EC5555
Economics Masters Refresher Course in Mathematics
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Lecture 8 – Integration

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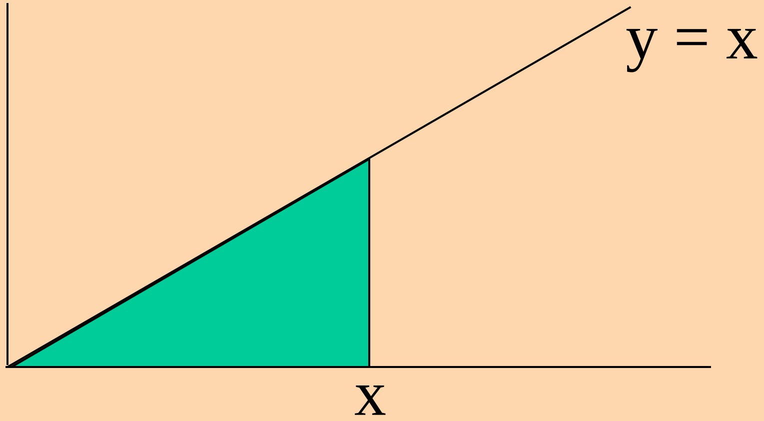
Introduction to Integration

Learning objectives. By the end of today's lecture you should:

- Understand the concept of integration and its relationship to areas under graphs.
- Know how to integrate some standard functions.

1. Introduction: Integration has two interpretations:
   - As the inverse of differentiation
     - E.g. what function of x differentiates to become y=x?
   - As a means of calculating areas under graphs.



   - Technically, in integration, the second interpretation provides the definition of integration. The first interpretation is then proved through the Fundamental Theorem of Calculus.

Inverse of differentiation.

E.g. what function of x differentiates to become y=x?

- We know x^2 differentiates to 2x,
- So $0.5x^2$ must differentiate to x.

- But, $0.5x^2 + 5$ also differentiates to x and so does $0.5x^2 - 5$
- In fact $0.5x^2 + c$ differentiates to x for any value of c.

- We call this the indefinite integral of x.
- We write this as,

$$\int x dx = \frac{x^2}{2} + c$$

- $\int$ is the integration sign (it is an old fashioned 's' for SUMMA which is latin for sum)
- In general we write

$$\int f(x)dx = F(x) + c \text{ where } \frac{dF(x)}{dx} = f(x)$$

Some standard indefinite integrals.

function	integrates to:
ax	$\frac{a}{2}x^2 + c$
$ax^n, n \neq -1$	$\frac{a}{n+1}x^{n+1} + c$
$\frac{a}{x}$	$a \ln x + c$
e^x	$e^x + c$
$\frac{f'(x)}{f(x)}$	$\ln(f(x)) + c$
$f'(x)e^{f(x)}$	$e^{f(x)} + c$

Rules of operation

Integral of a sum

The integral of a sum of functions is the sum of the integrals of those functions

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Example:

$$\int x^3 + x\, dx = \int x^3 dx + \int x\, dx =$$

$$= \frac{x^4}{4} + c_1 + \frac{x^2}{2} + c_2 =$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + c$$

Integral of a multiple

The integral of a sum of functions is the sum of the integrals of those functions

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Examples:

$$\int -f(x)dx = -\int f(x)dx$$

$$\int 2x^3 dx = 2\int x^3 dx = 2\left(\frac{x^4}{4} + c_1\right) = \frac{x^4}{2} + c$$

$$\int \frac{2}{x} + x^3\,dx = \int \frac{2}{x}dx + \int x^3 dx = 2\int \frac{1}{x}dx + \int x^3 dx =$$

$$= 2(\ln x + c_1) + \frac{x^4}{4} + c_2 =$$

$$= 2\ln x + \frac{x^4}{4} + c$$

Substitution

The integral of a sum of functions is the sum of the integrals of those functions

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + c$$

This rule is the counterpart of the chain rule

Consider $F(u)$ where $u = u(x)$, by chain rule

$$\frac{dF(u)}{dx} = \frac{dF(u)}{du}\frac{d(u)}{dx} = F'(u)\frac{d(u)}{dx} = f(u)\frac{d(u)}{dx}$$

$$\frac{dF(u)}{dx} = f(u)\frac{d(u)}{dx}$$

Therefore

$$\int f(u)\frac{du}{dx}\,dx = F(u) + c$$

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + c$$

Example

$$\int 2x(x^2+1)\,dx \;= \int (2x^3+2x)dx = \frac{x^4}{2} + x^2 + c$$

Let $u = x^2 + 1$,

then $\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

We replace u and dx in the integral and we get:

$$\int 2xu\frac{du}{2x} =$$

$$= \int u\,du = \frac{u^2}{2} + c_1 =$$

$$= \frac{(x^2+1)^2}{2} + c_1 =$$

$$= \frac{x^4+2x^2+1}{2} + c_1 = \frac{x^4}{2} + 2x^2 + c$$

Integration by parts.

- Example: how do we integrate y = log(x)?
  - (In fact the answer is xlog(x) – x.)
- To find this we use the product rule for differentiation:
- If f(x) = u(x)v(x) then

$$\frac{df}{dx} = \frac{du}{dx} v(x) + u(x) \frac{dv}{dx}$$

- It follows that:

$$\int \frac{df}{dx}\, dx = f(x) = u(x)v(x) = \int \frac{du}{dx} v(x)\, dx + \int u(x) \frac{dv}{dx}\, dx$$

- Rearranging:

$$\int \frac{du}{dx} v(x) dx = u(x)v(x) - \int u(x) \frac{dv}{dx} dx$$

- To use this method you have to separate your function into two components:
  - i. du/dx and ii. v(x)
- You need to choose these carefully so that you can integrate du/dx and u(x)dv/dx.
- (This is an art!)

Integration by parts - example.

- How do we integrate y = log(x)?
- We will let v = log(x) and set du/dx = 1
- So that u(x) = x (ignoring the constant of integration) and dv/dx = 1/x.
- It follows that: $\int \frac{du}{dx} v(x) dx = u(x)v(x) - \int u(x) \frac{dv}{dx} dx$

$$= x\log(x) - \int x \frac{1}{x} dx$$

$$= x\log(x) - \int 1 dx$$

$$= x\log(x) - x$$

Definite integrals

Given an indefinite integral

$$\int f(x)dx = F(x) + c$$

If we choose two values of x in the domain, say a and b ($a < b$) and we substitute in the RHS and form the difference we get:

$$[F(b) + c] - [F(a) + c] = F(b) - F(a)$$

We get a specific value (free of variable x)

It is called the *definite integral of f(x) from a to b*

a is the lower limit of integration

b is the upper limit of integration

We denote the definite integral in the following way:

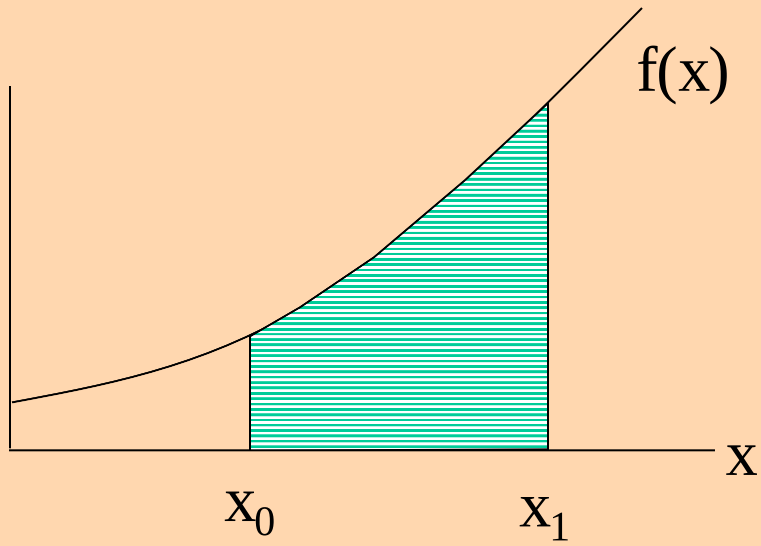
$$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$$

The Fundamental Theorem of Calculus.

- Part 1. Let f(x) be a continuous function and let $F(x) = \int_0^x f(x)dx$
- Then $f(x) = \dfrac{dF}{dx}$

- Part 2. Suppose f is a continuous function and F is the anti-derivative of f
- (i.e. $f(x) = \dfrac{dF}{dx}$), then $F(x) = \int_0^x f(x)dx$

Areas under graphs.



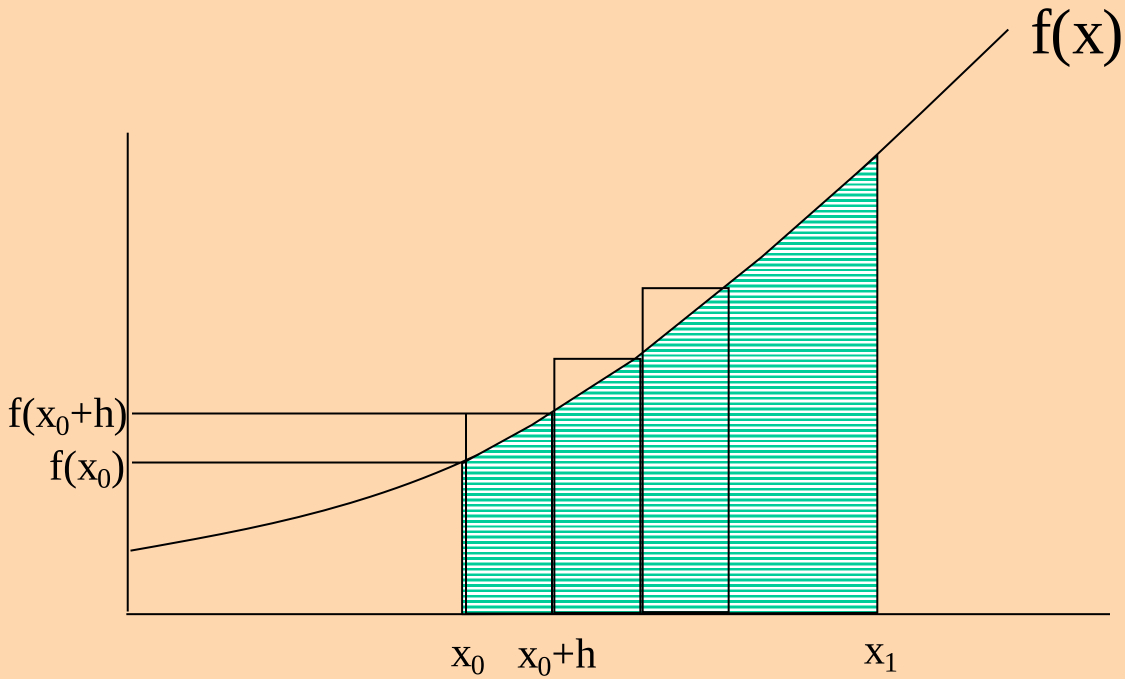
Suppose we wish to find the area under the function f(x) between x_0 and x_1

We write this as $F(x_1, x_0)$ or

$$F(x_1, x_0) = \int_{x_0}^{x_1} f(x)dx$$

One way to find an approximate answer is to divide x_1-x_0 into n intervals of width h (so that $h = (x_1-x_0)/n$).

Areas under graphs.



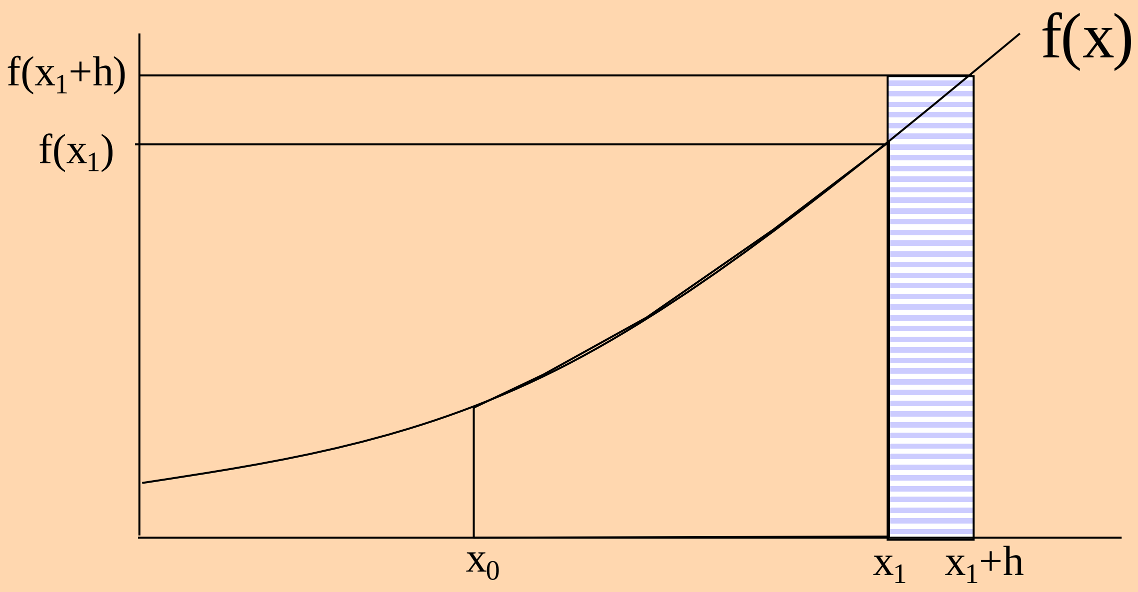
$$Approximate\ area\ within\ interval\ i\ =\ h[f(x_0+(i-1)h)+f(x_0+ih)]/2$$

$$Total\ approximation\ =\sum_{i=1}^{n} h[f(x_0+(i-1)h)+f(x_0+ih)]/2$$

The integral of f between x_0 and x_1 is then the limit of this sum as $h \rightarrow 0$

Note that one implication of this definition is that integrals are additive so that $F(x_1,0) = F(x_1,x_0)\ +\ F(x_0,0)$ or $F(x_1,x_0) =\ F(x_1,0)\ -\ F(x_0,0)$.

Usually we drop the zero part of this and just write $F(x_1)$ to mean the integral of x between 0 and x_1, then $F(x_1,x_0) =\ F(x_1)\ -\ F(x_0)$.



The area under the graph between x_0 and x_1 is then just $F(x_1) - F(x_0)$

Now consider the area between x_1 and x_1+h as h→0

This is $F(x_1 + h) - F(x_1)$.

We have $$F(x_1+h)-F(x_1) \approx h[f(x_1+h)+f(x_1)]/2 \approx hf(x_1)$$

or, $\frac{F(x_1+h)-F(x_1)}{h} \approx f(x_1)$ in the limit, as h→0, the left hand side of this expression is the derivative of F at x_1.

In other words, f is just the derivative of F.

Properties of definite ntegrals

1) The interchange of the limits of integration changes the sign of the definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2) A definite integral has value 0 when the two limit of integration are identical

$$\int_a^a f(x)dx = 0$$

3) A definite integral can be expressed as a sum of a finite number of sub integrals as follows

$$\int_a^d f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx$$

where $a < b < c < d$

4)

$$\int_a^b -f(x)dx = -\int_a^b f(x)dx$$

5)

$$\int_a^b kf(x)dx = k\int_b^a f(x)dx$$

6)

$$\int_a^b [f(x) + g(x)]dx = \int_b^a f(x)dx + \int_a^b g(x)dx$$

7) Integration by parts

$$\int_{x=a}^{x=b} v\,du = uv]_{x=a}^{x=b} - \int_b^a u\,dv$$

Improper integrals.

A definite integral is improper in one of two cases:

1) One or both of the limits of the integral is infinite.

An improper integral can be defined as

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

If this limit exists, then the integral is said to converge

2) When the integrand becomes infinite somewhere in the interval of integration $[a, b]$

$f(x) \to \infty \; as \; x \to p$ and $a < p \; < b$

$$\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b f(x)dx = \lim_{y\to p^-} \int_a^y f(x)dx + \lim_{y\to p^+} \int_y^b f(x)dx$$

If these limit exist, then the integral is said to converge

Examples

$$\lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big]_2^b = \lim_{b\to\infty} -\frac{1}{b}+\frac{1}{2} = \frac{1}{2}$$

$$\lim_{b\to\infty}\int_2^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big]_2^b = \lim_{b\to\infty} \ln(b)-\ln(2) = \infty$$

$$\int_a^3 \frac{1}{x}\,dx = \lim_{a\to 0}\int_a^3 \frac{1}{x}\,dx = \lim_{a\to 0} \ln(x)\Big]_a^3 = \lim_{a\to 0} \ln(3)-\ln(a) = -\infty$$

$$\int_{-1}^{1} \frac{1}{x^3}\,dx = \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_{0}^{1} \frac{1}{x^3}\,dx$$

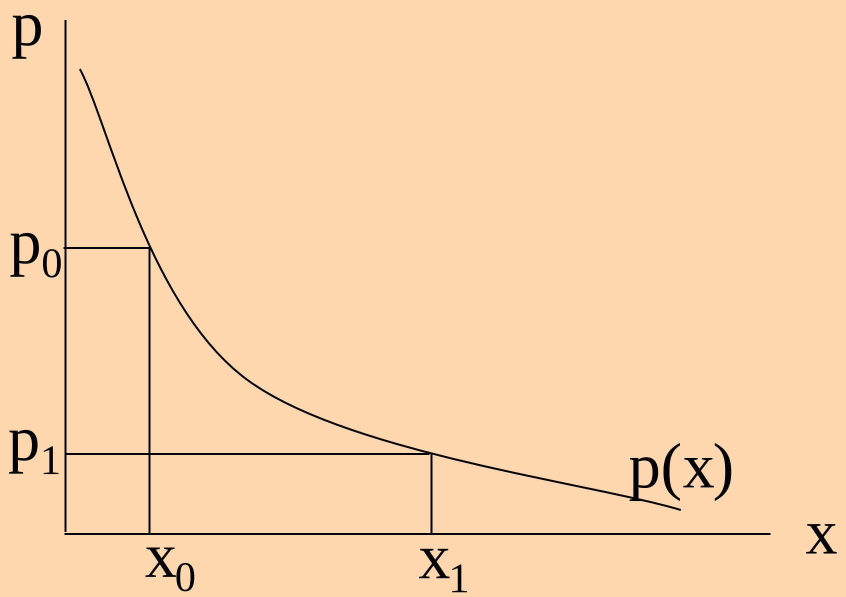
The integral is divergent because

$$\lim_{b\to 0^-}\int_{-1}^{b} \frac{1}{x^3}\,dx = \lim_{b\to 0^-}\left[-\frac{1}{2x^2}\right]_{-1}^{b} =$$

$$\lim_{b\to 0^-} -\frac{1}{2b^2} + 0.5 = -\infty$$

Example 1: Consumer Surplus

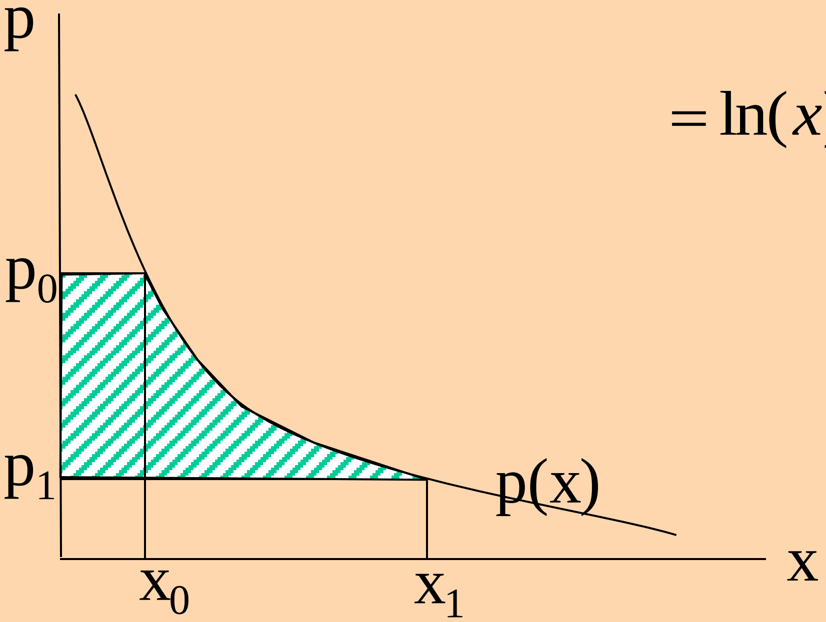
- Possibly the commonest application of integration in economics is in the calculation of consumer surplus. Mathematically this is straightforward, but it is confused by the way we put the ‘x’ variable on the vertical:



- Suppose demand function is p = 1/x. The price falls from p=2 to p=1. Find the increase in consumer surplus.
- Note that x rises from 0.5 to 1. It is tempting to find the surplus change by integrating the inverse demand function from 0.5 to 1.
- But this is wrong!

Example 1: Consumer Surplus

- The change in cs is the shaded area.
- i.e. write x = 1/p and integrate between p=1 and p=2: $\Delta CS = \int_1^2 \frac{1}{x}\,dx$

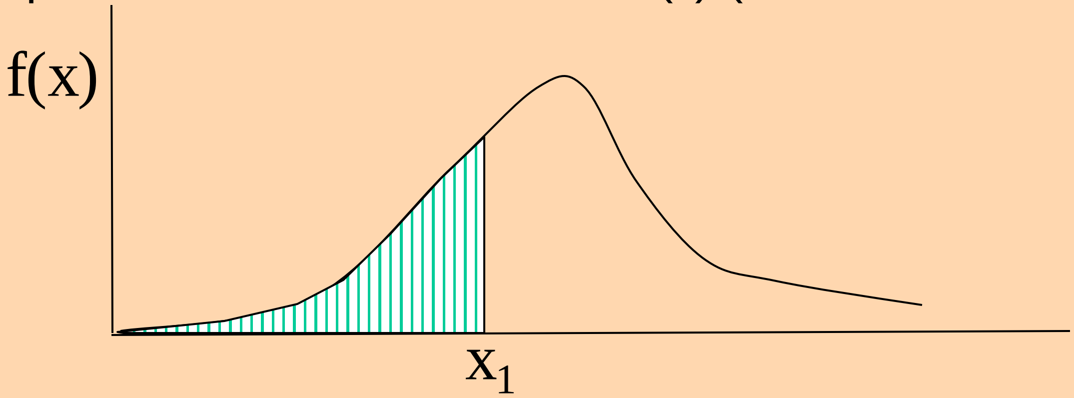


$$= \ln(x)]_1^2 = \ln(2) \approx 0.693$$

- Note that this is an example where CS is undefined (the relevant integral is improper and does not converge), but changes in CS are meaningful.

Example 2: Probability

- Integration is also useful in probability theory and statistics.
- Consider a continuous random variable– e.g. height or (roughly) income.
- The probability that the variable is less than or equal to x is called the cumulative density function, F(x).
- The probability density function (**pdf**), f(x) is the probability density that the variable equals x. It is the derivative of F(x) (where the derivative exists).



- That is,

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

- Note that: $\int_{-\infty}^{x} f(x)dx \geq 0; \int_{-\infty}^{\infty} f(x)dx = 1;$

- Note also that in some contexts x may not be defined everywhere on [-∞,∞]

Example 2: Probability

- The expected value of a random variable is its mean. It is calculated as:

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

- The variance is:

$$E\left[(x - E(x))^2\right] = \int_{-\infty}^{\infty} (x - E(x))^2 f(x) dx$$

- Example: at Egham Station the probability of queuing at least t minutes for a ticket is $e^{-0.3t}$. What is the expected waiting time?
- The probability of queuing less than t minutes = 1 – $e^{-0.3t}$
- The pdf is the derivative of this function, and is $0.3e^{-0.3t}$

$$E(t) = \int_{0}^{\infty} t f(t) dt = \int_{0}^{\infty} t 0.3 e^{-0.3t} dt$$

- Integrate by parts

Example 2: Probability

$$\int \frac{du}{dx} v(x) = u(x)v(x) - \int u(x)\frac{dv}{dx}$$

- Let v(x) = t and du/dx = 0.3 $e^{-0.3t}$

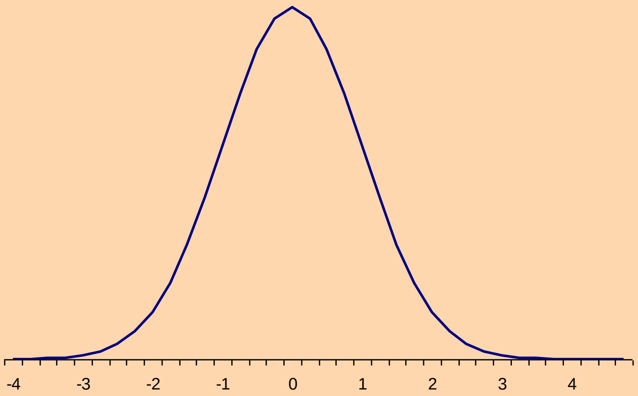
$$\int_0^\infty t0.3e^{-0.3t}dt = -te^{-0.3t}]_0^\infty - \int_0^\infty -e^{-0.3t}dt = 0 - \frac{e^{-0.3t}}{0.3}]_0^\infty \approx 3.33$$

- (Work it through to verify the answer)

Example 2: Probability II

- The probability density function for a normal distribution with mean 0 and variance 1 is:

$$f(x) = \frac{e^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}$$



- This is known as a standard normal distribution. The expected value is:

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} \frac{xe^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-(1/2)\left[x^2\right]}dx$$

- We can integrate:

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} \frac{xe^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}dx = \left.\frac{-e^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}\right|_{-\infty}^{\infty} = \left.\frac{-e^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}\right|_{-\infty}^{0} + \left.\frac{-e^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}\right|_{0}^{\infty}$$

$$= \frac{-1}{\sqrt{2\pi}} + \frac{--1}{\sqrt{2\pi}} = 0$$

Example 3: Capital Formation and Investment

- Capital formation is the process of adding to the capital stock by investment
- The capital stock at any time is given by K(t)
- The rate of capital formation is the change in the capital stock
- K'(t) = I(t)
- sometimes first derivative wrt time is written with a dot over the variable:

$$\dot{K}(t) \equiv K'(t)$$

- So capital stock is the sum of changes in capital formation over time

$$K(t) = \int I(t)dt$$

- Note the use of the t as the argument in the expression K(t) to denote the value of the integral at any time *t* over the time period defined by the integral

Integration and differential equations.

- Definition: a differential equation includes the derivative of a function as a variable to model the change in the level of a variable
- E.g.

$$\frac{dy}{dx} = a \quad or \quad \frac{dy}{dx} = ax^2 \quad or \quad \frac{dy}{dx} = y - ax^2$$

- These examples have only first derivatives. They are called first order differential equations. Second order differential equations include second derivatives and so on.

$$\frac{d^2y}{dx^2} = a + \frac{dy}{dx}$$

Differential equations.

- Linear equations have no powers in x and no interaction terms between y and x
- Homogeneous equations are ones such that if we multiply all variables by a constant, the differential equation is unchanged.

$$\frac{dy}{dx} = x \quad or \quad a\frac{dy}{dx} = ax \qquad homogeneous$$

$$\frac{dy}{dx} = x^2 \quad or \quad a\frac{dy}{dx} = (ax)^2 \quad non-\text{homogeneous}$$

- In the most common examples, x = time, usually indicated by ‘t’
- In economics, decisions are rarely taken continuously. Hence differential equations are usually approximations to processes that might be more exactly represented via difference equations:

$$\frac{dy}{dt} = ay \quad or \quad y_t - y_{t-1} = ay_{t-1}$$

- To solve a differential equation completely we also need information about y at some time t. E.g. starting point or end point.

Solving a simple homogeneous equation

- We are interested in finding the value of an underlying variable at any point in time when we are only given information on how the variable evolves over time (the differential)
- Example 1

$$\frac{dx}{dt} = f(t) \equiv \frac{dx}{dt} - f(t) = 0$$

- Integrate both sides wrt t

$$\int \frac{dx}{dt} dt = \int f(t) dt$$

$$\Rightarrow x = \int f(t) + c$$

- Eg. f(t) = t^2 – 1 = dx/dt
- Gives x = t^3 /3 – t + c
- So given t can work out the value of x in any period

Solving a simple homogeneous equation.

$$\frac{dy}{dt} = ay$$

- One way to approach the solution is to treat dy and dt as separate items and integrate:

$$\frac{1}{y} dy = adt$$

$$\int \frac{1}{y} dy = \int adt$$

$$\ln(y) = at + c$$

$$y = e^c e^{at} = Ae^{at}$$

Solving a simple homogeneous equation.

- Now $y(t) = Ae^{at}$
- (is an equation commonly used to describe the evolution of GDP over time where a = rate of growth)
- Note that A could be any value
- (in the growth literature A is used as a measure of technical progress)
- $y(t) = Ae^{at}$ is said to be the ***general solution*** to the differential equation

$$\frac{dy}{dt} = ay$$

- For a particular value of A this becomes a ***particular solution***
- If we start off the process y(0)=A, then this gives the ***definite solution***
- Note the solution ($y(t) = Ae^{at}$) is free of any derivative and is not a numerical value rather a function (or a “time path”) giving the value of y at any point in time

Solving a simple non-homogeneous equation

$$\frac{dy}{dt} = ay + b$$

- The solution to the related homogeneous equation is called the **complementary function (y_c)**
- We call the **particular integral (y_p)** any particular solution to the non-homogeneous equation)
- The sum of the complementary function and the particular integral constitutes the **general solution** of a 1st order linear non-homogenous differential equation

Solving a simple non-homogeneous equation.

(1) $$\frac{dy}{dt} = ay + b$$

To solve proceed in 2 steps:

Look for ***any*** value that satisfies the equation

the simplest is to let y(t) = k = a constant

then dy/dt = 0 and (1) becomes

- ay = b

and

y(t) = -b/a = k (a ≠ 0)

This solution to the non-homogeneous equation is called the particular integral

Solving a simple non-homogeneous equation.

Step 2:

We consider the homogeneous related equation

$$\frac{dy}{dt} = ay$$

And we have already seen that the solution to this type of homogenous differential equation is $y(t) = Ae^{at}$

Solving a simple non-homogeneous equation

- The solution to the related homogeneous equation (**complementary function (y_c)** in this case is

$$y(t) = Ae^{at}$$

- A particular solution to the non-homogeneous equation (**particular integral (y_p)**) in this case is

$$y(t) = -b/a$$

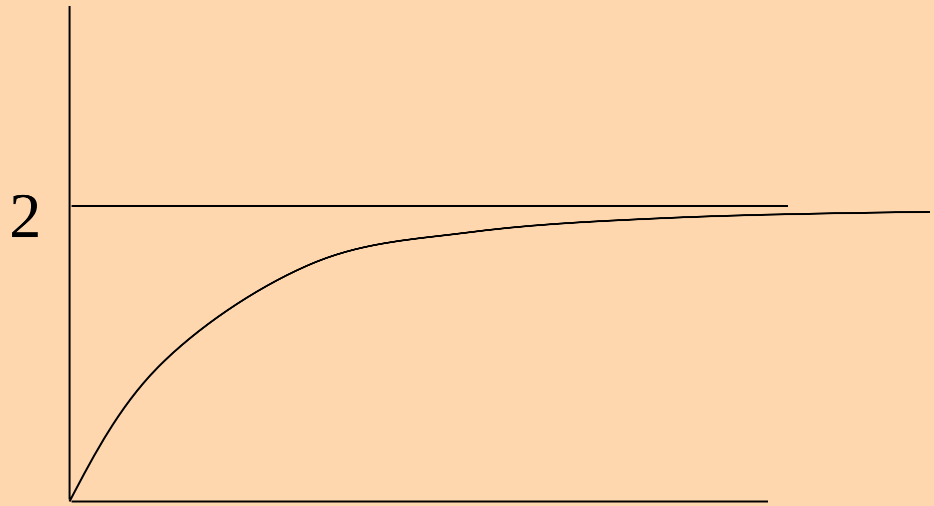
- The sum of the complementary function and the particular integral constitutes the **general solution** of a 1st order linear non-homogenous differential equation, in this case:

$$y(t) = y_c + y_p = Ae^{at} - b/a$$

- to get the definite solution we need to impose an initial condition:
- the value of y at t = 0
- In this example $y(0) = Ae^0 – b/a = A – (b/a)$,
- $A = y(0) + (b/a)$
- $y(t) = (y(0) + (b/a))e^{at} – (b/a)$

Example: a=-1, b =2=-K

$$y = 2 - 2e^{-t}$$



- Often in economics the particular integral will represent the equilibrium of a system (here y =2), while the complementary function supplies the dynamics to get there (from the initial condition value)

Solving a simple non-homogeneous equation

Example 2

Solve dy/dt +2y(t) = 6 with an initial condition y(0) = 10

The constant solution dy/dt = 0 gives the particular integral

$$y_p = 6/2 = 3$$

The complementary function is the solution to dy/dt = -2y(t) (with the constant set to zero)

- Or:

$$\frac{1}{y}\frac{dy}{dt} = -2$$

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int -2dt$$

- Integrating both sides wrt *t*:

$$\ln(y) + c_1 = -2t + c_2$$

$$\ln(y) = -2t + (c_2 - c_1)$$

$$y = e^{-2t}e^{k}; k = c_2 - c_1$$

$$y = Ae^{-2t}$$

Solving a simple non-homogeneous equation

So the general solution is:

$$y(t) = y_c + y_p = Ae^{-2t} + 3$$

The definite solution is to use the value for the initial conditions t=0, y(0) =10

$$y(0) = Ae^0 +3 = A + 3 \text{ so then } A = y(0) – 3 = 10 – 3 = 7$$

So then

$$y(t) = [10 – 3]e^{-2t} + 3 = 7e^{-2t} + 3$$

This is saying the system converges to the value 3, starting from an initial value of 10 as $t \rightarrow \infty$

Solving a simple non-homogeneous equation

- Sometimes solution is written down as

$$\frac{dy}{dt} + vy = z$$

$$\Rightarrow y(t) = e^{-\int v dt}\left(A + \int z e^{\int v dt} dt \right)$$

- where A is an arbitrary constant

$e^{-\int v dt} A$ is the complementary function

$e^{-\int v dt} \int z e^{\int v dt}$ is the particular integral

- The particular integral y_p gives the inter-temporal equilibrium level of y(t)
- The complementary function y_c gives the deviation from the equilibrium
- For y(t) to be dynamically stable y_c must approach 0 as t goes to infinity

Solving a simple non-homogeneous equation

- Eg: Find the general solution for dy/dt +4y =12
- Let v = 4 and z = 12
- Using

$$y(t) = e^{-\int v dt}\left(A + \int z e^{\int v dt} dt\right)$$

$$\Rightarrow y(t) = e^{-4t}\left(A + \int 12 e^{4t} dt\right) = e^{-4t}\left(A + 3e^{4t}\right)$$

$$\Rightarrow y(t) = Ae^{-4t} + 3$$

- The 1st term is the complementary function and the second is the particular integral
- As *t* goes to infinity the complementary function goes to zero and so y(t) approaches the level of the particular integral
- Often helpful to check the solution by differentiating backwards

Exact differential equations

- Recall from earlier lectures that a total differential of F(y,t) is:

$$dF = \frac{\partial F}{\partial y} dy + dt \frac{\partial F}{\partial t}$$

- If dF = 0, the result is called an exact differential equation. The general solution should be of the form F = c (c is a constant)
- Usually though we seek to find F given the differential.
- The statement that a differential equation of the form:

$$0 = Mdy + dtN$$

(M and N are functions)

is exact is equivalent to the statement that there exists F such that

$$M = \frac{\partial F}{\partial y}; N = \frac{\partial F}{\partial t}$$

- By Young's theorem:

$$\frac{\partial^2 F}{\partial y \partial t} = \frac{\partial^2 F}{\partial t \partial y}$$

- So F exists provided: $\dfrac{\partial M}{\partial t} = \dfrac{\partial N}{\partial y}$

Solving exact differential equations

- Step 1. Check that $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$ If it is then we have an exact equation.
- If it is not exact then we may still be able to solve (see later)

- Step 2. Integrate M partially with respect to y. The result is:

$$F = \int M dy + g(t)$$

- Note the term g(t) which we do not know yet.
- Step 3. Partially differentiate the result in 2 with respect to t to get N:

$$\frac{\partial F}{\partial t} = \frac{\partial \int M dy}{\partial t} + g'(t) = N$$

- Note that g' is the derivative of g with respect to t.
- Step 4. We know N so we can solve this equation to find g'. From that we can integrate g' to find g then use starting conditions to find the general solution.

Solving exact differential equations - example

- Solve $\frac{dy}{dt} = 4t$
- Step 1.
- We rewrite the differential equation: $0 = dy - 4tdt$
- So M =1 and N =-4t. $\frac{\partial M}{\partial t} = 0 = \frac{\partial N}{\partial y}$
- Step 2. Integrate M partially with respect to y. The result is:

$$F = \int M dy + g(t) = y + g(t)$$

- Step 3. Partially differentiate the result in 2 with respect to t to get N:

$$\frac{\partial F}{\partial t} = \frac{\partial y}{\partial t} + g'(t) = 0 + g'(t) = -4t$$

- Step 4. find g’. g’=-4t, so integrate g’ to find g: $\int g' dt = k - 2t^2$
- So, $F = k + y - 2t^2 \quad or \quad c = y - 2t^2 \quad or \quad y = 2t^2 + c$

Integrating factors

- Sometimes M and N mean that the differential equation is inexact, but we can still find another function of t and y which we can multiply through by to get an exact equation.
- Example. $0 = tdy + 2ydt$ $\quad \frac{\partial M}{\partial t} = 1 \neq 2 = \frac{\partial N}{\partial y}$

- Multiply through by t to get: $0 = t^2 dy + 2ytdt$
- Step 1. Check $\frac{\partial M}{\partial t} = 2t = \frac{\partial N}{\partial y}$
- Step 2. Integrate M partially with respect to y. The result is:

$$F = \int M dy + g(t) = yt^2 + g(t)$$

- Step 3. Partially differentiate the result in 2 with respect to t to get N:

$$\frac{\partial F}{\partial t} = \frac{\partial \int M dy}{\partial t} + g'(t) = 2ty + g' = N = 2ty$$

- Step 4. From this we get g' = 0 or g = k,

$$F = yt^2 + k \quad or \quad c = yt^2 \quad or \quad y = \frac{c}{t^2}$$