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Lecture 5 – Unconstrained Optimization and Quadratic Forms

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We consider the unconstrained optimization for the case of functions with many variables:

$$\max_x f(x) \text{ subject to } x \in S$$

where x is a vector

To face this topic we need some preliminariy notions:

- Quadratic forms
- Concavity and convexity of functions of many variables

Definition of quadratic forms

A form is a polynomial function in which each component has the same exponent sum:

- a linear form is f(x,y,z) = 4x – 9y + z

(each term has exponents that add to one (the “first degree”)

- a quadratic form is f(x y z) = $4x^2 + 2zy – xz + 2z^2$

(each term has exponents that add to two (the “second degree”)

A polynomial equation in which ***each term*** is of the 2nd degree (sum of the integer exponents = 2) is a quadratic form

Definition

A **quadratic form** in *n* variables is a function

$$Q(x_1, \dots . x_n) = b_{11}x_1^2 + b_{12}x_1x_2 + \cdots + b_{ij}\, x_i x_j + \cdots + b_{nn}x_n^2 =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j$$

where b_{ij} for $i = 1, \dots n$ and $j = 1, \dots n$ *n* are constants.

Example

The function

$$Q(x_1, x_2) = {x_1}^2 + 2x_1x_2 - 3x_2x_1 + 5{x_2}^2$$

is a quadratic form in two variables.

We can write it using matrices

$$Q(x_1, x_2) = (x_1 \quad x_2)\begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note: we can simplify this function

$$Q(x_1, x_2) = {x_1}^2 - x_2x_1 + 5{x_2}^2$$

That can be written as:

$$Q(x_1, x_2) = (x_1 \quad x_2)\begin{pmatrix} 1 & -0.5 \\ -0.5 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Where the matrix is symmetric.

In general we can in fact write any quadratic form as

$$Q(x) = x'Ax$$

where

- x is the column vector of x_i's and
- A is a symmetric $n \times n$ matrix for which the (i, j)th element is

$$a_{ij} = (1/2)(b_{ij} + b_{ji})$$

note that $x_i x_j = x_j x_i$ for any i and j, so that

$$b_{ij}x_i x_j + b_{ji}x_j x_i$$

can be written as

$$(b_{ij}+b_{ji})x_i x_j$$

or

$$\frac{1}{2}(b_{ij} + b_{ji})x_i x_j + \frac{1}{2}(b_{ij} + b_{ji})x_j x_i$$

Example

$$Q(x_1, x_2) = {x_1}^2 + a x_1 x_2 + b x_2 x_1 - c x_1 x_3 + 5{x_2}^2$$

$$Q(x_1, x_2) = (x_1 \quad x_2 \quad x_3) \begin{pmatrix} 1 & \frac{a+b}{2} & -\frac{c}{2} \\ \frac{a+b}{2} & 5 & 0 \\ -\frac{c}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x2 \\ x_3 \end{pmatrix}$$

Conditions for definiteness

With quadratic forms there are ways of establishing whether their signs are positive or negative and this will help determine whether the function of interest is concave or convex

Definition

Let $Q(x)$ be a quadratic form, and let A be the symmetric matrix that represents it (i.e. $Q(x) = x'Ax$).

Then the associated matrix A (and the quadratic form) is:

1. positive definite if $x'Ax > 0$ for all $x \neq 0$
2. negative definite if $x'Ax < 0$ for all $x \neq 0$
3. positive semidefinite if $x'Ax \geq 0$ for all x
4. negative semidefinite if $x'Ax \leq 0$ for all x
5. indefinite if it is neither positive nor negative semidefinite (i.e. if $x'Ax > 0$ for some x and $x'Ax < 0$ for some x).

Examples

1) $ax_1{}^2 + cx_2{}^2 = (x_1 \quad x_2)\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is positive definite for $a, c > 0$
because $ax_1{}^2 + cx_2{}^2 > 0$ for $a, c > 0$ and $(x_1 \quad x_2) \neq 0$

2) $x_1{}^2 + 2x_1x_2 + x_2{}^2 = (x_1 \quad x_2)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is positive semidefinite
because we can write it as $(x_1 + x_2)^2$ that is non negative for all x_1, x_2
It is not positive definite because for $x_1 = 1, x_2 = -1$ its value is 0.

3) Prove that the form $x_1{}^2 - x_2{}^2 + ax_1x_2$ is indefinite for any value of a

Positive or Negative definite matrices

To obtain conditions for an *n*-variable quadratic form to be positive or negative definite, we need to examine the determinants of some of its submatrices.

Definition:

The ***leading principal matrices*** of a nxn square matrix are the matrices found by deleting

1. The last n-1 rows and columns – to give D_1
2. The last n-2 rows and columns – to give D_2
3. …
4. and the original matrix – D_n

Definition:

The leading principal ***minors*** of a matrix are the determinants of these leading principal matrices.

Example:

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

the leading principal matrices are then

D_2 = A and (D_1 = 1)

and the determinants (principal minors) are

5 and 1

Example 2.

Find D_1 D_2 and D_3

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 2 & -1 & 0 \end{pmatrix}$$

If a square matrix is negative definite then the leading principal minors have the following signs

$$\left|D_1\right| < 0; \left|D_2\right| > 0; \left|D_3\right| < 0 \ldots$$

a positive definite matrix requires leading principal minors are **all** positive, i.e.

$$\left|D_1\right| > 0; \left|D_2\right| > 0; \left|D_3\right| > 0 \ldots$$

To check if a square matrix is negative semi-definite we have to compute all principal minors (not only the leading principal minors)

Positive or Negative semidefinite matrices

To obtain conditions for an *n*-variable quadratic form to be positive or negative semidefinite, we need to examine the determinants of some of its submatrices.

Definition:

The ***principal matrices*** of a nxn square matrix are the matrices found by deleting

1. n-1 rows and columns – in all possible combinations
2. n-2 rows and columns – – in all possible combinations
3. …
4. and the original matrix

Definition:

The principal ***minors*** of a matrix are the determinants of the principal matrices.

Let

$$A = \begin{matrix} a & b \\ b & c \end{matrix}$$

The first-order principal minors of A are a and c, and the second-order principal minor is the determinant of A, namely $ac - b^2$.

Let

$$A = \begin{matrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{matrix}$$

This matrix has 3 first-order principal minors, obtained by deleting

- the last two rows and last two columns
- the first and third rows and the first and third columns
- the first two rows and first two columns

which gives us simply the elements on the main diagonal of the matrix: 3, −1, and 2.

The matrix also has 3 second-order principal minors, obtained by deleting

- the last row and last column
- the second row and second column
- the first row and first column

which gives us −4, 2, and −11.

The matrix has one third-order principal minor, namely its determinant, −19.

Let A be an $n \times n$ symmetric matrix. Then:

A is positive semidefinite if and only if **all** the principal minors of A are nonnegative.

A is negative semidefinite if and only if all the kth order principal minors of A are ≤ 0 if k is odd and ≥ 0 if k is even.

Example

$$\begin{matrix} 1 & 0 \\ 4 & 0 \end{matrix}$$

The two first-order principal minors and 0 and 1, and the second-order principal minor is 0. Thus the matrix is positive semidefinite.

Procedures for checking the definiteness of a matrix

1. Find the leading principal minors and check if the conditions for positive or negative definiteness are satisfied. If they are, you are done.
2. the conditions are not satisfied, check if they are *strictly* violated. If they are, then the matrix is indefinite.
3. If the conditions are not strictly violated, find all its principal minors and check if the conditions for positive or negative semidefiniteness are satisfied.

Note that if a matrix is positive definite, it is certainly positive semidefinite, and if it is negative definite, it is certainly negative semidefinite

Three Variable Quadratic Forms

Can always write a quadratic form in 3 variables

$q = d_{11}x^2 + d_{12}xy + d_{13}xz + d_{21}yx + d_{22}y^2 + d_{23}yz + d_{31}zx + d_{32}zy\ d_{33}z^2$

in matrix form x'Ax where x = (x, y, z) and A is a symmetric 3 by 3 matrix

T

$$x'Ax = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

here are now 3 leading principal minors from the discriminants of A

$$|D_1| = |d_{11}|; |D_2| = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}; |D_3| = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

$$|D_1| = |d_{11}| = d_{11}$$

$$|D_2| = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} = d_{11}d_{22} - d_{21}d_{12} = d_{11}d_{22} - d_{21}{}^2$$

$$|D_3| = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} =$$

$$= d_{11}d_{22}d_{33} + d_{12}d_{23}d_{31} + d_{13}d_{21}d_{32} - d_{31}d_{22}d_{13} - d_{23}d_{32}d_{11} - d_{33}d_{12}d_{21}$$

$$= d_{11}d_{22}d_{33} + 2d_{12}d_{23}d_{13} - d_{22}d_{13}{}^2 - d_{11}d_{23}{}^2 - d_{33}d_{12}{}^2$$

Once again can convert into an expression where the 3 variables appear only as squared terms

$$q = d_{11}\left(x + \frac{d_{12}}{d_{11}} y + \frac{d_{13}}{d_{11}} z \right)^2 + \frac{d_{11}d_{22} - d_{12}^2}{d_{11}} \left(y + \frac{d_{11}d_{23} - d_{12}d_{13}}{d_{11}d_{22} - d_{12}^2} z \right)^2 +$$

$$\frac{d_{11}d_{22}d_{33} - d_{11}d_{23}^2 - d_{22}d_{13}^2 - d_{33}d_{12}^2 + 2d_{12}d_{13}d_{23}}{d_{11}d_{22} - d_{12}^2} (z)^2$$

And can show that q <0 (>0) iff the terms outside the brackets are all negative (positive)

and these terms are respectively:

$$|D_1|; \frac{|D_2|}{|D_1|}; \frac{|D_3|}{|D_2|}$$

If

$|D_1| < 0, |D_2| > 0, |D_3| < 0$

the matrix is said to be negative definite

if

$|D_1| > 0, |D_2| > 0, |D_3| > 0$

the matrix is said to be positive definite

A second test to check definiteness

Characteristic root test

Given an $n \ x \ n$: matrix D, we find a scalar r and an $n \ x$ 1vector $x \neq 0$ such that:

$$D\,x = r\,x$$

r is the characteristic root of matrix D (or eigenvalue)

x is the characteristic vector of matrix D (or eigenvector)

This equation is rewritten as:

$$(D - rI)\,x = 0$$

The condition that satisfies this is if the matrix
$(D - rI)$ is singular; i.e., its determinant is zero

The idea is to solve for r and then x

Example

$$D = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

$$D - rI = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-r & 2 \\ 2 & -1-r \end{bmatrix}$$

$$|D - rI| = \begin{vmatrix} 2-r & 2 \\ 2 & -1-r \end{vmatrix} = r^2 - r - 6 = 0$$

So the characteristic roots are $r_1 = 3$ and $r_2 = -2$

For $r_1 = 3$

$$(D - rI)x = 0 = \begin{bmatrix} 2-3 & 2 \\ 2 & -1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note that the rows of the matrix are linearly dependent – as expected for a singular matrix – giving an infinite number of solutions $x_1 = 2x_2$

To force out a unique solution, we need to ***normalise*** by imposing a restriction:

$$x_1^2 + x_2^2 = 1$$

and in general for n unknowns $\sum_{i=1}^{n} x_i^2 = 1$

- This is arbitrary but whichever rule is chosen, all subsequent values will be related

Then

$$x_1^2 + x_2^2 = (2x_2)^2 + x_2^2 = 5x_2^2 = 1$$

and

$$x_2 = 1/\sqrt{5}; x_1 = 2/\sqrt{5}$$

Thus, the 1st characteristic vector (eigenvector) is $x1 = \left(\frac{2}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \right)'$

and for $r = -2$, the 2nd characteristic vector (eigenvector) is $x2 = \left(\frac{-1}{\sqrt{5}} \quad \frac{2}{\sqrt{5}} \right)'$

Properties:

1) normalisation implies that the product of characteristic vectors, i.e. x1'x1 = 1
2) Each pair of characteristic vectors are orthogonal, i.e. x1' x2 = 0

Characteristic root test for the sign definiteness of a matrix D

1. D is positive definite if and only if every characteristic root is positive, i.e. > 0
2. D is negative definite if and only if every characteristic root is negative , i.e. < 0
3. D is positive definite if and only if every characteristic root is nonnegative, i.e. ≥ 0
4. D is negative definite if and only if every characteristic root is nonpositive, i.e. ≤ 0

Finding if a function with more variables is concave

When the function x consists of more than 1 variable the concavity condition is very similar – we replace the variable x by the **vector** x, and remember to get the order of multiplication right.

Now

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad (x' - x_0) = \begin{pmatrix} x_1' - x_{10} \\ \vdots \\ x_n' - x_{n0} \end{pmatrix}$$

We also need the ***vector*** of first partial derivatives of f,

and the ***matrix*** of second order partial derivatives, H

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}; \quad H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

J is called ***Jacobian*** of the function *f*

H is called the ***Hessian*** of the function *f*

The concavity condition is now:

$$f(x_0) + (x' - x_0)'\nabla f \geq f(x')$$

The Taylor approximation of $f(x')$ is now

$$f(x') \approx f(x_0) + (x' - x_0)'\nabla f + \frac{1}{2}(x' - x_0)'H(x' - x_0) + \cdots$$

Replacing in the first equation we get

$$f(x_0) + (x' - x_0)'\nabla f \geq f(x_0) + (x' - x_0)'\nabla f + \frac{1}{2}(x' - x_0)'H(x' - x_0) + \cdots$$

Simplifying we get

$$0 \geq (x' - x_0)'H(x' - x_0)$$

Then matrix H has to be a negative semi-definite matrix

Let f be a function of many variables with continuous partial derivatives of first and second order on the convex open set S and denote the Hessian of f at the point x by $H(x)$. Then

- f is concave **if and only if** $H(x)$ is negative semidefinite for all $x \in S$
- if $H(x)$ is negative definite for all $x \in S$ then f is strictly concave
- f is convex **if and only if** $H(x)$ is positive semidefinite for all $x \in S$
- if $H(x)$ is positive definite for all $x \in S$ then f is strictly convex.

Note to say that f is concave (convex) we need to prove that H(x) is negative (positive) definite

Putting it all together

So given a function f(x)

To find out whether the function is concave we need to know if

$$0 \geq (x' - x_0)' H (x' - x_0)$$

i.e. whether H is negative semi-definite

1.Find the Hessian matrix of second order derivatives, H

2.From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – written as D_1
2. The last n-2 rows and columns – written as D_2
3. …
4. The original matrix - D_n

3. Compute the determinants of these leading principal matrices

4. if the determinants have the following pattern (with not all zero): $|D_1| < 0, |D_2| > 0,$ $|D_3| < 0$ …….., then f is concave; if the determinants are all strictly positive then f is convex

5. if some condition is violated by equality you need to check the sign of all principal minors (condition or semidefiniteness)

6. if these conditions do not hold you've proved that the function is not concave or convex

Find whether the function $f(x) = -x_1x_2^2$ is concave

We need the Hessian matrix of second order derivatives, H

- The Jacobian is

$$\begin{pmatrix} \dfrac{df}{dx_1} \\ \dfrac{df}{dx_2} \end{pmatrix} = \begin{pmatrix} -x_2^2 \\ -2x_1x_2^2 \end{pmatrix}$$

- The Hessian is

$$H = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 0 & -2x_2 \\ -2x_2 & -2x_1 \end{pmatrix}$$

$$H = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 0 & -2x_2 \\ -2x_2 & -2x_1 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – written as $D_1 = (0)$
2. The last n-2 rows and columns – written as $D_2 = H$

Compute the determinants of these leading principal matrices.

1. Det. $D_1 = 0$
2. Det. $H = -4x_2^2$ which is negative

f is concave if the leading principal minors are $|D_1| < 0; |D_2| > 0;$

f is convex if the leading principal minors are $|D_1| > 0; |D_2| > 0;$

Leading principal minors do not have one of this patterns so f is not concave, not convex

UNCONSTRAINED OPTIMIZATION WITH MORE VARIABLE

We generalize the results for a single variable to the case of many variables

Consider the problem:

$$max_x\, f\,(x)\ subject\ to\ x\ \in S$$

where x is a vector

<u>Proposition</u>

Let *f* be a differentiable function of *n* variables defined on the set *S*. If the point *x* in the interior of *S* is a local or global maximizer or minimizer of *f* then

$$f_i{}'(x)\ =\ 0\ for\ i\ =\ 1, \ldots, n.$$

(Note, $f_i{}'$ means the i^th^ partial derivative of f)

Then the condition that all partial derivatives are equal to zero is a necessary condition for an interior optimum (and therefore for an optimum in an unconstrained optimization where each element of x could be any of the real numbers.

Conditions under which a stationary point is a local optimum

Let f be a function of n variables with continuous partial derivatives of first and second order, defined on the set S.

Suppose that x^* is a stationary point of f in the interior of S (so that $f_i'(x^*) = 0$ for all i).

If $H(x^*)$ is negative definite then x^* is a local maximizer.

If x^* is a local maximizer then $H(x^*)$ is negative semidefinite.

If $H(x^*)$ is positive definite then x^* is a local minimizer.

If x^* is a local minimizer then $H(x^*)$ is positive semidefinite.

where $H(x)$ denotes the Hessian of f at x.

Conditions under which a stationary point is a global optimum

Suppose that the function f has continuous partial derivatives in a convex set S and let x be in the interior of S.

1. if f is concave then x is a global maximizer of f in S if and only if it is a stationary point of f
2. if f is convex then x is a global minimizer of f in S if and only if it is a stationary point of f .

$H(z)$ is negative semidefinite for all $z \in S \Rightarrow$ [x is a global maximizer of f in S if and only if x is a stationary point of f]

$H(z)$ is positive semidefinite for all $z \in S \Rightarrow$ [x is a global minimizer of f in S if and only if x is a stationary point of f],

where $H(x)$ denotes the Hessian of f at x.

Example 1: Unconstrained Maximization with two variables

For example Utility = U(x, y) or Output = F(K, L)

Now try to find the values of x and y which maximise a function $f(x, y)$

Three steps:

1. Set **both** 1st order conditions equal to zero $f_x = 0$ and $f_y = 0$

(the slope of the function with respect to both variables must be simultaneously zero)

2. Solve the equations simultaneously for x and y

However this is a necessary but not sufficient condition (saddle points, points of inflection)

3. Second order conditions

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

$f_{xx} \leq 0,\ f_{yy} \leq 0$ and $f_{xx}f_{yy} - f^2_{xy} \geq 0$

$f(x,y) = 4x - 2x^2 + 2xy - y^2$

1. (i). $f_x = 4 - 4x + 2y = 0$
   (ii). $f_y = 2x - 2y = 0$

2. Solve: from (ii) we have x = y
   insert into (i) to get 4 – 4x + 2x = 0 or
   4 = 2x or x = 2
   so y = x = 2

3. $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -2 \end{pmatrix}$

The first order leading principal minor is $f_{xx} = -4 < 0$

The second order leading principal minor is $f_{xx}f_{yy} - f^2_{xy} = (-4)(-2) - (2)^2 = 4 > 0$

Then the matrix H is negative definite

f is (strictly) concave, so we have a maximum point where x = 2 and y = 2

Quiz

Maximize

1. $z = -y^2 - 2x^2 + xy$

2. $z = -y^2 - 2x^2 + 2xy$

Example 2

Maximize $f(x) = -x_1^2 - 2x_2^2$

The first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -4x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Is this a maximum? – it will be if function is concave

1. H is,

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – written as D_1 = (-2)
2. The last n-2 rows and columns – written as D_2 = H

Compute the determinants of these leading principal matrices.

1. $|D_1| = -2$
2. $|H| = 8$

Then the matrix H is negative definite

f is (strictly) concave

the values of x which satisfy FOC (0 and 0) give a maximum.

Example 3

Total revenue R = $12q_1 + 18q_2$

Total Cost = $2q_1^2 + q_1q_2 + 2q_2^2$

Find the values of q_1 and q_2 that maximise profit

Profit = revenue – cost = $12q_1 + 18q_2 - (2q_1^2 + q_1q_2 + 2q_2^2)$

The first order conditions are:

$$\begin{pmatrix} \dfrac{\partial \pi}{\partial q_1} \\ \dfrac{\partial \pi}{\partial q_2} \end{pmatrix} = \begin{pmatrix} 12 - 4q_1 - q_2 \\ 18 - q_1 - 4q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving for q_1 and q_2 gives $q_1 = 2$ and $q_2 = 4$

Is this a maximum? –it will be if function is concave

The Hessian is

$$H = \begin{pmatrix} \dfrac{\partial^2 \pi}{\partial q_1^2} & \dfrac{\partial^2 \pi}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 \pi}{\partial q_2 \partial q_1} & \dfrac{\partial^2 \pi}{\partial q_2^2} \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – written as $D_1 = (-4)$
2. The last n-2 rows and columns – written as $D_2 = H$

Compute the determinants of these leading principal matrices.

1. $|D_1| = -4$
2. $|H| = (-4) * (-4) - 1 = 15$

So H is negative definite, then f is (strictly) concave and the values for q_1 and q_2 maximise profits

Example with three variables

Maximize $f(x) = -x_1^2 - 2x_2^2 - x_3^2$

The first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -4x_2 \\ -2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Hessian is:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – $D_1 = (-2)$
2. The last n-2 rows and columns – $D_2 = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$
3. The last 0 rows and columns – $D_3 = H$

1. Compute the determinants of these leading principal matrices.

1. $|D_1| = -2,$
2. $|D_2| = 8$
3. $|H| = -16$

H is negative definite, then f is (strictly) concave

Summing up – two variable maximization

1. Differentiate f(x) and solve the first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. Check concavity of f to see if the conditions represent a maximum.
   a. We compute the Hessian
   b. We check if it is negative definite
   c. i.e. check if, for all x_1 and x_2, $\frac{\partial^2 f}{\partial x_1^2} < 0$

and $$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} or \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} > 0$$

3. If these conditions hold, H is negative definite, f is strictly concave and the stationary point is a maximum

4. If these conditions are violated by equality, i.e. are equal to zero, check the conditions for semi definiteness

$$\frac{\partial^2 f}{\partial x_1^2} \leq 0 \qquad \frac{\partial^2 f}{\partial x_2^2} \leq 0 \qquad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} \geq 0$$

5. If these conditions hold, H is negative semidefinite, f is concave and the stationary point is a maximum

6. If these conditions are violated, we need further investigation

Summing up – 3 variable maximization

1. Differentiate f(x) and solve the the first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2. Check concavity of f to see if the conditions represent a maximum.

   a. We compute the Hessian

   b. We check if it is negative definite

$$\frac{\partial^2 f}{\partial x_1^2} < 0 \qquad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} or \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} > 0$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = f_{11} \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} - f_{12} \begin{vmatrix} f_{21} & f_{23} \\ f_{31} & f_{33} \end{vmatrix} + f_{13} \begin{vmatrix} f_{21} & f_{22} \\ f_{31} & f_{32} \end{vmatrix} < 0$$

3. If these conditions hold, H is negative definite, f is strictly concave and the stationary point is a maximum

4. If these conditions are violated by equality, i.e. are equal to zero, check the conditions for semi definiteness

$$f_{11} \leq 0, f_{22} \leq 0\ f_{33} \leq 0$$

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \geq 0 \begin{vmatrix} f_{11} & f_{13} \\ f_{31} & f_{33} \end{vmatrix} \geq 0 \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} \geq 0$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \leq 0$$

5. If these conditions hold, H is negative semidefinite, f is concave and the stationary point is a maximum

6. If these conditions are violated, we need further investigation