EC5555
Economics Masters Refresher Course in Mathematics
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Lecture 3 – Differentiation

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Rationale for Differentiation

Much of economics is concerned with optimisation

(maximise profits, minimise costs etc) or the comparison of different equilibrium states associated with a different set of parameters or exogenous variables

Underlying this is the concepts of calculus and in particular differentiation

(Note: these are topics you should already know so some of this will be revision)

e.g.

- Crime = f(unemployment, poverty, family structure, morals…)

  What happens to crime if unemployment rises?

- Tax revenue = f(VAT, income tax, state of the economy…)

- Sales of BMW X5 = f(own price, Price of Audi Q5,…)

We may be interested in the effect of a change in just one of these variables

But just the change is not enough, often we need to know

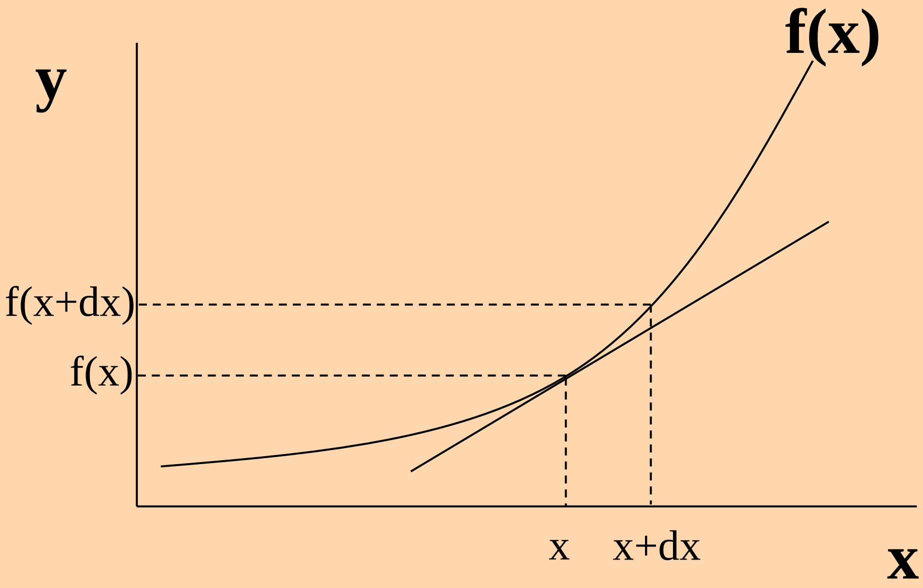
1) The direction of change (positive or negative)
2) The magnitude of change (by how much)

The outcome variable (dependent variable) changes in response to a change in one (or more) of the inputs (also called exogenous variables)

And this is where the role of the ***derivative*** comes in

Geometrically.

The derivative is the instantaneous slope of a function: y = f(x)



As the size of the change in the exogenous variable, dx, becomes smaller and smaller, the slope of a tangent at x comes closer and closer to

$$\frac{f(x+dx)-f(x)}{dx}$$

which measures the average rate of change in the value of the function in a small interval

More Formally: differentiation.

Let y = f(x)

i.e. y is a function of x

dx = a (small) change in x

Lim = ‘in the limit’

Lim dx→0 means ‘in the limit as dx approaches 0’

Then the ***derivative*** is defined as,

$$f'(x) = \frac{dy}{dx} = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$

(“derived” from the primary function y=f(x))

E.g. y = 2x

Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx} = \frac{2(x+dx) - 2x}{dx} = \frac{2dx}{dx} = 2$$

Could in principle find the derivative using this idea of a small difference

Given

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $y = f(x) = 3x^2 - 4$

$$f(x+\Delta x) = 3(x+\Delta x)^2 - 4$$

and

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{3(x+\Delta x)^2 - 4 - (3x^2 - 4)}{\Delta x} =$$

$$= \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = 6x + 3\Delta x$$

So if $$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x$$

and x= 2 and Δx= 3

then the average rate of change of y as x goes from 2 to 5 will be

6(2) + 3*3 = 21

ie 21 units of y for every unit change in x

Since the derivative is concerned with the change in y following a very small change in x then need to evaluate this as $\Delta x \rightarrow 0$

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (6x + \Delta x) = 6x$$

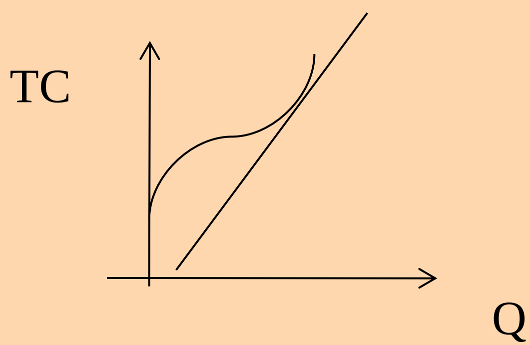
Remember also that the idea of change is analogous to the idea of marginal differences used regularly in Economics

Eg think of y = f(x) as capturing the relationship between Total Costs (y) and Output (x)

$$TC = f(Q)$$

Then the slope of this function (the derivative) gives the Marginal Cost (the change in Total costs resulting from a small change in output)

$$MC = f'(Q) = dTC/dQ$$



Eg Total Cost = $Q^3 - 4Q^2 + 10Q + 75$

Fixed Cost ?

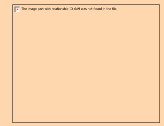
Marginal Cost ?

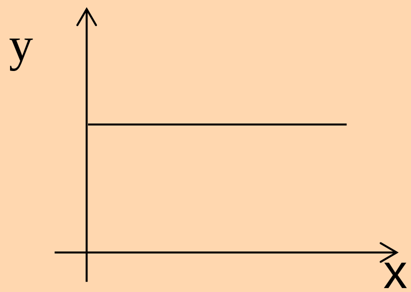
(might help if draw diagram)

Recalling rules.

Generally it is not necessary to work out the derivative from first principles – simple rules can be used instead

1. Constants y = a





2. Linear equations y = a + bx $\frac{dy}{dx} = b$

e.g. y = 2 + 4x

3. Power Functions $y = ax^n$

multiply by power, take one off the power: $\frac{dy}{dx} = nax^{n-1}$

4. Sum/Difference

If y = f(x) + g(x),

then $\frac{dy}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx} = f'(x) + g'(x)$

5. Product Rule

y = f(x)g(x)

$$\frac{dy}{dx} = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx} = f'(x)g(x) + g'(x)f(x)$$

eg 1 $y = x^2(2x+1)$

eg 2 AR (Average Revenue) = 15 – Q. Find Marginal Revenue

Since AR= TR/Q then TR=AR*Q = f(Q)g(Q) = -1(Q) + 1(15-Q) = 15-2Q

(Or just expand to find TR:

TR =15Q - Q^2 so MR = TR'(Q) = 15-2Q)

6. Quotient Rule: y = f(x)/g(x)

$$\frac{dy}{dx} = \frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Eg. Given Total Cost = C(Q) show that the slope of the AC curve will be positive iff the MC curve lies above the AC curve

Consider the rate of change in (slope of) the Average Cost Curve AC =C(Q)/Q

$$\frac{d}{dQ}\left(\frac{C(Q)}{Q}\right) = \frac{C'(Q)*Q - 1*C(Q)}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right]$$

$$\frac{d}{dQ}\left(\frac{C(Q)}{Q}\right) > 0$$

$$iff\ \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right] > 0$$

$$\Rightarrow C'(Q) > \frac{C(Q)}{Q}$$

7. Chain Rule (function of a function)

If we have a function y = f(x) and x is in turn a function of another variable z, x=g(z)

Then y = f(g(z))

and

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = f'(x)g'(z)$$

(intuitively if z changes there must be a change in x and if x changes there will be a change in y – a "chain" reaction)

Eg 1 $y = 3x^2$ and x = 2z + 5

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = f'(x)g'(z) = 6x * 2 = 12x = 12(2z+5)$$

Chain Rule (function of a function)

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = f'(x)g'(z)$$

Eg 2 The marginal revenue product of labour is defined as
MRPL = Marginal Revenue *Marginal Product of Labour

If total revenue is a function of the level of output, R = f(Q)
and output Q is in turn a function of the amount of labour employed L,
Q = g(L)

then

$$\frac{dR}{dL} = \frac{dR}{dQ}\frac{dQ}{dL} = f'(Q)g'(L)$$

Partial differentiation

Recall motivating examples

- Crime = f(unemployment, poverty, family structure, morals…)
  What happens to crime if U rises?

- Tax revenue = f(VAT, income tax, state of the economy…)

- Sales of the sun = f(own price, Mirror's price,…)

Real world outcomes consist of functions of more than one variable

Partial differentiation is the technique used to find the effect on the function of an infinitesimal change in one of the variables, keeping the values of all other variables constant.

More Formally

Let z = f(x,y)

i.e. z is a function of x and y.

Then the partial derivative of z with respect to x is given by,

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

(note use of δ rather than d when using partial derivatives)

What happens to y?

- Nothing

So treat it as a constant (since we are interested in how z changes as x changes holding the value of y constant)

$$\frac{\partial z}{\partial x} = \lim_{dx \to 0} \frac{f(x+dx, y) - f(x, y)}{dx}$$

E.g.1 $z = 3x^2 + xy + 4y^2$

Find, $\frac{\partial z}{\partial x} = \lim_{dx \to 0} \frac{3(x+dx)^2 + (x+dx)y + 4y^2 - (3x^2 + xy + 4y^2)}{dx}$

$$\frac{\delta z}{\delta x} = \lim_{dx \to 0} \frac{3(x^2 + 2xdx + dx^2) + (xy + dxy) + 4y^2 - (3x^2 + xy + 4y^2)}{dx}$$

$$= \lim_{dx \to 0} \frac{6xdx + dxy}{dx} = \lim_{dx \to 0} 6x + y$$

Informally, just treat y like a constant and differentiate wrt x using sum/difference rule (in this case)

$$\frac{\partial z}{\partial x} = 6x + y$$

Eg 2

A simple 2 input production function Q = F(K, L)

Then the partial derivative of output wrt labour gives the marginal product of labour – the change in the level of output wrt a small change in the amount of labour input, *holding the level of capital constant*

$$MP_{labour} = \frac{\partial Q}{\partial L}$$

Recalling rules

The rules are much the same as for differentiation:

1. Linear equations $z = ax + by + c$

   e.g. $yx + 4y + c$

2. Powers – multiply by power, take one off the power.

   e.g. $Z = 2x^2 \cdot y$

   e.g $z = y/x^2$

1. Products $z = g(x,y)h(x,y)$ $$\frac{\partial z}{\partial x} = \frac{\partial g}{\partial x} h + g \frac{\partial h}{\partial x}$$

   e.g. $Z = x^2(2x+y)$

2. Chain rule (function of a function) if $z = g(f(x,y))$ $$\frac{\partial z}{\partial x} = \frac{\partial g}{\partial f} \frac{\partial f}{\partial x}$$

Examples

1. $Z = 4xy - 2y$

   $\delta z / \delta x = 4y$

2. $Z = (2x+y)^3$

   $\delta z / \delta x = 6(2x+y)^2$

3. $Z = 4y/x^2$

   $\delta z / \delta x = -8y/x^3$

Your turn:

4. $Z = 400 + x + y$
5. $Z = 3y/x$
6. $Z = x^{0.4}y^{0.6}$

Digression 1: Logs and Exponentials

Introduction

(i) Consider £1 invested and receiving a rate of x in interest per annum (expressed in decimals, so x = 10% = 0.1)

At the end of one year, the payoff is

$$(1+x)^1 = 1+x$$

(ii) If instead x/2 is the rate of interest and it is paid half-yearly the payoff after one year is

$$(1+x/2)^2 = 1 + x + x^2/4$$

(iii) If instead x/n is the rate of interest and it is paid n times each year, the payoff after one year is,

$$\left(1+\frac{x}{n}\right)^n$$

(iv) the function e^x is defined as

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

(v) e is called 'exponential' and e = exp = 2.718

The exponential function

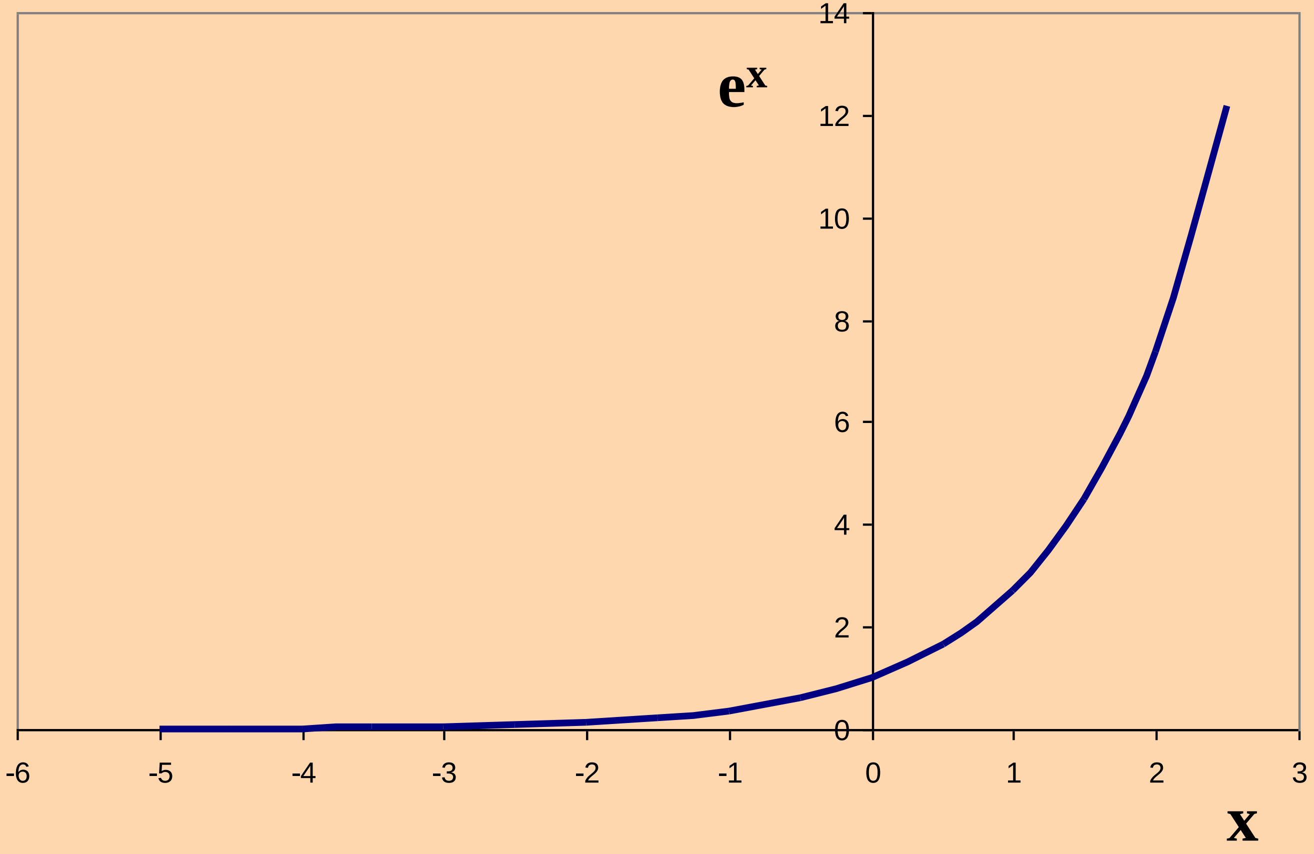
Strictly increasing

Always positive

$e^0=1$

$e^{a+b} = e^a e^b$

(This is a general property of powers – just the same as $x^2 x^4 = x^{2+4} = x^6$)



Quick Quiz

1. Sketch $y = e^{-t}$

2. Sketch $y = -e^{-t}$

For t = 0, 0.5 1 & 2

To what values do the two functions converge?

The exponential function

t = 0

Y = e(0) = 1 y= -e(-0) = -1

t = 0.5

Y = e(-0.5) = .606 y= -e(-0.5) = -.606

t = 1

Y = e(-1) = .368 y= -e(-0.5) = -.368

t = 2

Y = e(-2) = .135 y= -e(0.5) = -.135

Both asymptote to zero (one from above, one from below)

Economic Interpretation of *e*

Remember that e can be defined as $e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = 2.71828..$

$f(1) = \left(1 + \frac{1}{1}\right)^1 = 1 \quad f(2) = \left(1 + \frac{1}{2}\right)^2 = 2.25 \quad \ldots\ldots.$

which is also the value of £1 invested at an interest rate 100% (r=1) split (compounded) an infinite number of times during one year

$$V(m) \ = \ 1\left(1 + \frac{1}{m}\right)^m$$

and this can be generalised to any interest rate r and principal sum A over t years

$$V(m) = \ A\left(1 + \frac{r}{m}\right)^{mt}$$

To give the expression for continuous exponential growth, write the above expression as

$$V(m) = \ A\left[\left(1 + \frac{r}{m}\right)^{m/r}\right]^{rt} = A\left[\left(1 + \frac{1}{w}\right)^{w}\right]^{rt}$$

$$V = \lim_{m \to \infty} V(m) = Ae^{rt}$$

(and A =V/e^{rt} = Ve^{-rt} , making e^{-rt} the expression for the discount factor)

The log functions

If

$$b^y = x$$

then we can write

$$\log_b x = y$$

b is called “the base”

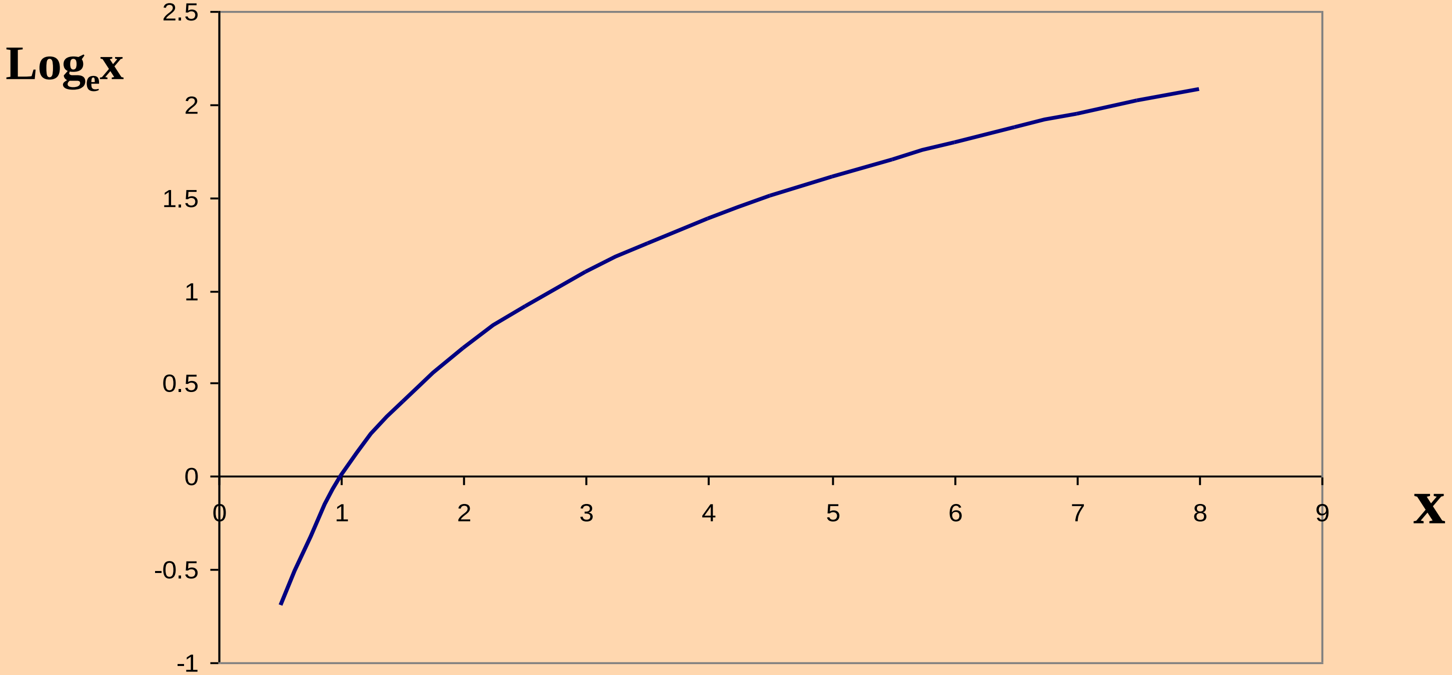
Two numbers are commonly chosen as bases, e and 10

If b=10 we use either $\log_{10}$ or $\log$

If b=e we use either $\ln_e$ or $\ln$

Log functions are often used in economics

- to define growth rates,
- estimate elasticities) and
- were particularly important in the days before calculators because one property of using logs means that multiplication can be turned into addition if one has a table of the log function



Remember also that Ln(e) = 1, and Ln(1) = 0

Rules on Logarithmic & Exponential Functions

$$y = \log_e x = \text{Ln } x$$

y is the "natural" log of x - the power to which e must be raised to get x

Rules of logs (the following rules are still valid if we replace ln by $\log_b$)

1. Log Product: $\ln(XY) = \ln(X) + \ln(Y)$

2. Log Quotient: $ln(X/Y) = ln(X) - ln(Y)$

3. Log Power: $ln(XY) = Yln(X)$

In economics logs are useful for transforming utility functions, in calculating elasticities and in transforming functions for econometric estimation

E.g. suppose $z = x^a y^b$. This is non linear and cannot be estimated using simple econometric techniques. But $\log(z) = a\log(x) + b\log(y)$. Thus log z is linear in $\log(x)$ and $\log(y)$ and can estimated using simple econometrics.

Proof of Log Product

Since $\ln(X)$ is the power to which e must be raised to get X then $e^{ln(X)} = X$

Similarly for any exponential raised to any log value $e^{ln(XY)} = XY$

But: 1) $e^{ln(X)}e^{ln(Y)} = YX$ 2) $e^{ln(X)}e^{ln(Y)} = e^{ln(X)+ln(Y)}$

Then: $e^{ln(X)+ln(Y)} = XY$

Then: $e^{ln(X)+ln(Y)} = e^{ln(XY)}$

Taking logs of both sides

$$ln(X) + ln(Y) = ln(XY)$$

(It also follows that if a is a constant then $ln(aX) = ln(a) + ln(X)$)

Differential of Logarithmic & Exponential Functions

Log-Function Rule:

Given $y = \log_e x$

$$\frac{dy}{dx} = \frac{d}{dx}\log_e x = \frac{1}{x}$$

Exponential-function rule:

Given $y = e^x$

$$\frac{dy}{dx} = \frac{d}{dx}e^x = e^x$$

Proof: if $y = e^x$ then $x = \log_e y$ and $dx/dy = 1/y$ (from above)

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{1/y} = y = e^x$$

Log Chain Rule

Given $y = \log_e f(x)$

$$\frac{dy}{dx} = \frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}$$

Proof:

Let $y = \ln(u)$ and $u = f(x)$ - so that $y = \ln f(x)$

By the chain rule $$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} = \frac{d\log_e u}{du} * \frac{df(x)}{dx} = \frac{1}{u} * f'(x) = \frac{f'(x)}{f(x)}$$

Exponent Chain Rule

Given $y = e^{f(x)}$

$$\frac{dy}{dx} = e^{f(x)} f'(x)$$

Quiz

Differentiate the following

1. y = ln(3x)

Ans: use log function rule $\frac{dy}{dx} = \frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}$ to get 1/x

2. y = e^{4x}

Ans: use exp function rule $\frac{dy}{dx} = e^{f(x)} f'(x)$ to get 4 e^{4x}

3.y = ln(4 –x^2)

Ans: use log function rule $\frac{dy}{dx} = \frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}$ to get -2x/(4 –x^2)

Now you try differentiating wrt x

1. y = ln(ax)
2. e^{rx}
3. Ln(1-x)

Quiz

- Differentiate wrt x:

1. $z = 0.5\log(x)$
2. $Z = x\log(x)$
3. $Z = 4e^{2x}$

- Partially differentiate wrt x:

1. $z = y\log(x)$
2. $Z = 0.4\log(x) + 0.6\log(y)$
3. $Z = e^{-2x}+4y$

Second Derivatives: slope of the slope.

Aside on notation: $\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x^2}, \quad f_{xx}$

all mean the same thing:

the second partial derivative of f with respect to x

(differentiate twice)

And the cross-partial derivative, $\frac{\partial^2 z}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad f_{xy}$

Means differentiate with respect to x and then with respect to y

Total differentials and marginal rate of substitution.

Often we want to know the change in the variable of interest resulting from a change in ***all*** of the variables that influence it.

In this case we need the total differential:

Eg Savings = s(Income, interest rates) =s(Y, r)

We know that the effect of a change in income can be found from the partial derivative

$$\frac{\partial S}{\partial Y}$$

If change in savings following a small change in Y is given by the partial derivative then the total change in savings following a larger change in income dY is given by

$$\frac{\partial S}{\partial Y} * dY$$

Similarly for the effect on savings of a change in interest rates

So the total change (the “total differential”)

$$dS = \frac{\partial S}{\partial Y} dY + \frac{\partial S}{\partial r} dr$$

This also generalises to a function of n variables $U = U(x_1, x_2, ...x_n)$

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + ... + \frac{\partial U}{\partial x_n} dx_n$$

A special case of interest: Consider a utility function U = U(x, y)

Along an indifference curve dU = 0, so: $0 = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$

Or

$$\frac{dy}{dx} = -\frac{\partial U}{\partial x} \Big/ \frac{\partial U}{\partial y}$$

This is the slope of the indifference curve = Marginal Rate of Substitution and is equal to the ratio of marginal (partial) utilities of the 2 inputs

E.g. U = 0.4 ln(x) + 0.6 ln(y)

$$0 = \frac{0.4}{x} dx + \frac{0.6}{y} dy, \quad \text{So} \quad \frac{dy}{dx} = -\frac{0.4}{x} \Big/ \frac{0.6}{y}$$

$$\frac{dy}{dx} = -\frac{2y}{3x}$$

Note that we can differentiate totally any number of times

So that, for example, the 2nd-order total differential

$$d^2U = d(dU) = \frac{\partial(dU)}{\partial x_1}dx_1 + \frac{\partial(dU)}{\partial x_2}dx_2 + \ldots.\frac{\partial(dU)}{\partial x_n}dx_n$$

$$d^2U = \frac{\partial(U_{x_1}dx_1 + U_{x_2}dx_2 + \ldots)}{\partial x_1}dx_1 + \frac{\partial(U_{x_1}dx_1 + U_{x_2}dx_2 + \ldots)}{\partial x_2}dx_2 + \ldots.\frac{\partial(dU_{x_1}dx_1 + U_{x_2}dx_2 + \ldots)}{\partial x_n}dx_n$$

Digression 2:
Uses of Derivatives - Taylor’s theorem.

1. Introduction

What is Taylor’s expansion?

- It’s a means of approximating a general function in terms of its derivatives.

Why do we need it?

- Useful when we need an approximation: e.g. forecasting, maximizing ***complicated*** functions, programming optimization routines.
- Useful for stability analysis and for comparative statics

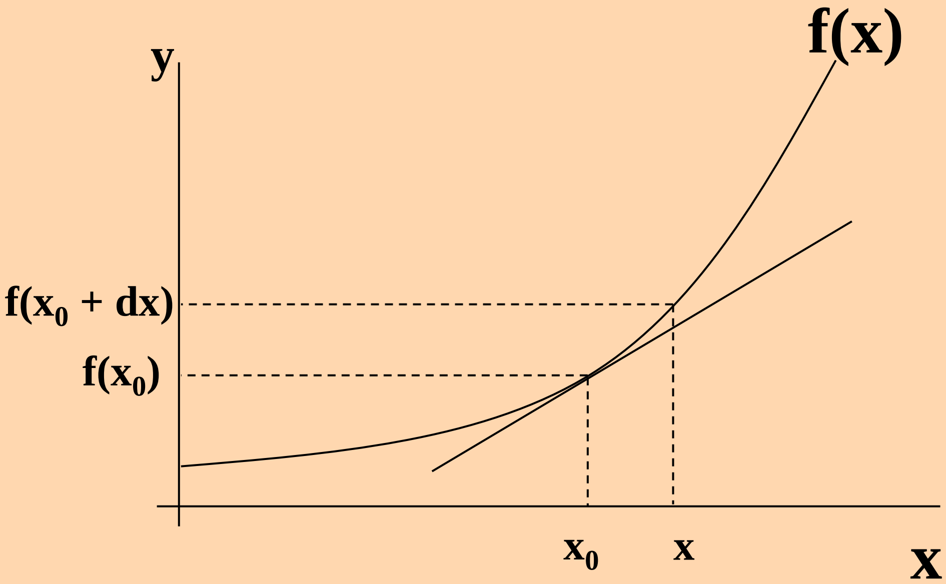
- Linear functions are easier to manipulate

Taylor's series.

Suppose we know y=f(x) at the value x_0 and its associated derivatives.

Suppose that for a small change in x, $dx = x - x_0$

We wish to estimate $f(x_0 + dx) = f(x_0 + (x-x_0))$



Since we know that

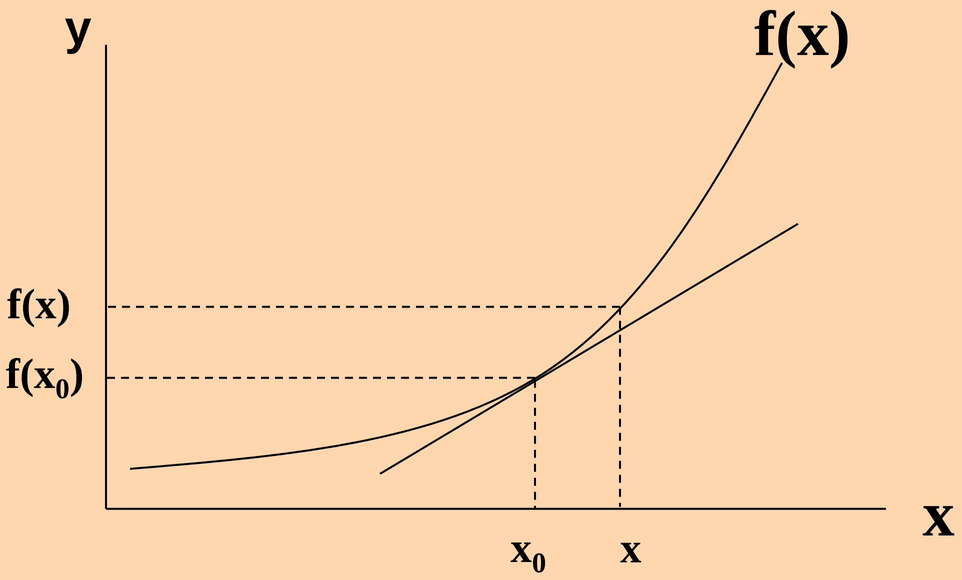
$$f\,'(x_0) = \frac{df}{dx} = \lim_{dx\to 0} \frac{f(x_0 + dx) - f(x_0)}{dx}$$

For $dx = x - x_0$ we can approximate the difference in the value of the function as: (difference in x *slope = difference in y)

$$(x - x_0)\, f\,'(x_0) \approx f(x) - f(x_0) \quad or \quad f(x) \approx f(x_0) + (x - x_0)\, f\,'(x_0)$$

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0)$$

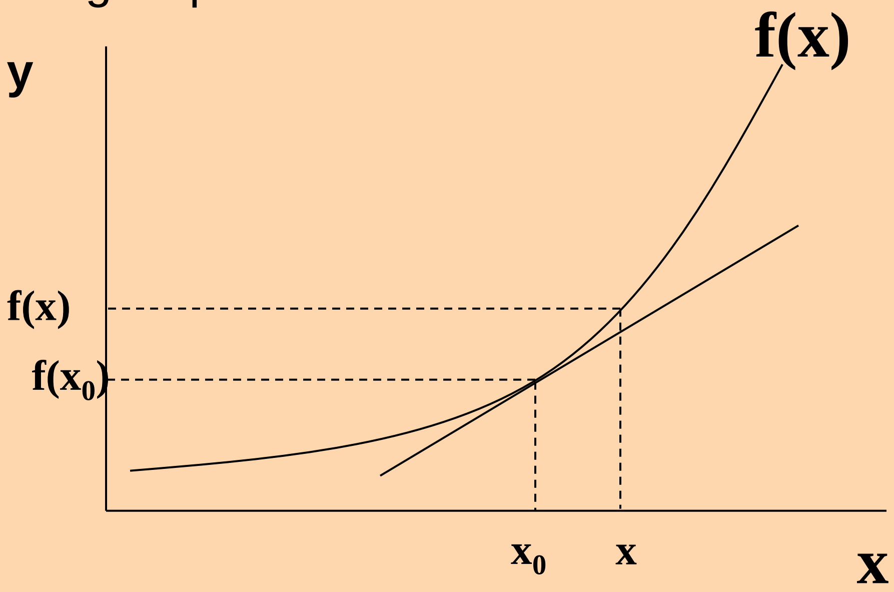
This is a linear function (the 2nd term is the equation of a tangent to the function at the point x_0). It is not the actual change in the value of the function, rather the change that would occur if y continued to change at a rate f'(x) for every change in x and the change in x was $x - x_0$



$$f(x) \approx f(x_0) + (x - x_0) f'(x_0)$$

has a number of properties:

1. When $dx = x - x_0 = 0$ the value of the function equals $f(x_0)$
   (and the values of the 1st derivatives also converge)
1. When $x - x_0 \neq 0$, the value of the function diverges from $f(x_0)$ (see diagram) – the approximation gets poorer.



So we might think that a better approximation can be had if we also matched the ***second derivative*** of f ie account for the changing direction of the slope 42 the essence of a Taylor series expansion

Taylor Expansion around x = 0:
The Maclaurin Expansion

More formally the ***expansion*** of a series means to evaluate (transform) a function as a polynomial in which the coefficients on the terms are the derivatives $f'(x_0)$, $f'(x_0)$, $f''(x_0)$ etc

Consider the polynomial function $f(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$ …..(1)

Follows that $f'(x) = a_1 + 2a_2x + \ldots na_nx^{n-1}$

$f''(x) = 2a_2 + \ldots n(n-1)a_nx^{n-2}$

:

$f^n(x) = n(n-1)(n-2)\ldots(3)(2)(1)$

Each successive differentiation reduces the number of terms by one

Now consider evaluating these derivatives at a particular value x = 0

At x = 0

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + ..na_nx^{n-1}$ $f'(0) = a_1$

$f''(x) = 2a_2 + ... n(n-1)a_nx^{n-2}$ $f''(0) = 2a_2$

$f'''(x) = (3)2a_3$ $f'''(0) = 3(2)a_3$

:

$f^n(x) = n(n-1)(n-2)...(3)(2)(1)a_n$ $f^n(0) = n(n-1)(n-2)...(3)(2)(1)a_n$

Let n! be 'n factorial' meaning n*(n-1)*(n-2)*….*1

E.g. 4! = 4x3x2x1 = 24 Note 1! = 1 and 0!=1

Then $a_1 = \frac{f'(0)}{1!}; a_2 = \frac{f''(0)}{2!}; a_3 = \frac{f'''(0)}{3!} ....a_n = \frac{fn(0)}{n!}$ ……………(2)

Insert (2) into (1) to give

$$f(x) = a_0 + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots \frac{f^n(0)}{n!}x^n$$

which since f(0) = a_0 gives the equivalent function expressed as derivatives evaluated at x = 0

- This is called a Maclaurin series

Eg the Maclaurin series of the function f(x) = 2+4x+3x^2

f(0) = 2

f'(x) = 4 + 6x so f'(0) = 4

f''(x) = 6 so f''(0) = 6

And

f(x) = f(0) +f'(0)x +(f''(0)/2)x^2 = 2 + 4x + 6x^2

Taylor's Series of a polynomial

A Taylor series expansion of a polynomial follows the same procedure but for any value of x

Usual to express this value as a deviation from a given point x_0

So $x = x_0 + dx$

Eg $f(x) = 2+4x+3x^2$ becomes $f(x) = 2 + 4(x_0 + dx) + 3(x_0 + dx)^2$

$f'(x) = 4 + 6(x_0 + dx)$

$f''(x) = 6$

Writing like this with a fixed point x_0 means that only dx is variable so f(x) is a function of dx , say $f(x) = g(dx) = 2 + 4(x_0 + dx) + 3(x_0 + dx)^2$

and $g'(dx) = f'(x) = 4+ 6(x_0 + dx)$ $g''(dx) = 6 = f''(x)$

We now know how to expand g(dx) around dx=0 (& $dx = 0$ implies $x = x_0$)

$$g(dx) = g(0) + \frac{g'(0)}{1!}dx + \frac{g''(0)}{2!}dx + \ldots$$

With g(0) = f(x_0) = 2 g''(0) = f(x_0) = 4 etc

Hence

$$g(dx) = f(x) = f(x_0) + \frac{f'(x_0)}{1!}dx + \frac{f'(x_0)}{2!}dx^2 + ..$$

$$\equiv g(x - x_0) = f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f'(x_0)}{2!}(x - x_0)^2 + ..$$

This is the expression for a Taylor series expansion of a polynomial

Approximations using Taylor's theorem

We can use the approximation to estimate values of functions

$$f(x) \approx f(x_0) + \sum_{n=1}^{n=N} \frac{(x-x_0)^n}{n!} f^n(x_0)$$

Eg1. Find the 2nd order Taylor series approximation for the value of the function:

$f(x) = 2 + 4x + 3x^2$ at the point $x_0 + h$

Using

$$f(x) \approx f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2 f''(x_0)}{2}$$

$f(x+h) = (2 + 4x + 3x^2) + h(4+6x) + h^2(6)/2$

$= 2 + 4(x+h) + 3(x^2 + h^2)$

Eg 2. Approximate (expand) the value of the function

$$f(x) = \frac{1}{1+x}$$

using a 4th order approximation around the value $x_0 = 1$

So

$$f(x) \approx f(x_0) + \sum_{n=1}^{n=4} \frac{(x-x_0)^n}{n!} f^n(x_0)$$

then

$f(x_0) = 1/(1+x_0)$	so $f(1) = 1/(1+1) = 1/2$
$f'(x_0) = -1/(1+x_0)^2$	so $f'(1) = -1/(1+1)^2 = -1/4$
$f''(x_0) = 2/(1+x_0)^3$	so $f''(2) = 2/(1+1)^3 = 1/4$
$f'''(x_0) = -6/(1+x_0)^4$	so $f''(2) = 2/(1+1)^3 = -3/8$
$f^4(x_0) = 24/(1+x_0)^5$	so $f''(2) = 2/(1+1)^3 = 3/4$

So

$$f(x) \approx \frac{1}{2} + \left(1 * \frac{-1}{4}(x-x_0)\right) + \left(\frac{1}{2} * \frac{1}{4}(x-x_0)^2\right) + \left(\frac{1}{6} * \frac{-3}{8}(x-x_0)^3\right) + \left(\frac{1}{24} * \frac{3}{4}(x-x_0)^4\right)$$

$$f(x) \approx \frac{1}{2} - \left(\frac{1}{4}(x-1)\right) + \left(\frac{1}{8}(x-1)^2\right) - \left(\frac{1}{16}(x-1)^3\right) + \left(\frac{1}{32}(x-1)^4\right)$$

$$f(x) \approx \frac{31}{32} - \frac{13}{16}x + \frac{1}{2}x^2 - \frac{3}{16}x^3 + \frac{1}{32}x^4$$

Useful applications: Taylor's series for exponentials.

Consider $y = f(x) = e^x$

We can use the theorem to estimate e^h by using the fact that $e^0=1$ and that the derivative of e^x with respect to h is just e^x .

Use
$$f(x) \approx f(x_0) + \sum_{n=1}^{n=N} \frac{(x-x_0)^n}{n!} f^n(x_0)$$

and expand around $x_0 = 0$

So,

$$\begin{aligned} f^1 &= e^x & \qquad f^1(0) &= 1 \\ f^2 &= e^x & \qquad f^2(0) &= 1 \\ &\vdots & \qquad f^3(0) &= 1 \\ f^n &= e^x & \qquad f^n(0) &= 1 \end{aligned}$$

Making the series for e^x
$$\exp(x) \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$$

Note that this series converges (reaches a finite value) for any value of x

So when x =1 we can calculate $e = e^1$
$$\exp(1) \approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \ldots \approx 2.718$$

Useful applications: Taylor's series for logs

We can use the series to calculate log(1+x)

$$f(x) \approx f(x_0) + \sum_{n=1}^{n=N} \frac{(x-x_0)^n}{n!} f^n(x_0)$$

Let f(x) = log(1+x) and expand the series around x_0=0

$$f^1 = \frac{f'(x)}{f(x)} = \frac{1}{1+x}$$

$$f^2 = -\frac{1}{(1+x)^2}$$

$$f^3 = \frac{2}{(1+x)^3}$$

$$f^n = \frac{(n-1)!(-1)^{n-1}}{(1+x)^n}$$

$$f^1(0) = 1$$

$$f^2(0) = -1$$

$$f^3(0) = 2$$

$$f^n(0) = (n-1)!(-1)^{n-1}$$

So,

$$\log(1+x) \approx \log(1) + x - \frac{x^2}{2} + \frac{x^3}{3} - \ \dots . = x - \frac{x^2}{2} + \frac{x^3}{3} - \ \dots .$$

Note that this means the series won't converge to a finite number if the ***absolute value*** of x is 1 or larger.

Quiz

1. Use Taylor's series to calculate a 2nd order approximation for log(1.5) (if you have a calculator check your accuracy)

2. Write down a second order Taylor's series for the function

$$f(x) = x^2 + x - 4.$$

How exact is the approximation?

Conclusion:

1. Logs, exponential and Taylor’s theorem appear frequently in economic analysis.
2. By now you should know what log and exponential functions look like
3. You should know their main properties
4. And their derivatives.
5. For Taylor’s theorem, you should memorise the formula for a second order approximation and
6. Know how to calculate an approximation given a differentiable function.

OPTIONAL: Taylor's Series with Remainder.

Suppose we write

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0) + a(x - x_0)^2$$

And ask what is the value of the constant "a" such that when h = x – x_0 =0, (the limit) the second derivative of this function with respect to h is equal to f''?

Differentiate once:
$$\frac{d(f(x_0) + hf'(x_0) + ah^2)}{dh} = f'(x_0) + 2ah$$

And again:
$$\frac{d^2(f(x_0) + hf'(x_0) + ah^2)}{dh^2} = 2a$$

If 2a = f'', then a = f''/2. i.e. our better approximation is:

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2 f''(x_0)}{2}$$

Can continue by increasing the order of this polynomial

Let f^n be the n^{th} derivative of the function f with respect to x

It turns out that if we wish the nth term of the approximation to be equal to the $f^n(x)$ when $h = x – x_0 = 0$, then the nth term must be:

$$\frac{(x-x_0)^n}{n!} f^n(x_0)$$

where n! is ‘n factorial’ meaning n*(n-1)*(n-2)*….1

We have not actually proven that adding more terms makes the approximation better and better, but that is what Taylor’s theorem says.

The infinite sum

$$\sum_{n=0}^{n=\infty} \frac{(x-x_0)^n}{n!} f^n(x_0)$$

is called a Taylor's series

The finite sum $$\sum_{n=0}^{n=N} \frac{(x-x_0)^n}{n!} f^n(x_0)$$

is called a Taylor's series of order N

Taylor's theorem states that,

If f(x) is any function (not just a polynomial) that is differentiable at least N+1 times, then for any x-x_0 , there exists an approximation

$$f(x) = f(x_0) + \sum_{n=1}^{n=N} \frac{(x-x_0)^n}{n!} f^n(x_0) + \frac{a^{N+1}}{N+1!} f^{N+1}(x_0) \qquad 0 \le |a| \le |x|$$

$$f(x) = f(x_0) + \sum_{n=1}^{n=N} \frac{(x - x_0)^n}{n!} f^n(x_0) + \frac{x^{N+1}}{N+1!} f^{N+1}(a)$$

The last term is the "residual" or remainder term evaluated at some point between zero and x $0 \le |a| \le |x|$

Note that $|a|, \quad |x|$ *means the absolute value*

So, if x = 0.1, a must lie between 0 and 0.1

If x = -0.2, then a must lie between -0.2 and 0.

In general, this remainder term can be ignored since usually only the approximation is interesting