# EC5555
# Economics Masters Refresher Course in Mathematics
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Lecture 9 – Differential equations

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# Integration and differential equations.

- Definition: a differential equation includes the derivative of a function as a variable to model the change in the level of a variable
- E.g.

$$\frac{dy}{dx} = a \quad or \quad \frac{dy}{dx} = ax^2 \quad or \quad \frac{dy}{dx} = y - ax^2$$

- These examples have only first derivatives. They are called first order differential equations. Second order differential equations include second derivatives and so on.

$$\frac{d^2y}{dx^2} = a + \frac{dy}{dx}$$

## Differential equations.

- Linear equations have no powers in x and no interaction terms between y and x
- Homogeneous equations are ones such that if we multiply all variables by a constant, the differential equation is unchanged.

$$\frac{dy}{dx} = x \quad or \quad a\frac{dy}{dx} = ax \qquad homogeneous$$

$$\frac{dy}{dx} = x^2 \quad or \quad a\frac{dy}{dx} = (ax)^2 \quad non-\mathrm{homogeneous}$$

- In the most common examples, x = time, usually indicated by ‘t’
- In economics, decisions are rarely taken continuously. Hence differential equations are usually approximations to processes that might be more exactly represented via difference equations:

$$\frac{dy}{dt} = ay \quad or \quad y_t - y_{t-1} = ay_{t-1}$$

- To solve a differential equation completely we also need information about y at some time t. E.g. starting point or end point.

# Solving a simple homogeneous equation

- We are interested in finding the value of an underlying variable at any point in time when we are only given information on how the variable evolves over time (the differential)
- Example 1

$$\frac{dx}{dt} = f(t) \equiv \frac{dx}{dt} - f(t) = 0$$

- Integrate both sides wrt t

$$\int \frac{dx}{dt} dt = \int f(t) dt$$

$$\Rightarrow x = \int f(t) + c$$

- Eg. f(t) = $t^2$ – 1 = dx/dt
- Gives x = $t^3$ /3 – t + c
- So given t can work out the value of x in any period

## Solving a simple homogeneous equation.

$$\frac{dy}{dt} = ay$$

- One way to approach the solution is to treat dy and dt as separate items and integrate:

$$\frac{1}{y}dy = adt$$

$$\int \frac{1}{y}dy = \int adt$$

$$\ln(y) = at + c$$

$$y = e^c e^{at} = Ae^{at}$$

## Solving a simple homogeneous equation.

- Now y(t) = $Ae^{at}$
- (is an equation commonly used to describe the evolution of GDP over time where a = rate of growth)
- Note that A could be any value
- (in the growth literature A is used as a measure of technical progress)
- y(t) = $Ae^{at}$ is said to be the ***general solution*** to the differential equation

$$\frac{dy}{dt} = ay$$

- For a particular value of A this becomes a ***particular solution***
- If we start off the process y(0)=A, then this gives the ***definite solution***
- Note the solution (y(t) = $Ae^{at}$ ) is free of any derivative and is not a numerical value rather a function (or a “time path”) giving the value of y at any point in time

## Solving a simple non-homogeneous equation

$$\frac{dy}{dt} = ay + b$$

- The solution to the related homogeneous equation is called the **complementary function ($y_c$)**
- We call the **particular integral ($y_p$)** any particular solution to the non-homogeneous equation)
- The sum of the complementary function and the particular integral constitutes the **general solution** of a $1^{st}$ order linear non-homogenous differential equation

## Solving a simple non-homogeneous equation.

(1) $$\frac{dy}{dt} = ay + b$$

To solve proceed in 2 steps:

Look for ***any*** value that satisfies the equation
the simplest is to let y(t) = k = a constant
then dy/dt = 0 and (1) becomes

- ay = b

and

y(t) = -b/a = k (a ≠ 0 )

This solution to the non-homogeneous equation is called the particular integral

## Solving a simple non-homogeneous equation.

Step 2:

We consider the homogeneous related equation

$$\frac{dy}{dt} = ay$$

And we have already seen that the solution to this type of homogenous differential equation is $y(t) = Ae^{at}$

## Solving a simple non-homogeneous equation

- The solution to the related homogeneous equation (**complementary function ($y_c$)** in this case is

$$y(t) = Ae^{at}$$

- A particular solution to the non-homogeneous equation (**particular integral ($y_p$)**) in this case is

$$y(t) = -b/a$$

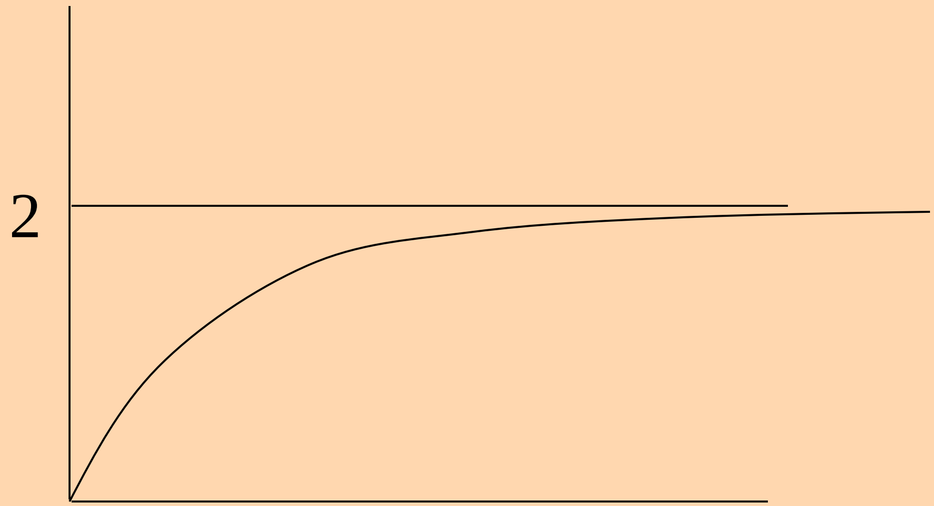
- The sum of the complementary function and the particular integral constitutes the **general solution** of a 1st order linear non-homogenous differential equation, in this case:

$$y(t) = y_c + y_p = Ae^{at} - b/a$$

- to get the definite solution we need to impose an initial condition:
- the value of y at t = 0
- In this example $y(0) = Ae^0 – b/a = A – (b/a)$,
- $A = y(0) + (b/a)$
- $y(t) = (y(0) + (b/a))e^{at} – (b/a)$

## Example: a=-1, b =2=-K

$$y = 2 - 2e^{-t}$$




- Often in economics the particular integral will represent the equilibrium of a system (here y =2), while the complementary function supplies the dynamics to get there (from the initial condition value)

## Solving a simple non-homogeneous equation

**Example 2**

Solve dy/dt +2y(t) = 6 with an initial condition y(0) = 10

The constant solution dy/dt = 0 gives the particular integral

$$y_p = 6/2 = 3$$

The complementary function is the solution to dy/dt = -2y(t) (with the constant set to zero)

- Or:

$$\frac{1}{y}\frac{dy}{dt} = -2$$

- Integrating both sides wrt *t*:

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int -2dt$$

$$\ln(y) + c_1 = -2t + c_2$$

$$\ln(y) = -2t + (c_2 - c_1)$$

$$y = e^{-2t}e^{k}; k = c_2 - c_1$$

$$y = Ae^{-2t}$$

## Solving a simple non-homogeneous equation

So the general solution is:

$$y(t) = y_c + y_p = Ae^{-2t} + 3$$

The definite solution is to use the value for the initial conditions t=0, y(0) =10

$$y(0) = Ae^0 +3 = A + 3 \text{ so then } A = y(0) – 3 = 10 – 3 = 7$$

So then

$$y(t) = [10 – 3]e^{-2t} + 3 = 7e^{-2t} + 3$$

This is saying the system converges to the value 3, starting from an initial value of 10 as $t \rightarrow \infty$

## Solving a simple non-homogeneous equation

- Sometimes solution is written down as

$$\frac{dy}{dt} + vy = z$$

$$\Rightarrow y(t) = e^{-\int v dt}\left( A + \int z e^{\int v dt} dt \right)$$

- where A is an arbitrary constant

$e^{-\int v dt} A$ is the complementary function

$e^{-\int v dt} \int z e^{\int v dt}$ is the particular integral

- The particular integral $y_p$ gives the inter-temporal equilibrium level of y(t)
- The complementary function $y_c$ gives the deviation from the equilibrium
- For y(t) to be dynamically stable $y_c$ must approach 0 as t goes to infinity

## Solving a simple non-homogeneous equation

- Eg: Find the general solution for dy/dt +4y =12
- Let v = 4 and z = 12
- Using

$$y(t) = e^{-\int v dt}\left(A + \int z e^{\int v dt} dt\right)$$

$$\Rightarrow y(t) = e^{-4t}\left(A + \int 12e^{4t} dt\right) = e^{-4t}\left(A + 3e^{4t}\right)$$

$$\Rightarrow y(t) = Ae^{-4t} + 3$$

- The 1st term is the complementary function and the second is the particular integral
- As *t* goes to infinity the complementary function goes to zero and so y(t) approaches the level of the particular integral
- Often helpful to check the solution by differentiating backwards

# Exact differential equations

a total differential of F(y,t) is:

$$dF = \frac{\partial F}{\partial y} dy + dt \frac{\partial F}{\partial t}$$

- If dF = 0, the result is called an exact differential equation. The general solution should be of the form F = c (c is a constant)
- Usually though we seek to find F given the differential.
- The statement that a differential equation of the form:

$$0 = Mdy + dtN$$

(M and N are functions)

is exact is equivalent to the statement that there exists F such that

$$M = \frac{\partial F}{\partial y}; N = \frac{\partial F}{\partial t}$$

- By Young's theorem: $$\frac{\partial^2 F}{\partial y \partial t} = \frac{\partial^2 F}{\partial t \partial y}$$

- So F exists provided: $$\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$$

## Solving exact differential equations

- Step 1. Check that $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$ If it is then we have an exact equation.
- If it is not exact then we may still be able to solve (see later)
- Step 2. Integrate M partially with respect to y. The result is:

$$F = \int M dy + g(t)$$

- Note the term g(t) which we do not know yet.
- Step 3. Partially differentiate the result in 2 with respect to t to get N:

$$\frac{\partial F}{\partial t} = \frac{\partial \int M dy}{\partial t} + g'(t) = N$$

- Note that g' is the derivative of g with respect to t.
- Step 4. We know N so we can solve this equation to find g'. From that we can integrate g' to find g then use starting conditions to find the general solution.

## Solving exact differential equations - example

- Solve $\frac{dy}{dt} = 4t$
- Step 1.
- We rewrite the differential equation: $$0 = dy - 4tdt$$
- So M =1 and N =-4t. $\frac{\partial M}{\partial t} = 0 = \frac{\partial N}{\partial y}$
- Step 2. Integrate M partially with respect to y. The result is:

$$F = \int M dy + g(t) = y + g(t)$$

- Step 3. Partially differentiate the result in 2 with respect to t to get N:

$$\frac{\partial F}{\partial t} = \frac{\partial y}{\partial t} + g'(t) = 0 + g'(t) = -4t$$

- Step 4. find g'. g'=-4t, so integrate g' to find g: $\int g' dt = k - 2t^2$
- So, $F = k + y - 2t^2 \quad or \quad c = y - 2t^2 \quad or \quad y = 2t^2 + c$

# Integrating factors

- Sometimes M and N mean that the differential equation is inexact, but we can still find another function of t and y which we can multiply through by to get an exact equation.
- Example. $0 = tdy + 2ydt$ $\quad \frac{\partial M}{\partial t} = 1 \neq 2 = \frac{\partial N}{\partial y}$
- Multiply through by t to get: $0 = t^2 dy + 2ytdt$
- Step 1. Check $\frac{\partial M}{\partial t} = 2t = \frac{\partial N}{\partial y}$
- Step 2. Integrate M partially with respect to y. The result is:

$$F = \int M dy + g(t) = yt^2 + g(t)$$

- Step 3. Partially differentiate the result in 2 with respect to t to get N:

$$\frac{\partial F}{\partial t} = \frac{\partial \int M dy}{\partial t} + g'(t) = 2ty + g' = N = 2ty$$

- Step 4. From this we get g' = 0 or g = k,

$$F = yt^2 + k \quad or \quad c = yt^2 \quad or \quad y = \frac{c}{t^2}$$