EC5555
Economics Masters Refresher Course in Mathematics
September 2013

Lecture 7 – Optimization with inequality constraints

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Optimization with inequality constraints: the Kuhn-Tucker (KT) conditions

The **KT conditions** for the problem

$$\max_x \; f(x) \text{ subject to } g_j(x) \le c_j \text{ for } j = 1, \ldots, m$$

are

$L_i'(x) = 0$ for $i = 1, \ldots, n$

$\lambda_j \ge 0$, $g_j(x) \le c_j$ and $\lambda_j[g_j(x) - c_j] = 0$ for $j = 1, \ldots, m$,

where

$$L(x) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j).$$

Necessity and sufficiency of KT conditions

A) The KT conditions are both necessary and sufficient if the objective function is concave and

- **either** each constraint is linear
- **or** each constraint function is convex and some vector of the variables satisfies all constraints strictly.

B) Suppose that the objective function is twice differentiable and quasiconcave and every constraint is linear.

- If x^* solves the problem then there exists a unique vector λ such that (x^*, λ) satisfies the Kuhn-Tucker conditions, and

- if (x^*, λ) satisfies the Kuhn-Tucker conditions and $f'_i(x^*) \neq 0$ for $i = 1, ..., n$ then x^* solves the problem.

Example

$$\max_{x_1, x_2} [-(x_1 - 4)^2 - (x_2 - 4)^2]$$

$$\text{s.t. } x_1 + x_2 \leq 4 \text{ and } x_1 + 3x_2 \leq 9$$

The objective function is concave and the constraints are both linear, so the solutions of the problem are the solutions of the Kuhn-Tucker conditions.

Kuhn Tucker conditions are

$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$

$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$

$x_1 + x_2 \leq 4$, $\lambda_1 \geq 0$, and $\lambda_1(x_1 + x_2 - 4) = 0$

$x_1 + 3x_2 \leq 9$, $\lambda_2 \geq 0$, and $\lambda_2(x_1 + 3x_2 - 9) = 0$

To solve this system of condition we have to consider all possibilities about the values of lambdas

We have to consider the following 4 cases:

1) $\lambda_1 = \lambda_2 = 0$
2) $\lambda_1 > 0$ $\lambda_2 = 0$
3) $\lambda_1 = 0$ $\lambda_2 > 0$
4) $\lambda_1 > 0$ $\lambda_2 > 0$

Kuhn Tucker conditions are

$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$

$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$

$x_1 + x_2 \le 4$, $\lambda_1 \ge 0$, and $\lambda_1(x_1 + x_2 - 4)= 0$

$x_1 + 3x_2 \le 9$, $\lambda_2 \ge 0$, and $\lambda_2(x_1 + 3x_2 - 9)= 0$

Case 1: $\lambda_1 = \lambda_2 = 0$

KT conditions are

$-2(x_1 - 4) = 0$

$-2(x_2 - 4) = 0$

$x_1 + x_2 \le 4$,

$x_1 + 3x_2 \le 9$

Then $x_1 = 4$ and $x_2 =4$

It not a solution because the last two inequalities are not satisfied

Kuhn Tucker conditions are

$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$

$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$

$x_1 + x_2 \leq 4$, $\lambda_1 \geq 0$, and $\lambda_1(x_1 + x_2 - 4) = 0$

$x_1 + 3x_2 \leq 9$, $\lambda_2 \geq 0$, and $\lambda_2(x_1 + 3x_2 - 9) = 0$

Case 2: $\lambda_1 > 0$ $\lambda_2 = 0$

KT conditions are

$-2(x_1 - 4) - \lambda_1 = 0$

$-2(x_2 - 4) - \lambda_1 = 0$

$x_1 + x_2 - 4 = 0$

$x_1 + 3x_2 \leq 9$,

From the first 2 equations $x_1 = x_2$

Using the third equation we get $x_1 = x_2 = 2$

It is a solution because the last inequality is satisfied

Kuhn Tucker conditions are

$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$

$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$

$x_1 + x_2 \leq 4$, $\lambda_1 \geq 0$, and $\lambda_1(x_1 + x_2 - 4)= 0$

$x_1 + 3x_2 \leq 9$, $\lambda_2 \geq 0$, and $\lambda_2(x_1 + 3x_2 - 9)= 0$

Case 3: $\lambda_1 =0$ $\lambda_2 > 0$

KT conditions are

$-2(x_1 - 4) - \lambda_2 = 0$

$-2(x_2 - 4) - 3\lambda_2 = 0$

$x_1 + x_2 \leq 4$

$x_1 + 3x_2 - 9= 0$

From the first 2 equations $x_2=3\,x_1-8$

Using the last equation we get $x_1 = 3.3$

It not asolution because it not satisfy the inequality

Kuhn Tucker conditions are

$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$

$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$

$x_1 + x_2 \le 4$, $\lambda_1 \ge 0$, and $\lambda_1(x_1 + x_2 - 4) = 0$

$x_1 + 3x_2 \le 9$, $\lambda_2 \ge 0$, and $\lambda_2(x_1 + 3x_2 - 9) = 0$

Case 4: $\lambda_1 > 0$ $\lambda_2 > 0$

KT conditions are

$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$

$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$

$x_1 + x_2 - 4 = 0$

$x_1 + 3x_2 = 0$

Using the last two equation we get $x_1 = 1.5$ and $x_2 = 2.5$

Replacing in the first two equation we get the values of lambdas

$\lambda_1 = 6$ $\lambda_2 = -1$

This is not a solution because it violates the condition $\lambda_2 \ge 0$.

Optimization with inequality constraints: non negativity constraints

The general form of such a problem is:

$$\max_x f(x) \text{ subject to}$$
$$g_j(x) \leq c_j \text{ for } j = 1, ..., m \text{ and}$$
$$x_i \geq 0 \text{ for } i = 1, ..., n.$$

Lagrangean is

$$L(x) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j) - \sum_{j=1}^{n} \lambda_{m+j}(-x_j)$$

It is a special case of the general maximization problem with inequality constraints: the nonnegativity constraint on each variable is simply an additional inequality constraint.

Specifically, if we define the function g_{m+i} for $i = 1, ..., n$ by $g_{m+i}(x) = -x_i$ and let $c_{m+i} = 0$ for $i = 1, ..., n$, then we may write the problem as

$$\max_x f(x) \text{ subject to}$$
$$g_j(x) \leq c_j \text{ for } j = 1, ..., m+n$$

and solve it using the Kuhn-Tucker conditions

Optimization with inequality constraints: non negativity constraints

Approaching the problem in this way involves working with $n + m$ Lagrange multipliers, which can be difficult if n is large.

Then we can use an alternative approach, the ***modified Lagrangean***

Consider the following problem:

$$\max_x f(x) \text{ subject to}$$
$$g_j(x) \leq c_j \text{ for } j = 1, ..., m \text{ and}$$
$$x_i \geq 0 \text{ for } i = 1, ..., n.$$

The *modified Lagrangean is:*

$$M(x) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j)$$

Kuhn-Tucker conditions for the modified Lagrangean:

$M_i'(x) \leq 0$, $x_i \geq 0$, and $x_i \cdot M_i'(x) = 0$ for $i = 1, ..., n$

$g_j(x) \leq c_j$, $\lambda_j \geq 0$, and $\lambda_j \cdot [g_j(x) - c_j] = 0$ for $j = 1, ..,m$.

in any problem for which the original Kuhn-Tucker conditions may be used, we may alternatively use the conditions for the modified Lagrangean.

For most problems in which the variables are constrained to be nonnegative, the Kuhn-Tucker conditions for the modified Lagrangean are easier than the conditions for the original Lagrangean

Example.

Consider the problem

$\max_{x,y} xy$ subject to $x + y \leq 6$, $x \geq 0$, and $y \geq 0$

Function xy is twice-differentiable and quasiconcave and the constraint functions are linear, so the Kuhn-Tucker conditions are necessary and if $((x^*, y^*), \lambda^*)$ satisfies these conditions and no partial derivative of the objective function at (x^*, y^*) is zero then (x^*, y^*) solves the problem.

Solutions of the Kuhn-Tucker conditions at which all derivatives of the objective function are zero may or may not be solutions of the problem

We try to solve it

1) using the lagrangean
2) Using the modified lagrangean

1) Using Lagrangean

$$L(x, y) = xy - \lambda_1(x + y - 6) - \lambda_2(-x) - \lambda_3(-y)$$

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) = 0$$

We have to consider the following 8 cases:

1) $\lambda_1 = 0$ $\lambda_2 = 0$ $\lambda_3 = 0$
2) $\lambda_1 > 0$ $\lambda_2 = 0$ $\lambda_3 = 0$
3) $\lambda_1 = 0$ $\lambda_2 > 0$ $\lambda_3 = 0$
4) $\lambda_1 > 0$ $\lambda_2 > 0$ $\lambda_3 = 0$
5) $\lambda_1 = 0$ $\lambda_2 = 0$ $\lambda_3 > 0$
6) $\lambda_1 > 0$ $\lambda_2 = 0$ $\lambda_3 > 0$
7) $\lambda_1 = 0$ $\lambda_2 > 0$ $\lambda_3 > 0$
8) $\lambda_1 > 0$ $\lambda_2 > 0$ $\lambda_3 > 0$

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \qquad (-x) \leq 0, \qquad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \qquad (-y) \leq 0, \qquad \lambda_3(-y) = 0$$

Case 1: $\lambda_1 = 0$ $\lambda_2 = 0$ $\lambda_3 = 0$

Kuhn Tucker conditions are:

$$y = 0$$

$$x = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \qquad (-x) \leq 0, \qquad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \qquad (-y) \leq 0, \qquad \lambda_3(-y) = 0$$

All conditions are satisfied, but the first derivatives of the objective function, evaluated at x=y=0 are equal to zero. Then this could be a solution.

Kuhn Tucker conditions are: $y - \lambda_1 + \lambda_2 = 0$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \qquad (-x) \leq 0, \qquad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \qquad (-y) \leq 0, \qquad \lambda_3(-y) = 0$$

Consider now $\lambda_1 = 0$

Kuhn Tucker conditions are:

$$y + \lambda_2 = 0$$

$$x + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \qquad (-x) \leq 0, \qquad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \qquad (-y) \leq 0, \qquad \lambda_3(-y) = 0$$

Then $\lambda_2 = -y$ and $x = -\lambda_3$. If λ_2 (λ_3) is strictly positive, then y (x) is strcitly negative and does not satisfy the last two conditions.

This allows us to eliminate all combinations where $\lambda_1 = 0$ and at least one among λ_2 and λ_3 is strictly positive, then combinations 3, 5, 7

Then we have to check only the combinations 2, 4, 6, 8

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \qquad (-x) \leq 0, \qquad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \qquad (-y) \leq 0, \qquad \lambda_3(-y) = 0$$

Case 2) $\lambda_1 > 0$ $\lambda_2 = 0$ $\lambda_3 = 0$

Kuhn Tucker conditions are:

$$y - \lambda_1 = 0$$

$$x - \lambda_1 = 0$$

$$\lambda_1 \geq 0, x + y = 6,$$

$$(-x) \leq 0,$$

$$(-y) \leq 0,$$

From the first 3 conditions we have that x = y = 3 and λ_1=3

These values satisfy the last conditions and it the derivatives of objective function evaluated in this point are different from zero.

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \qquad (-x) \leq 0, \qquad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \qquad (-y) \leq 0, \qquad \lambda_3(-y) = 0$$

Case 4) $\lambda_1 > 0$ $\lambda_2 > 0$ $\lambda_3 = 0$

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

$$(-x) = 0, \qquad \lambda_2(-x) = 0$$

$$(-y) \leq 0$$

From condition in the 4^{th} line we have x =0,

replacing in the second line we get $\lambda_1 = 0$, a contradiction with the initial assumption of $\lambda_1 > 0$

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \qquad (-x) \leq 0, \qquad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \qquad (-y) \leq 0, \qquad \lambda_3(-y) = 0$$

Case 6) $\lambda_1 > 0$ $\lambda_2 = 0$ $\lambda_3 > 0$

The first two conditions are

$$y - \lambda_1 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

and the last is implies y=0.

Replacing it in the first line we find that $\lambda_1 = 0$, a contradiction with the initial assumption of $\lambda_1 > 0$

Kuhn Tucker conditions are: $$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) = 0$$

Case 8) $\lambda_1 > 0$ $\lambda_2 > 0$ $\lambda_3 > 0$

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$x + y = 6$$

$$x = 0 \quad y = 0$$

From the last three conditions one contradiction arises

Two possible solutions

1) x = 0 and y = 0

2) x = 3 and y = 3

The second one produces the higher value of the objective function, then it is the solution of the problem

2) Using the modified lagrangean

$$M(x, y) = xy - \lambda_1(x + y - 6)$$

Kuhn-Tucker conditions for the modified Lagrangean:

$$x \geq 0, \qquad y - \lambda_1 \leq 0 \qquad x(y - \lambda_1) = 0$$

$$y \geq 0 \qquad x - \lambda_1 \leq 0 \qquad y(x - \lambda_1) = 0$$

$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

Kuhn-Tucker conditions for the modified Lagrangean:

$$x \geq 0, \qquad y - \lambda_1 \leq 0 \qquad x(y - \lambda_1) = 0$$
$$y \geq 0 \qquad x - \lambda_1 \leq 0 \qquad y(x - \lambda_1) = 0$$
$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

Consider a case where x=0 and y=0, then:

$$-\lambda_1 \leq 0$$
$$-\lambda_1 \leq 0$$
$$\lambda_1 \geq 0, \qquad x + y \leq 6, \qquad \lambda_1(x + y - 6) = 0$$

These conditions are satisfied only for $\lambda_1 = 0$

Then x=0 y=0 is a candidate to the solution (the derivatives of the objective function are equal to zero in this point)

Consider a case where x>0 and y=0, then:

Replacing these values in the first condition we get $\lambda_1 = 0$

Replacing $\lambda_1 = 0$ in the second condition we get $x \leq 0$

A contradiction with the initial assumption x>0.

Kuhn-Tucker conditions for the modified Lagrangean:

$$x \geq 0, \quad y - \lambda_1 \leq 0 \quad x(y - \lambda_1) = 0$$

$$y \geq 0 \quad x - \lambda_1 \leq 0 \quad y(x - \lambda_1) = 0$$

$$\lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) = 0$$

Consider a case where x=0 and y>0, then:

Replacing these values in the second condition we get $\lambda_1 = 0$

Replacing $\lambda_1 = 0$ in the first condition we get $y \leq 0$

A contradiction with the initial assumption y>0.

Consider the case x>0 and y>0

$$y - \lambda_1 = 0$$

$$x - \lambda_1 = 0$$

$$\lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) = 0$$

Then $y = x = \lambda_1 > 0$ then the last condition implies$x + y = 6$ and then x=y=3

As in the procedure of the KT condition