, 2, ...\}$:

	1	2	3	4	5	...
1	6,6	0,0	5,5	0,0	5,5	
2	0,0	0,0	5,5	0,0	5,5	
3	5,5	5,5	5,5	0,0	5,5	
4	0,0	0,0	0,0	0,0	5,5	
5	5,5	5,5	5,5	5,5	5,5	
⋮						⋱

Table 5

[4]This is clearly a 'constitutional' rule, not a property of public goods! It is a generalisation of the 'unanimity rule' whereby everybody can pose a veto on the decision to provide a public good which can only be provided in quantities 0 or 1.

[5]Which shows how my theorem could be slightly strengthened with a somewhat more pedantic layout of the assumptions.

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