Problem Set 4: Constrained Optimisation

1. Draw a function that is quasi-concave but not concave and explain why.

2. Sketch the function $2x^2$ and work out if the function is quasi-concave or quasi-convex

Lagrangeans

1. Optimise the following function subject to the given constraint

$$Z = 4x^2 - 2xy + 6y^2 \quad \text{st} \quad x + y = 72$$

2. For each case, identify the objectives and the constraints.
Write down the constraint function and indicate whether it is a constraint that must hold with equality or not.
Finally, write down the Lagrangean.

Example: A new business employs L workers and wishes to maximize profits $2L-L^2$. There are only 12 workers available for hire

Answer: The objective function is $2L-L^2$. There are two constraints: $L \geq 0$ and $12 - L \geq 0$. Neither needs to hold with equality. The Lagrangean is $2L-L^2 + \lambda(12-L) + \mu L$ whereμ and λ are the Lagrangean multipliers

(i) A government wishes to choose its spending on health (H) and education (E) expenditure levels to maximize its popularity $P = E + 2H + EH^2$. Its annual budget is £40bn and it is not allowed to borrow money

(ii) A consumer's utility function is: $u = x^{0.25}y^{0.75}$ where x and y are two goods. Total income is £10,000 and the prices of the two goods are £4 and £6 respectively

3. A monopolist faces a demand curve of $q = 20 - p$, where q is output and p is its price. Total costs are given by 4q

(i). Write down an expression for the monopolist's profits as a function of its output
(ii). Find the profit-maximizing level of output and the associated level of profits

(iii). A fire in one of its factories means that the maximum output for the firm is constrained to 6. Use constrained optimization to find the profit maximizing output for the firm under these conditions

(iv). What are the firm's profits?

(v). What is the value of the lagrangean multiplier?

(vi). The firm can build extra production capacity at a cost of 5. Should it ?

Linear & Non-Linear Programming

1. Draw a diagram to show the constraints and the constraint set.

(i) $x \geq 0$, $y \geq 0$, $x + 2y \leq 10$
(ii) $x \geq 0$, $y \geq 0$, $2x + y \leq 16$, $y + x \geq 4$
(iii) consumption of x is not negative, consumption of y is not negative, the price of x is 4, the price of y is 6 and the consumer has 600 pounds to spend. The consumer cannot buy more than 30 units of y.

2. Maximise the following objective function using linear programming

$$2x_1 + 3x_2$$
$$s.t.$$
$$x_1 + 4x_2 \leq 160$$
$$3x_1 + x_2 \leq 135$$
$$x_1 + x_2 \leq 50$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

3. A consumer's utility function is: $u = x^{0.25}y^{0.75}$ where x and y are two goods.

(i) Suppose total income is £10,000 and the prices of the two goods are £4 and £6 respectively. Use constrained optimisation to find the consumer's demand for both goods.
(ii) Now replace the price of the second good with p. Find a formula for the consumer’s demand for this good. Draw the demand curve and comment on its properties
(iii) What is the own-price elasticity of demand for the first good?

Bordered Hessians

1. Optimise the function

$$z = 4x^2 + 3xy + 6y^2$$

subject to $x + y = 56$

using the bordered Hessian to check the second order conditions

2. Find the second order conditions that guarantee maximisation of the following utility function

$U = q_1q_2$

Subject to the constraint $Y = p_1q_1+p_2q_2$

Where Y = income and U = utility