EC5555
Economics Masters Refresher Course in Mathematics
September 2013

Lecture 1

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Aims

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- Teaching Assistant: Tanya Wilson

- Aim: to refresh your maths and statistical skills, to the level needed for success in the Royal Holloway Economics and Finance masters.
- Lectures 10:00 – 12:00 h and Tutorials 13:00 – 14:00 h each day.

- In the afternoons and evening you will be expected to:

  - Read the text and practise
  - Do the daily problem set.
  - Prepare to present your answers to the class the next day.

Lecture 1. Vectors and matrices

Learning objectives. By the end of this lecture you should:

- Understand the concept of vectors and matrices
- Understand their relationship to economics
- Understand vector products and the basics of matrix algebra

Introduction

We are often interested in analysing the economic relationship between **several** variables

Use of vectors and matrices can make the analysis of complex **linear** economic relationships simpler and quicker

Matrices

1. Introduction

- Matrices are tables where the order of columns and rows matters
  - E.g. marks in a course test for each question and each student.

	Andy	Ahmad	Anka
Qn. 1	44	74	65
Qn. 2	23	56	48
Qn. 3	8	24	44

Some common types of matrices in applied economics.

1. Storing data

	US GDP	UK GDP	PRC GDP
1999	100	80	11
2000	103	82	12
2001	103	84	13

2. Input-output Tables

	Per kg iron	Per kg coal
Kg iron	0.1	0.01
Kg coal	1	0
Hours labour	0.01	0.001

3. Transition matrices.

(U = unemployed, E = employed, prob. = probability)

	Prob. U in 2011	Prob E in 2011
U in 2010	0.3	0.7
E in 2010	0.1	0.9

DEFINITIONS.

Dimensions of a matrix

is always defined by the number of rows followed by the number of columns

So $A_{m\times n}$

is a matrix with *m* rows and *n* columns.

Example
$$A=\begin{pmatrix}4 & -1 & 3 & 3 & 44\\ 7 & 23 & 6 & 0 & 1\end{pmatrix}$$

So A is a 2 x 5 matrix

$$B=\begin{pmatrix}4 & 7\\ -1 & 23\\ 3 & 6\\ 3 & 0\\ 44 & 1\end{pmatrix}$$

B is 5x2 matrix

Sometimes we wish to refer to ***<u>individual elements in a matrix</u>***

E.g. the number in the third row, second column.

We use the notation a_{ij} (or b_{ij} etc.) to indicate the appropriate element

i refers to the row

j refers to the column

Example

$$A = \begin{pmatrix} 4 & -1 & 3 & 3 & 44 \\ 7 & 23 & 6 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{pmatrix}$$

So $a_{13} = 3$ and $a_{21} = 7$

If

$$B = \begin{pmatrix} 4 & 7 \\ -1 & 23 \\ 3 & 6 \\ 3 & 0 \\ 44 & 1 \end{pmatrix}$$

What is b_{21} ?

The ***transpose*** of a matrix A is obtained by by turning rows into columns and vice versa

swapping a_{ij} for a_{ji} for all i and j

We write the transpose as A' or A^T (A "prime")

$$A = \begin{pmatrix} 1 & 0 & 6 \\ 2 & -1 & 7 \\ 0 & 3 & 0 \end{pmatrix} \quad so \quad A' = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 6 & 7 & 0 \end{pmatrix}$$

A ***symmetric matrix*** is one where A' = A.

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} = A'$$

Note that (A')' = A

i.e. ehe transpose of a transpose is the original matrix

vectors are special cases of matrices

a r*ow vector* is a 1 x n matrix

b = (1 0 -1) is a row vector (1 x 3 matrix)

a column vector is an n x 1 matrix

$$c = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

So c is a 3 x 1 matrix

Important notes:

1) the content of c is the same as b'. This means that you can store the same information in different ways)

2) usually we denote matrices by capital letters and vectors by lowercase letters.

3) A row vector is distinguished from a column vector by the use of a primed symbol

Some useful Definitions

A positive matrix is one where **all** of the elements are strictly positive.

A **non-negative** matrix is one where all of the elements are either positive or zero

A **negative** matrix is one where **none** of the elements are positive.

A **strictly negative** matrix is one where all of the elements are strictly negative

$$A = \begin{pmatrix} 1 & 0 & 6 \\ 2 & -1 & 7 \\ 0 & 3 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & -3 \\ -3 & -1 \\ -1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$$

C is strictly positive and symmetric; B is negative; A is neither positive nor negative.

The null matrix or zero matrix is a matrix consisting entirely of zeros

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A square matrix is one where the number of rows equals the number of columns i.e. nxn

Eg.

$$\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$$

For a square matrix***, the main (or leading) diagonal*** is all the elements

a_{ij} where i = j

So in the above the main diagonal is (2 1)

The identity matrix is a square matrix consisting of zeros except for the leading diagonal which consists of 1s:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We write I for the identity matrix.

If we wish to identify its size (number of rows or columns) we write I_n

A ***<u>diagonal matrix</u>*** *is a* square matrix with $a_{ik} = 0$ *whenever* $i \neq k$

And so consists of zeros everywhere except for the main diagonal which consists of non-zero numbers

(so the identity matrix is a special case of a diagonal matrix since it has just ones along the main diagonal)

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{pmatrix}$$

4. Mini quiz

1. What are the dimensions of A?

$$A = \begin{pmatrix} 4 & -1 & 3 & 3 \\ -7 & 3 & 0 & 0 \end{pmatrix}$$

2. What is a_{21}?

3. Is B a square matrix?

$$B = \begin{pmatrix} 1 & 0 & 6 \\ 2 & -1 & 7 \\ 0 & 3 & 0 \end{pmatrix}$$

4. What is the largest element on the main diagonal of B?

5. What is the value of the largest element on the main diagonal of B?

MATRIX OPERATIONS

Adding matrices

- add each element from the corresponding place in the matrices.

i.e. if A and B are m x n matrices, then A+B is the m x n matrix where

$c_{ij} = a_{ij} + b_{ij}$ for i = 1,..,m and j = 1,…,n.

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 & 6 \\ 2 & -1 & 7 \\ 0 & 3 & 0 \end{pmatrix}; \quad A + B = \begin{pmatrix} -1 & 0 & 6 \\ 4 & -2 & 7 \\ 0 & 4 & 0 \end{pmatrix}$$

You can only add two matrices if they have the same dimensions.

e.g. you cannot add A and B

$$A = \begin{pmatrix} 4 & -1 & 3 & 3 \\ -7 & 3 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 6 \\ 2 & -1 & 7 \\ 0 & 3 & 0 \end{pmatrix}$$

.

Note: (A + B)’ = A’ + B’ The transpose of a sum is the sum of the transposes

Multiplying by a scalar

When you multiply by a scalar (e.g. 3, 23.1 or -2), then you multiply each element of the matrix by that scalar.

Example 1: what is 4*A* if $A = \begin{pmatrix} 4 & -1 & 3 & 3 \\ -7 & 3 & 0 & 0 \end{pmatrix}$

$$4A = \begin{pmatrix} 16 & -4 & 12 & 12 \\ -28 & 12 & 0 & 0 \end{pmatrix}$$

Example 2: what is *xB* if $B = \begin{pmatrix} 1 & 0 & 6 \\ 2 & -1 & 7 \\ 0 & 3 & 0 \end{pmatrix}$

$$xB = \begin{pmatrix} x & 0 & 6x \\ 2x & -x & 7x \\ 0 & 3x & 0 \end{pmatrix}$$

Matrix Multiplication

In general multiplication of 2 or more matrices has some special rules

1. The first rule is that the order of multiplication matters.

In general AxB (or AB) is not the same as BA

(so this is very different to multiplying numbers where the order doesn't matter – e.g. 3x4 = 4x3 = 12)

Aside:

- Addition, multiplication, matrix multiplication etc. are examples of operators
- An operator is said to be ***commutative*** if x *operator* y = y *operator* x for any x and y (the order of multiplication does not matter)

- Addition is ***commutative***: x+y = y+x; multiplication is commutative; subtraction is not commutative (2-1 ≠ 1-2)

- and matrix multiplication is also **not** commutative

2. The second rule is that you can only multiply two matrices if they are **conformable**

- Two matrices are **conformable** if the number of **columns** for the first matrix is the same as the number of **rows** for the second matrix

- If the matrices are not conformable they cannot be multiplied.

- Example 1: does AB exist?

$$A = \begin{pmatrix} 4 & -1 & 3 & 3 \\ -7 & 3 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 6 \\ 2 & -1 & 7 \\ 0 & 3 & 0 \end{pmatrix}$$

– Answer: A is a 2x4 matrix. B is a 3x3.
– So A has 4 columns and B has 3 rows.
– Therefore AB does not exist. A and B are not conformable.

- Example 2: does AB exist?

$$A = \begin{pmatrix} 4 & -1 & 3 & 3 & 0 \\ 7 & 2 & 6 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 0 \\ -1 & 3 \\ 3 & 6 \\ 3 & 0 \\ 1 & 1 \end{pmatrix}$$

– Answer: A is a 2x5 matrix. B is a 5x2. So A has 5 columns and B has 5 rows. A and B are conformable. Therefore AB exists.

Multiplying two matrices.

Finding AB

$$A = \begin{pmatrix} 4 & -1 & 3 & 3 & 0 \\ 7 & 2 & 6 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 0 \\ -1 & 3 \\ 3 & 6 \\ 3 & 0 \\ 1 & 1 \end{pmatrix} \qquad C = AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

– A and B are conformable. So C = AB exists and will be a 2 x 2 matrix.

To calculate it:

i. To get the first element on the first row of C take the first row of A and multiply each element in turn against its corresponding element in the first column of B. Add the result.

– Example: c_{11} = (4x4) + (-1x-1) + (3x3) + (3x3) + (0x1) = 35

ii. To get the remaining elements in the first row: repeat this procedure with the first row of A multiplying each column of B in turn.

So top left hand element is

$$AB = \begin{pmatrix} 4 & -1 & 3 & 3 & 0 \\ 7 & 2 & 6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -1 & 3 \\ 3 & 6 \\ 3 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} (4)(4)+(-1)(-1)+(3)(3)+(3)(3)+(0)(1) & . \\ . & . \end{pmatrix}$$

And top right hand element is

$$AB = \begin{pmatrix} 4 & -1 & 3 & 3 & 0 \\ 7 & 2 & 6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -1 & 3 \\ 3 & 6 \\ 3 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 35 & (4)(0)+(-1)(3)+(3)(6)+(3)(0)+(0)(1) \\ . & . \end{pmatrix}$$

General rule

Suppose A is an mxn matrix and and B is an nxr matrix with typical elements a_{ik} and b_{kj} respectively

then AB =C where element c_{ij} is :

$$c_{ij} = \sum_{k=1}^{k=n} a_{ik} b_{kj}$$

Note that the result is an mxr matrix

i.e. the result is a matrix with a number of row of the first matrix and a number of columns of the second matrix

Note: (AB)’ = B’A’

The transpose of a product is the product of the transposes *in reverse order*

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$ *compute* (AB)’ and B’A’

Another example.

- Example 2: calculate AB

$$a' = \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix}$$

  – First we note that a’ is 1x4 and B is 4x2
  – so C= a’ B exists and is a 1x2 matrix
  – The first element c_{11} = (1)(1)+(0)(2)+(2)(1)+(0)(0)= 3
  – The second element c_{12} = (1)(0)+(0)(0)+(2)(3)+(0)(1)= 6
  – So c’ = (3   6)

Quiz.

Can you multiply the following matrices? If so, what is the dimension of the result?

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

1. BA
2. BC
3. AA^T
4. A^TA
5. CC

Finding CC (sometimes written C^2).

$$CC = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2.2 + 2.1 & 2.2 + 2.1 \\ 1.2 + 1.1 & 1.2 + 1.1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 3 & 3 \end{pmatrix}$$

Associative, Commutative and Distributive laws

Using scalars

Commutative law of addition: $a + b = b + a$

Commutative law of multiplication: $a \cdot b = b \cdot a$

Associative law of addition: $(a + b) + c = a + (b + c)$

Associative law of multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive law: $a \cdot (b + c) = a \cdot b + a \cdot c$

Most, but not all, of these laws also apply to matrix operations.

Using matrices

Commutative law of addition: $A + B = B + A$

Commutative law of multiplication: $A \cdot B \neq B \cdot A$

(matrix multiplication is **not** commutative)

Associative law of addition: $(A + B) + C = A + (B + C)$

Associative law of multiplication: $(A \cdot B) \cdot C = A \cdot (B \cdot C) = A \cdot B \cdot C$

Distributive law: $A \cdot (B + C) = A \cdot B + A \cdot C$

$(B + C) \cdot A = B \cdot A + C \cdot A$

Properties of transposes

1. $(A')' = A$
2. $(A + B)' = A' + B'$
3. $(AB)' = B'A'$

Try to verify these properties using the following two matrices

$$A = \begin{matrix} 4 & 1 \\ 9 & 0 \end{matrix} \qquad B = \begin{matrix} 2 & 0 \\ 7 & 1 \end{matrix}$$

Use of matrices in economic theory

Eg consider a simple demand and supply system for a good

Supply: Q= - 2 + 3P

Demand: Q=10 – 2P

Re-arranging

Supply: Q – 3P = - 2

Demand: Q + 2P = 10

Or in short-hand Ax = b

where

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \qquad x = \begin{bmatrix} Q \\ P \end{bmatrix} \qquad b = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

In general any system of m equations with n variables (x_1, x_2, ...x_n)

$$a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ...a_{mn}x_n = b_m$$

Can be written in matrix form Ax = b

where

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ \vdots & & \vdots & \\ a_{m1} & a_{m2} & .. & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

Multiplying square matrices by the identity matrix

Recall:

1. A square matrix has the same number of rows and columns – nxn

2. The identity matrix I is a square matrix with 1s in the leading diagonal and 0s everywhere else.

E.g.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The usefulness of the identity matrix is similar to that of the number 1 in number algebra

Since IA = AI = A

- if multiply a matrix by the identity matrix the product is the original matrix

Eg

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (1*1)+(0*3) & (1*2)+(0*4) \\ (0*1)+(1*3) & (0*2)+(1*4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

(leave it to you to show AI=A)

A general result for square matrices

1. If A is a square matrix then IA = AI = A.

$$If \quad A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \quad IA=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

$$=\begin{pmatrix} 1a_{11}+0a_{21}+\cdots 0a_{n1} & \cdots & 1a_{1n}+0a_{12}+\cdots 0a_{nn} \\ \vdots & & \vdots \\ & & \\ 0a_{11}+0a_{21}+\cdots 1a_{n1} & \cdots & 0a_{1n}+0a_{12}+\cdots 1a_{nn} \end{pmatrix}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}=A$$

NB. This only applies to square matrices

This result can be useful sometimes to help solve matrix algebra

Since if AI = A

Then

AIB = (AI) B = A (BI) = AB

-The inclusion of the identity matrix does not affect the matrix product result (like multiplying by "1")

(will see example of this in econometrics classes)

Also note that

$$I_n^2 = I_n$$

An identity matrix squared is equal itself

Definition:
A matrix A is said to be **idempotent** if

AA = A

Note on vector operations

Adding vectors

General rule

If $a' = (a_1, a_2 \ldots\ldots a_n)$ and $b' = (b_1, b_2 \ldots\ldots b_n)$

then $a'+b' = (a_1+ b_1, a_2 + b_2, \ldots\ldots a_n + b_n)$

e.g. (0 2 3) + (1 0 4) = (0+1, 2+0, 3+4) = (1 2 7)

(same rule for addition of column vectors)

Note that the result is a vector with the same dimension 1 x n

Note you can only add two vectors if they have the same dimensions

e.g. you cannot add (0 2 3) and (0 1)

Note also that sometimes adding two vectors may be mathematically ok, but economic nonsense:

adding factor prices together does not give total input price

Multiplying vectors by a number (a ‘scalar’)

Let $x = (x_1, \ldots, x_n)$ and a be a scalar, then $ax = (ax_1, ax_2, \ldots, ax_n)$

e.g. $a = 2$, $x = (1\ \ 2\ \ 3)$

$ax = (2\ \ 4\ \ 6)$

Multiplying vectors

In general you cannot multiply two row vectors or two column vectors together.

But there are some special cases where you can

And you can often multiply a column vector by a row vector and vice versa

Suppose a' is 1 x n and b is n x 1

a' b = c where c is a scalar (1 x 1)

a b' = D where D is n x n

If a is 1 x n and b is m x 1

a' b' = F where F is n x m

Vector products – a special case of vector multiplication

Definition: Vector product, also known as the **dot product** or **inner product**

Let x = $(x_1, x_2, \ldots, x_n)$, y = $(y_1, \ldots, y_n)$.

The vector product is written x.y and equals $x_1y_1 + x_2y_2 + \ldots x_ny_n$.

Or

$$x.y = \sum_{i=1}^{i=n} x_i y_i$$

Note:

1) that vector products are only possible if the vectors have the same dimensions.
2) Is like to multiply a row vector with a column vector a' b where the number of column of the row vector is the same that the number of columns of the column vector.

Example

$$x.y = \sum_{i=1}^{i=n} x_i y_i$$

Miki buys two apples and three pears from the Spar shop.

Apples cost £0.50 each; pears cost £0.40 each.

How much does she spend in total?

Answer: This can be seen as an example of a vector product

- Write the prices as a vector: p=(0.5, 0.4)
- Write the quantities as a vector q=(2, 3)

- Total Expenditure (=Price*Quantity) is the sum of expenditures on each good
- found by multiplying the first element of the first vector by the first element of the second vector and multiplying the second element of the first vector by the second element of the second vector & adding the results
- Spending = 2x0.5 + 3x0.4 = £2.20

Instant Quiz.

1. What are the dimensions of the following?

$$i.\begin{pmatrix}1 & 2 & 4 & 0 & -1\end{pmatrix} \qquad ii.\begin{pmatrix}0\\0\\0\end{pmatrix} \quad iii.\begin{pmatrix}0 & 1\\2 & 3\\2 & 2\end{pmatrix}$$

2. Can you add the following (if you can, provide the answer)?

$$i.\begin{pmatrix}4 & 1\end{pmatrix} \& \begin{pmatrix}-1 & 0\end{pmatrix} \qquad ii.\begin{pmatrix}0\\0\\0\end{pmatrix} \& \begin{pmatrix}4 & 4 & 1\end{pmatrix} \quad iii.\begin{pmatrix}1 & 3 & 4\end{pmatrix} \& \begin{pmatrix}1 & 2\end{pmatrix}$$

3. Find the dot product

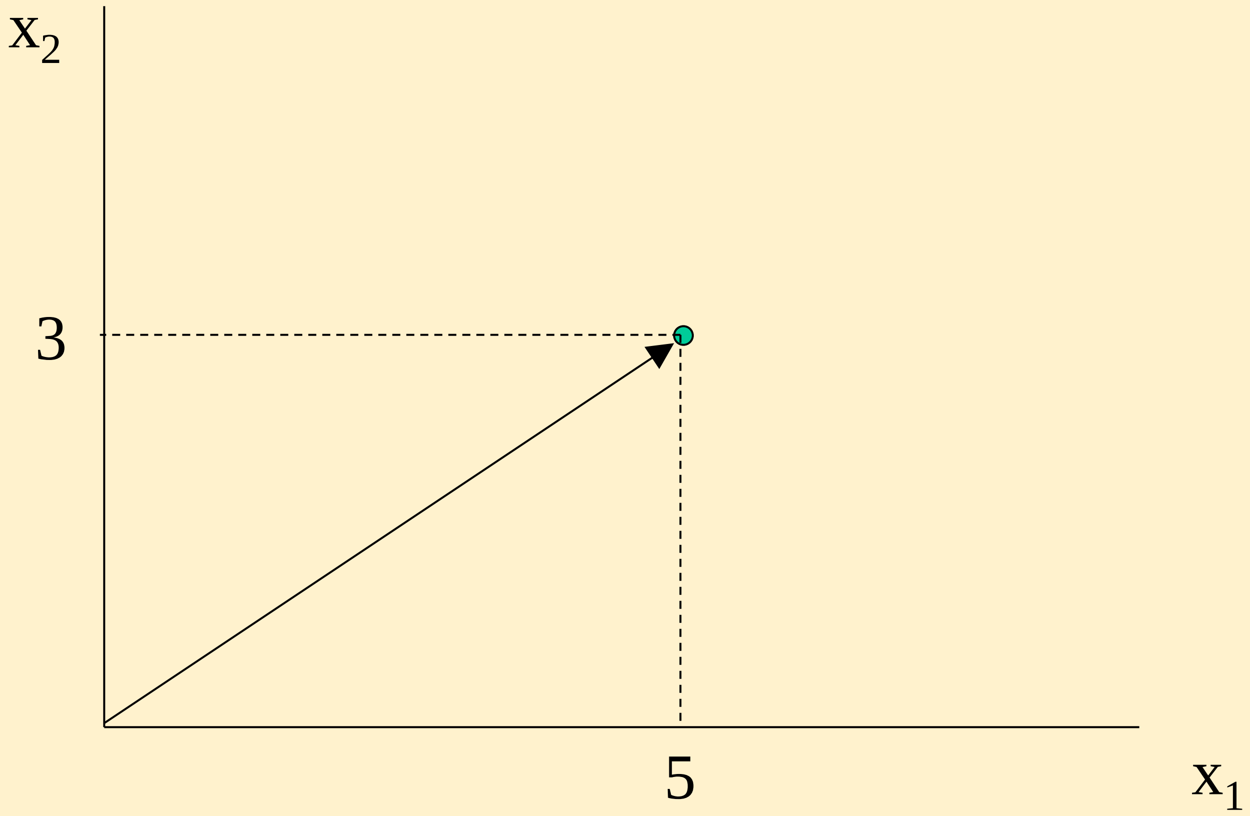
$$i.\begin{pmatrix}4 & 1\end{pmatrix} \& \begin{pmatrix}-1 & 0\end{pmatrix} \qquad ii.\begin{pmatrix}4\\0\\1\end{pmatrix} \& \begin{pmatrix}2\\1\\0\end{pmatrix} \quad iii.\begin{pmatrix}1 & 3 & 4\end{pmatrix} \& \begin{pmatrix}-3 & 1 & 0\end{pmatrix}$$

Geometry of the vectors

Idea: vectors can also be thought of as co-ordinates in a graph.

A *n x 1* vector can be a point in n–dimensional space

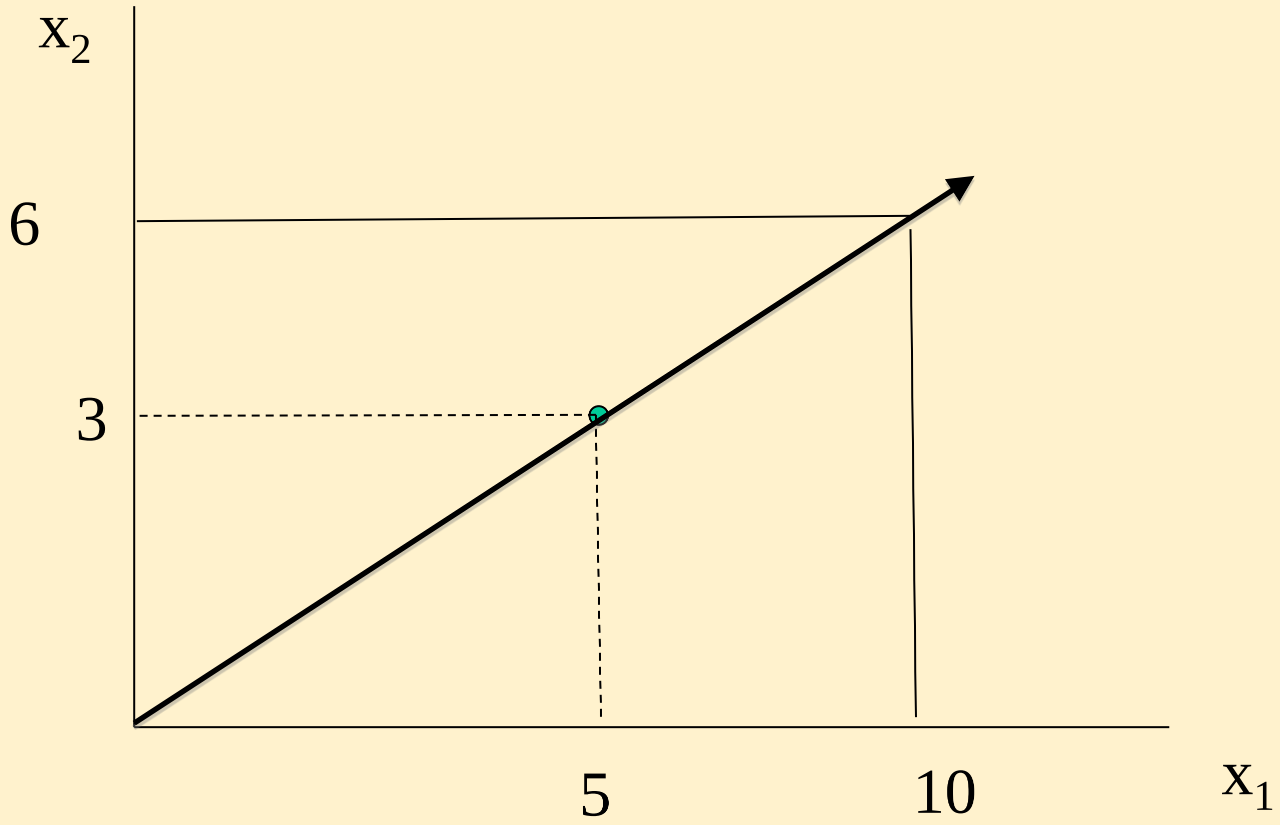
E.g. x = (5 3) – which might be a consumption vector C = (x_1, x_2)



A straight line is drawn out from the origin with definite length and definite direction is called a *radius vector*

Using this idea can give a geometric interpretation of scalar multiplication of a vector, vector addition or a "linear combination of vectors"

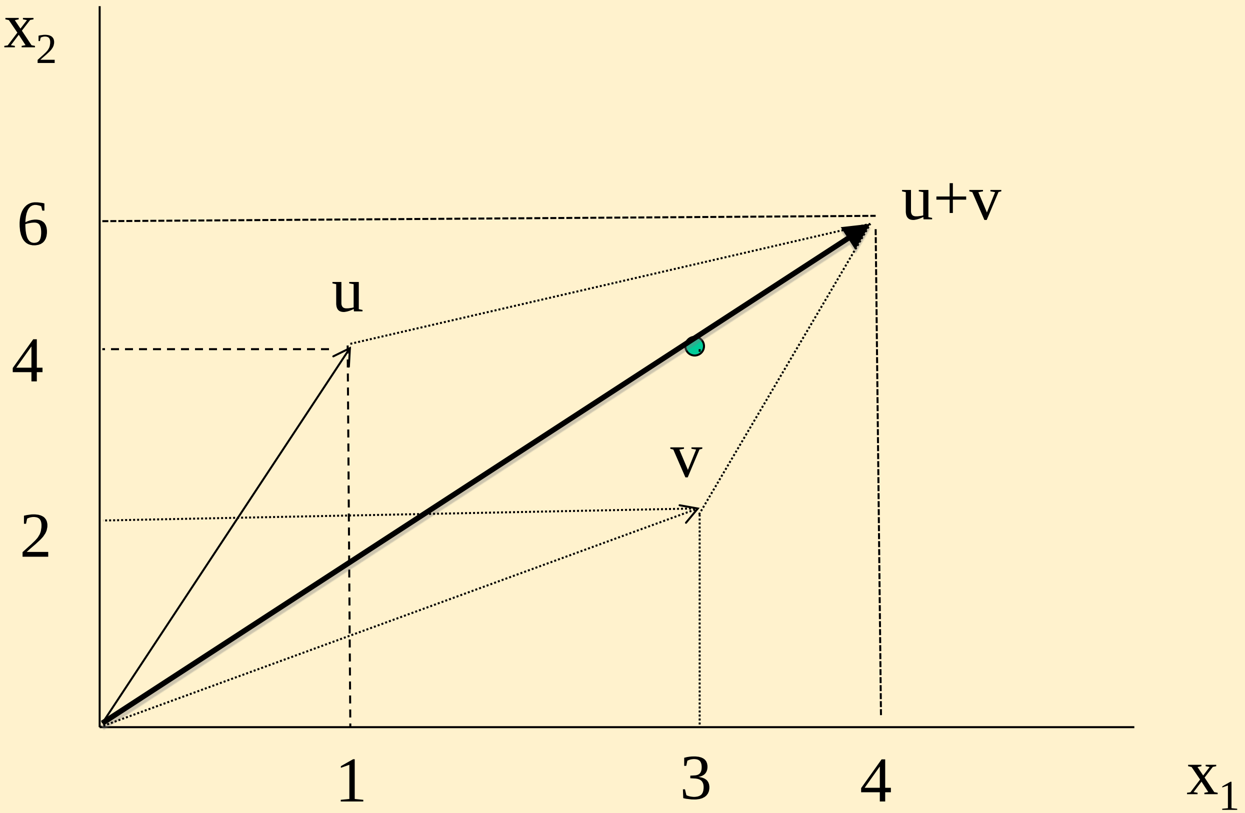
Eg If x = (5 3) then 2x = (10 6) and the resulting radius vector will overlap the original but will be twice as long



Similarly multiplication by a negative scalar will extend a radius vector in the opposite quadrant

Vector addition generates a new radius vector between the 2 original vectors

Eg If u = (1 4) and v = (3 2) then u + v = (4, 6) and the resulting radius vector will look like this



Note that this forms a parallelogram with the 2 vectors as two of its sides

Can also think of a geometric representation of vector inner (dot) product

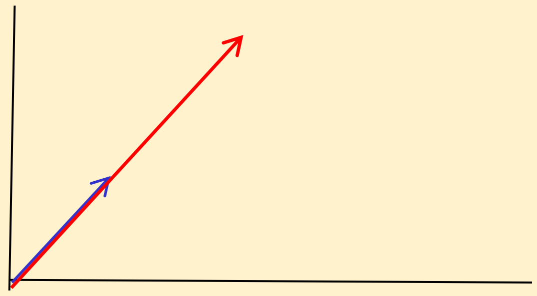
Remember

$$x.y = \sum_{i=1}^{i=n} x_i y_i$$

A geometric representation of this is that the dot product measures how much the 2 vectors lie in the same direction

Special Cases

a) If x.y=constant then the radius vectors overlap



Eg x=(1 1) y =(2 2)

So x.y= (1*2 + 1*2) = 4

Can also think of a geometric representation of vector inner (dot) product

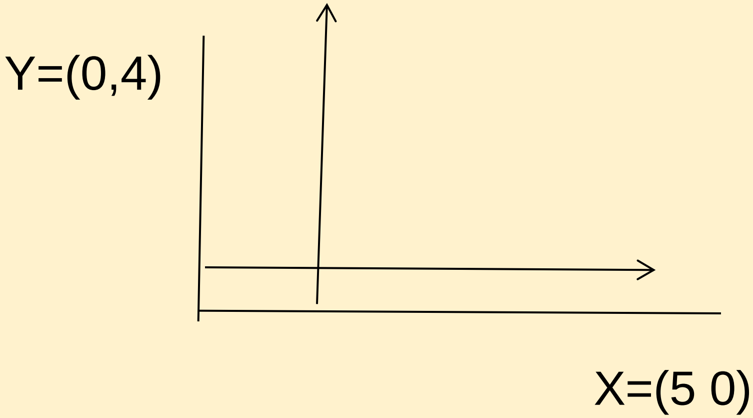
Remember

$$x.y = \sum_{i=1}^{i=n} x_i y_i$$

A geometric representation of this is that the dot product measures how much the 2 vectors lie in the same direction

Special Cases

b) If x.y= 0 then the radius vectors are perpendicular (***orthogonal***)



Eg x=(5 0) y =(0 4)

So x.y= (5*0 + 0*4) = 0

$$x \perp y$$

A group of vectors are said to be ***linearly dependent*** iff one of them can be expressed as a linear combination of the other

If not the vectors are said to be linearly independent

Equally linear dependence means that there is a linear combination of them involving non-zero scalars that produces a null vector

E.g. $x = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}; y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are linearly dependent

Proof. Because x=2y or x – 2y = 0

But $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; y = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ are linearly independent

Proof. Suppose x – ay = 0 for some a or other. In other words,

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} - a \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad or \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} a \\ 3a \end{pmatrix} = \begin{pmatrix} 2-a \\ 1-3a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then 2 – a = 0 and 1 – 3a = 0. So a = 2 and a = 1/3 – a contradiction

Generalising, a group of m vectors are said to be linearly dependent if there is a linear combination of (m-1) of the vectors that yields the m^{th} vector

Formally, the second definition:

Let x^1, x^2, …x^m be nx1 vectors.

If for some scalars a^1, a^2, …a^{m-1}

$a^1x^1 + a^2x^2 + \ldots a^{m-1} x^{m-1} = x^m$

then the group of vectors are linearly dependent,

But also

$a^1x^1 + a^2x^2 + \ldots a^{m-1} x^{m-1} - x^m = 0$ which is the first definition

Summary. To prove linear dependence find a linear combination that produces the null vector. If you try to find such a linear combination but instead find a contradiction then the vectors are linearly independent

Quiz II

A group of vectors are said to be linearly independent if there is no linear combination of them that produces the null vector

1. $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix};\ y = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$ are linearly independent. Prove it.

2. $x = \begin{pmatrix} 2 \\ 2 \end{pmatrix};\ y = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ are linearly independent. Prove it.

3. What about $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix};\ y = \begin{pmatrix} 0 \\ 1 \end{pmatrix};\ z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

Rank

The rank of a group of vectors is the maximum number of them that are linearly independent

$$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \; y = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

are linearly independent. So the rank of this group of vectors is 2

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \; y = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

are linearly dependent, (2x=y) So the rank is 1

$$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \; y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Any two vectors in this group are independent (x = y + 2z) so the rank is 2

An odd example.

- In Cafeland there are only two goods: x_1 = latte, x_2 = muffin.
- Peculiarly, it is not possible to buy and sell the goods separately.
- Instead the following combinations are available
  - x: Latte lover : $x_1 = 2$, $x_2 = 1$.
  - y: Muffintopia: $x_1 = 0$, $x_2 = 3$.

Any fraction of these combinations can be bought and sold

Joan wishes to own and consume exactly one latte and one muffin. Can she buy to achieve her goal?

Answer:

She buys 1/2 of latte lover combo and 1/6 of muffintopia combo.

This mix of vectors is called a linear combination.

More formally, if x and y are n x 1 vectors and a and b are scalars,

ax + by is a linear combination.

Joan's purchase is (1/2)x + (1/6)y

The example again

- In Cafeland there are only two goods: x_1 = latte, x_2 = muffin.
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Any fraction of these combinations can be bought and sold

Can Joan construct any combination of latte and muffin out of x and y?

$$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \; y = \begin{pmatrix} 0 \\ 3 \end{pmatrix};$$

Any combination: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ So formally, is there an a and a b such that:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax + by = a\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2a \\ a + 3b \end{pmatrix}$

The example again

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2a \\ a+3b \end{pmatrix}$$

I.e. two equations in two unknowns:

$x_1 = 2a$ and

$x_2 = a+3b$

Or

$a = 0.5x_1$

and so

$x_2 = a + 3b = 0.5x_1 + 3b$ or

$x_2 - 0.5x_1 = 3b$ or

$b = (x_2 - 0.5x_1)/3$

For instance if $x_1 = 1$ and $x_2 = 1$, then $a = 0.5$ and $b = 1/6$