EC5555
Economics Masters Refresher Course in Mathematics
September 2013

Lecture 2 – Matrix Inversion and Linear Equations

Francesco Feri

Matrix inversion and linear equations

Learning objectives.

- Matrix inversion
- Know Cramer's rule
- Understand how linear equations can be represented in matrix form
- Know how to solve linear equations using matrices and Cramer's rule

Inverses for (square) matrices

Idea:

In standard multiplication every number has an inverse
(except maybe zero unless you count infinity)

The inverse of 3 is 1/3; the inverse of 27 is 1/27, the inverse of -1.1 = -1/1.1

Also a number times its inverse equals 1: $x(1/x) = 1$

and the inverse times the number equals 1: $(1/x)x = 1$

and the inverse of the inverse is the original number $1/(1/x) = x$

Inverses for (square) matrices

The rules for the inverse of a matrix are similar (but not identical)

If A is an nxn matrix then the inverse of A, written A^{-1} , is an nxn matrix such that:

1. $AA^{-1} = I$
2. $A^{-1}A = I$

Notes:

1. This means that A is the inverse of A^{-1}
2. But...
3. A^{-1} may not always exist

Example

consider a simple demand and supply system for a good

Supply: Q= - 2 + 3P

Demand: Q=10 – 2P

We can write this using matrices

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \qquad x = \begin{bmatrix} Q \\ P \end{bmatrix} \qquad b = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

If A^{-1} exists, we multiply both sides of the equation by A^{-1}

$$A^{-1} Ax = A^{-1} b$$

Using results $A^{-1}A = I$ we get the solution.

$$x = A^{-1} b$$

Therefore computing the inverse of A we get the solution of the equations

An Inverse matrix example

1. Suppose $A = \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1/3 & 0 \\ 1/3 & -1 \end{pmatrix}$

Then

$$AB = \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 0 \\ 1/3 & -1 \end{pmatrix} = \begin{pmatrix} (3)(1/3) + (0)(1/3) & (3)(0) + (0)(-1) \\ (1)(1/3) + (-1)(1/3) & (1)(0) + (-1)(-1) \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So given $AA^{-1}=I$ it must be that in this case $B=A^{-1}$

Properties of inverted matrices

1. *If* A^{-1} *exists,* A *can be regarded as the inverse of* A^{-1}
2. *If* A *is* n *x* n *then* A^{-1} *must also be* n *x* n
3. *If an inverse exists, then it is unique*
4. $(A^{-1})^{-1} = A$
5. $(AB)^{-1} = B^{-1}A^{-1}$
6. $(A')^{-1} = (A^{-1})'$

In general we need to introduce some more terminology before can invert a matrix

1. Only square matrices can be inverted.

   Not all square matrices can be inverted however

2. A matrix that can be inverted is said to be **nonsingular**

   (so ***squareness*** is a necessary but not ***sufficient*** condition to invert)

3. The sufficient condition is that the columns (or rows since it is square) be **linearly independent**

   - think of this as being separate equations so the equations must be independent (n equations and n unknowns) if a solution is to be found

Example

$$Ax = d \Rightarrow \begin{bmatrix} 10 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\Rightarrow$$

$$10x_1 + 4x_2 = d_1$$

$$5x_1 + 2x_2 = d_2$$

So that the 1st row of A is twice that of the 2nd row and there is linear dependence

One equation is redundant (no extra information) and the system reduces to a single equation with 2 unknowns

So no unique solution for x_1 and x_2 exists

Rank of a matrix

The idea of vector rank can be easily extended to a matrix

The rank of a matrix is the maximum number of linearly independent rows or columns

If the matrix is square the maximum number of independent rows must be the same as the maximum number of independent columns

If the matrix is not square then the rank is equal to the smaller of the maximum number of rows or columns, $\rho <= \min(\text{rows}, \text{cols})$

If a matrix of order n is also of rank n, the matrix is said to be of **full rank**

Important: Only full rank matrices can be inverted

Matrix ranks are closely linked to the concept of **determinants**

Determinants

Let A be an nxn matrix then the determinant of A is a unique number (scalar), defined as:

(1) $$\det(A) = |A| = \sum_{j=1}^{j=n} (-1)^{1+j} a_{1j} \det(A^{1j})$$

Notes: In each term there are three components:

1. $(-1)^{1+j}$
2. a_{1j}
3. $\mathrm{Det}(A^{1j})$
4. What does this mean?

Start with a 2 x 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

which gives a single number (scalar) as the answer – as do all determinants

Can you see how this relates to equation (1) ?

Eg

$$A = \begin{bmatrix} 10 & 4 \\ 8 & 5 \end{bmatrix}$$

$$|A| = (10*5) - (4*8) = 18$$

What is the determinant of

$$B = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$$

So matrices that are not full rank – have linear dependent rows/columns - have zero determinants (will come back to this) and are singular

The determinant of a matrix is defined iteratively

1. An nxn is calculated as the sum of terms involving the determinants of n, (n-1)x(n-1) matrices

2. Each (n-1)x(n-1) matrix determinant is the sum of terms involving n-1 determinants of (n-2)x(n-2) matrices and so on

3. Since we know how to calculate the determinant of a 2x2 matrix we can always use this definition to find the determinant of an nxn matrix

4. In practice we shall not go above 3x3 matrices (unless using a computer program) but we need to know the general formula for an inverse

General properties of determinants

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

1) If B = A', then det. B = det. A

2) If B is constructed from A by swapping two rows, then det. B = -det. A

3) If B is constructed from A by swapping two columns, then det. B = -det. A

4) If B is constructed from A by multiplying one row (or column) by a constant, c, then det. B = c det. A

5) If B is constructed from A by adding a multiple of one row to another, then det. B = det. B

Determinants of triangular matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

Are examples of – respectively – an upper triangular and a lower triangular matrix

(zeros below or above the main diagonal)

The determinant of either an upper or lower triangular matrix is equal to
the product of the elements on the main diagonal

*Eg det.A = 1(24-0) - 2(0) + 3(0) = 24 = 1*4*6*

Determinants

For a 3 x 3 matrix, using

$$\det(A) = |A| = \sum_{j=1}^{j=n} (-1)^{1+j} a_{1j} \det(A^{1j})$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Question: What is the determinant of

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

Method 1: Laplace expansion of an n x n matrix.

Can generalise this rule for the determinant of any nxn matrix

$$\left|A\right| = \sum_{j=1}^{j=n} (-1)^{1+j} a_{1j} \left|M_{1j}\right| = \sum_{j=1}^{j=n} a_{1j} \left|C_{ij}\right|$$

As part of this method, you need to know the following:

1. **Minor M**
2. **Co-factor C**

(which are also essential to invert a matrix)

nxn Matrix inversion - minors

There is a minor M_{ij} for each element a_{ij} in the square matrix.

1. To find it construct a new matrix by deleting the row i and deleting the column j.
2. Then find the determinant of what is left
3. E.g. What is M_{11}

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}; \quad so\ M_{11} = \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

4. M_{12}

	1	2	3
A=	4	5	6
	7	8	9

Delete row and column

1	2	3
4	5	6
7	8	9

$$i.e. \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}; \quad so\ M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36 - 42 = -6.$$

N xn matrix inversion: Co-factor and adjoint matrices

The **cofactor** C_{ij} is a minor with a pre-assigned algebraic sign given to it

1. For each element a_{ij}, work out the minor

2. Then multiply it by $(-1)^{i+j}$

3. In simpler language: if i+j is even then $C_{ij} = M_{ij}$

4. If i+j is odd, then $C_{ij} = -M_{ij}$

5. The co-factor matrix is then

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

6. And the **adjoint** matrix is C' – i.e. the transpose of C.

Co-factor, adjoint matrices and the inverse matrix

1. The inverse matrix, A^{-1} is just

$$A^{-1} = \frac{1}{|A|} adj.A = \frac{1}{|A|} C'$$

So

i) find the determinant

– if it is non-zero, the matrix is non-singular so its inverse exists

ii) Find the minor and then the cofactors of all the elements of A and arrange them in the cofactor matrix

iii) Transpose this matrix to get the adjoint matrix

iv) Divide the adjoint matrix by the determinant to get the inverse

Example 1 (2 x 2 matrix)

1. If $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ find A^{-1}

Use the formula $A^{-1} = \frac{1}{|A|} adj.A = \frac{1}{|A|} C'$

$$A^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}$$

First find the determinant $|A| = (a_{11}a_{22} - a_{12}a_{21}) = (2)(1) - (1)(0) = 2$

which is non-zero so can continue

Now find matrix of cofactors, which in the 2 x 2 case is a set of 1 X 1 determinants

$$C = \begin{pmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj.A = \frac{1}{|A|} C'$$

Now transpose the matrix of cofactors to get the adjoint matrix

$$adj.A = C' = \begin{pmatrix} |C_{11}| & |C_{21}| \\ |C_{12}| & |C_{22}| \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

Now using the formula above

$$|A| = \left(a_{11}a_{22} - a_{12}a_{21}\right) = (2)(1) - (1)(0) = 2$$

which is non-zero so can continue

then $$A^{-1} = \frac{1}{(2)} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1 \end{bmatrix}$$

NB. Always check that the answer is right by looking if $AA^{-1} = I$

$$AA^{-1} = \frac{1}{(2)} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \frac{1}{(2)} \begin{pmatrix} 2+0 & -2+2 \\ 0+0 & 0+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 2: 3 x 3 matrix

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}; \quad so\ M = \begin{pmatrix} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} & \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 7 & 9 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 7 & 8 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} \end{pmatrix}$$

$$M = \begin{pmatrix} 45-48 & 36-42 & 32-35 \\ 18-24 & 0-21 & 0-14 \\ 12-15 & 0-12 & 0-8 \end{pmatrix} = \begin{pmatrix} -3 & -6 & -3 \\ -6 & -21 & -14 \\ -3 & -12 & -8 \end{pmatrix};$$

$$C = \begin{pmatrix} -3 & 6 & -3 \\ 6 & -21 & 14 \\ -3 & 12 & -8 \end{pmatrix}; C' = \begin{pmatrix} -3 & 6 & -3 \\ 6 & -21 & 12 \\ -3 & 14 & -8 \end{pmatrix};$$

So

$$|A| = a_{11}c_{11} + a_{12}c_{12} \cdots + a_{1n}c_{1n} = (0)(-3) + (2)(6) + (3)(-3) = 3$$

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}; \quad so\ A^{-1} = \left(\frac{1}{3}\right) \begin{pmatrix} -3 & 6 & -3 \\ 6 & -21 & 12 \\ -3 & 14 & -8 \end{pmatrix}$$

Quiz

1. In each case find the matrix of minors

2. Find the determinant

3. Find the inverse and check it.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

Some more jargon

$$B = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

- The term $(a_{11}a_{22} - a_{12}a_{21})$ is called the determinant of A, often written det.(A). Vertical lines surrounding the original matrix entries also means ‘determinant of A’.

$$\det.A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- The matrix part of the solution is called the ‘adjoint of A’ written adj. A. The elements of the adj.A are called co-factors. So,

$$adj.A = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$B = \frac{1}{\det.A} adj.A$$

Linear Equations

Recall that linear equations are those where the variables enter in a linear form:

e.g. $4 = x_1 + x_2$

is linear (variables appear in additive form)

$4 = x_2 x_1 + x_2$

is not linear because of the $x_2 x_1$ term

Linear equations can be written a more concise form

Eg. $4 = x_1 + x_2$

can be written in terms of vector products

$$4 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix form for linear equations

If there are several simultaneous linear equations, they can be stacked in matrix form

e.g.

$4 = x_1 + x_2$
$3 = 2x_1 + x_2$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

In general, a system of ***m*** linear equations in ***n*** unknowns can be represented as

$$Ax = b$$

where ***A*** is an mxn matrix, ***x*** is a nx1 matrix (or column vector) and ***b*** is an mx1 matrix

Conversely Ax= b can be interpreted as a system of m linear equations in n unknowns

Solving linear equation systems

In general we want to find the solution to the equation system   Ax=b

There are three possible cases:

1) There is no solution
2) There is exactly one solution
3) There is more than one solution (typically an infinite number)

It is not always obvious which case applies

Example 1. No solution

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This says that $4 = x_1 + x_2$ **and** $3 = x_1 + x_2$ simultaneously

Thus 3 =4, which is nonsense   Hence there is no solution

Example 2. One solution

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This says that

$4 = x_1 + x_2$

and $3 = 2x_1 + x_2$.

Thus $-1 = x_1$ and $x_2 = 5$

Example 3. Many solutions

$$\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This says that

$4 = x_1 + x_2$

and $8 = 2x_1 + 2x_2$

So, all we can say is that $x_2 = 4-x_1$ If $x_1 = 0$, then $x_2 = 4$; if $x_1 = 1$ then $x_2 = 3$, etc...

Solving when there are n equations and n unknowns- Cramer's Rule

This (n & n) is a special case

(in general we may have more equations than unknowns or vice versa)

If n unknowns then m = n so in the equation Ax = b, A is a square matrix.

Suppose det. A ≠ 0 so that A^{-1} exists.

If Ax = b then $A^{-1}Ax = A^{-1}b$ or $x = A^{-1}b$

and we have our solution

Example: $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\quad A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

So,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}b = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4+3 \\ 8-3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

2. No solution. Suppose b ≠ 0 then if there is no solution A^{-1} does not exist – i.e. det. A = 0.

Example:
$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since

$$|A| = (1*1) - (1*1) = 0$$

solution A^{-1} does not exist and can't solve Ax=b for x

3. Many solutions. Suppose b ≠ 0 then if there are multiple solutions A^{-1} does not exist – i.e. det. A = 0.

Example:

$$\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So any $x_2 = 4 - x_1$ is a solution. In this case. Det. A = 2-2=0.

Compare 2) and 3). When there are no solutions det. A = 0 and when there are multiple solutions, det. A = 0. So when det. A = 0 all we know is that there isn't one solution. We don't know if we're in the no solution or the multiple solutions case.

To find out, we need to check to see if the equations are consistent. In case 2. the equations are inconsistent – hence there are no solutions. In case 3, the equations are consistent – hence there are many solutions.

Quiz II.

1. Use matrix inversion to find the solution to the following set of equations.

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Why in this case is it quicker not to use matrix inversion?

Part II Cramer's rule

1. Introduction: Cramer's rule

- Often when faced with $Ax=b$ we are not interested in a complete solution for x

- We may only wish to find x_1 or x_4

- Cramer's rule is a short cut for finding a particular x_i
  It is particularly useful when A is 3x3 or bigger

- It is not sensible to use it if you need to find several x_i
  – finding A^{-1} is generally quicker

Suppose you have the system of equations, Ax = b

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Define the matrix A_i as the matrix where the i^{th} column of A has been replaced with column vector b

Example 1.

$$A_n = \begin{pmatrix} a_{11} & \cdots & b_1 \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_n \end{pmatrix}$$

Replaces the n^{th} column with the column vector b

Example 2.

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad so \quad A_1 = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$$

Replaces the 1^{st} column with the column vector b

Suppose you have the system of equations   Ax = b, then, if det. A≠ 0,

(1)

$$x_i = \frac{|A_i|}{|A|}$$

So to find the value of the j^th variable, replace the j^th column of A by the vector of constants b

- Intuitively the solution

$$x = A^{-1}b = \frac{1}{|A|}(adj.A)*b$$

effectively does (1)

Example 2. (Recall that the solution to this system was x_1 = -1, x_2 = 5)

$$\det.A = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \quad \det A_1 = \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 1; \quad \det A_2 = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = -5$$

So x_1 = 1/-1 = -1 and x_2 = -5/-1 = 5

Quiz

Suppose,

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Use Cramer's rule to find x_1 and x_2

Cramer’s rule in macroeconomics

Many macroeconomic models involve a system of linear equations

Cramer’s rule can be used to solve for one particular variable

Example. Consider a simple macro model of the economy

$Y = C + I + G$
$C = a + bY \qquad 0 < b < 1$
$I = I_0$
$G = G_0$

Write this system in matrix form then use Cramer’s rule to find consumption, C

Step 1: identify the endogenous variables and the exogenous variables

The endogenous variables correspond to the x vectors in the previous example

The exogenous variables are equivalent to the parameters of the system and correspond to the b vector

Example: here C and Y are endogenous. I_0 and G_0 are the exogenous variables

Step 2: Simplify the system of equations if possible then write down the system in such a way that all the endogenous variables are on one side of the equation and all the exogenous variables are on the other side

Example: simplify the equations

$Y = C + I_0 + G_0$

$C = a + bY.$

Rewrite:

$Y - C = I_0 + G_0$

$C – bY = a$

Step 3. Put into matrix form

Matrix form:

$$\begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \end{pmatrix} = \begin{pmatrix} I_0 + G_0 \\ a \end{pmatrix}$$

Step 4. then use Cramer's rule

$$\begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix}\begin{pmatrix} Y \\ C \end{pmatrix} = \begin{pmatrix} I_0 + G_0 \\ a \end{pmatrix}$$

So, to find C we replace the second column of the matrix with the column vector of parameters.

$$C = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + b(I_0 + G_0)}{1 - b}$$

Quiz II. Find Y using the same procedure.

Some guidance on solving mxn equation systems

Often the set of unknowns and equations to solve them are not equal

The general problem involves m equations and n unknowns.

Many systems of equations involve fewer equations than variables, m<n

Some involve more equations than variables, n < m.

In either case you cannot use matrix inversion to characterise the solution (if it exists).

Example.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

When m ≠ n we seek to do two things:

1. Find out if any solution exists.
2. If at least one solution exists, identify its features.

The rank of a matrix provides a guide to the number of solutions

Reminder: The rank of a matrix is the largest number of linearly independent rows or columns.

•Note that the column rank and the row rank will be the same.

•Note that the rank cannot be larger than the smaller of m and n. i.e. if A is an mxn matrix rank(A) ≤ min(m,n)

Example.

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

The rank of this matrix is at most 2, but in fact rank(A) = 1.

Note that for an nxn matrix (det A = 0) ↔ rank(A) < n.

We can see ← from the properties of determinants. If rank(A) < n we can add and subtract rows to create a row of zeros. The determinant of this new matrix is therefore 0, but by property 5 adding and subtracting rows does not change the determinant. So det(A) = 0.

Find the rank of the system. Note that the maximum possible rank is min(m,n). Let’s suppose this is n

i. If rank(A) = n, then there may be a unique solution
ii. If rank(A) < n then there cannot be a unique solution

Example:

$$Ax = b = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This has a rank of 2. We can take the second row away from the first row to create:

$$\begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Or $x_1 = 1-2x_3$ and $x_2 = 1+3x_3$. This is as far as we can go in defining a solution. One variable is free to take on any value which then determines the value of the other two variables. For example, given any x_3 we can calculate the other two variables.

In general, if the equation system is consistent, then the number of free variables is n-rank(A).

Summary

skills you should be able to do

- Use Cramer's rule to solve for a variable
- Write down a linear macroeconomic system in matrix form
- Work out the rank of a matrix
- Characterise the solution of Ax=b.

Practical tips on finding solutions

If the matrix is square then the first issue is whether the determinant is zero.

So if the matrix is square you need to first find its determinant.

Example:

$$\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\det.A = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = (2)(0-0) + (1)(-1)(-1-0) + (0)(-1-0) = 1$$

So, A^{-1} exists. In fact,

$$A^{-1} = \frac{1}{1} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

Thus

$$x = A^{-1}b = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ -5 \end{pmatrix}$$

What's the solution to this set of linear equations?

$$\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Obviously this is related to the last set of equations - x_1 has been relabelled as x_3 and vice versa. So,

$$A^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad x = \begin{pmatrix} -5 \\ 6 \\ -1 \end{pmatrix}$$

Note that the new inverse is obtained from the old inverse by swapping columns – not rows.

Recall, general properties of determinants.

1) If B = A^T, then det. B = det. A.
2) If B is constructed from A by swapping two rows, then det. B = -det. A
3) If B is constructed from A by swapping two columns, then det. B = -det. A
4) If B is constructed from A by multiplying one row (or column) by a constant, c, then det. B = c det. A
5) If B is constructed from A by adding a multiple of one row to another, then det. B = det. B.

Some implications of properties of determinants.

1) You can use any row to calculate the determinant (often there are easy rows to use).

$$A = \begin{pmatrix} 12 & -4 & 6 \\ 0 & 1 & 0 \\ 3 & 2 & 2 \end{pmatrix}$$

Using row 1, det. A = 12(2-0)+4(0+0)+6(0-3) = 6

Using row 2, det. A = 1 (12.2-3.6)= 6

2) If one row (column) is a multiple of another row (column) then det. A = 0.

3) If one row (column) can be constructed by adding/subtracting multiples of other rows (columns) then det. A = 0.

Quiz III.

1. Show that all the determinants equal 0

$$A = \begin{pmatrix} 12 & 45 & 2 & 2 \\ 12 & 2 & -11 & 7 \\ 0 & 0 & 0 & 0 \\ 15 & 3.2 & 4 & -5 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 8 & 0 & 0 \\ 3 & 4 & 8 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ -1 & 3 & -5 \end{pmatrix}$$

Markov Chains

A common application of matrices in macroeconomics is based on the idea of modelling changes over time as a ***Markov process*** – where the levels of any variables of interest are modelled as

$$x_t = M_t x_{t-1}$$

Eg the vector

$$x_t = \begin{bmatrix} Employed_t \\ Unemployed_t \end{bmatrix}$$

and the transition matrix

$$M_t = \begin{bmatrix} ee & ue \\ eu & uu \end{bmatrix} = \begin{bmatrix} .95 & .30 \\ .05 & .70 \end{bmatrix}$$

Gives the set of probabilities of moving between the (two) states during time t-1 and t

Eg

eu=No. flowing from E to U/No. in employment at time t-1

Note ee = 1 – eu (column probabilities add to one)

Note that

$$x_t = M_t x_{t-1} \Rightarrow \begin{bmatrix} Employed_t \\ Unemployed_t \end{bmatrix} = \begin{bmatrix} ee & ue \\ eu & uu \end{bmatrix} \begin{bmatrix} Employed_{t-1} \\ Unemployed_{t-1} \end{bmatrix}$$

$$\begin{bmatrix} Employed_t \\ Unemployed_t \end{bmatrix} = \begin{bmatrix} ee * Employed_{t-1} + ue * Unemployed_{t-1} \\ eu * Employed_{t-1} + uu * Unemployed_{t-1} \end{bmatrix}$$

So that the stock now = proportion staying in that state

+ proportion moving in out of other state

If these transition matrices hold at every point in time then can calculate a ***steady state*** stock as

$x_t = M^n x_{t-n}$

(ie the constant transition matrix raised to the power n)

Eg in period 2

$$x_2 = M^2 x_0$$

$$\begin{bmatrix} employed_2 \\ unemployed_2 \end{bmatrix} = \begin{bmatrix} .95 & .3 \\ .05 & .7 \end{bmatrix}^2 \begin{bmatrix} employed_0 \\ unemployed_0 \end{bmatrix} = \begin{bmatrix} .9175 & .495 \\ .0825 & .505 \end{bmatrix} \begin{bmatrix} employed_0 \\ unemployed_0 \end{bmatrix}$$

Note that as number of time periods increases the matrix M^n will converge to a particular set of values where the columns are identical

$$M^{10} = \begin{bmatrix} .859 & .845 \\ .140 & .154 \end{bmatrix}$$

– the steady state transition probabilities that ensure the stocks of employment and unemployment are stable over time

Note that you can calculate the inflows and outflows from each state as

eu*E and ue*U

in a steady state these flows should be equal