The Complexity of the Constraint Satisfaction Problem.

ISG-CS Mini-Conference

Professor David Cohen

Department of Computer Science
Royal Holloway College,
University of London

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Outline

1. Introducing the CSP
   - What is a Constraint?
   - What is the Constraint Satisfaction Problem (CSP)?

2. CSP Complexity
   - Hardness
   - Tractable Languages
   - Structural Tractability

3. Hybrid Tractability
   - Tractable Independent Set Problem
   - A General Theory

4. CSP Becomes VCSP
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What is a **Constraint**?

A constraint has two parts:

- A *list of variables* that are constrained
- A *set of allowed combinations of values* for these variables
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  - which we call the **scope**
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Examples of Constraints

Two constraints:

- with their scopes
- and relations

A traditional graph theory constraint

Variables $x$ and $y$ must be different colours

An important temporal constraint

The CSP talk must finish before the next talk
## Examples of Constraints

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- with their **scopes**
- and relations

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CSP Becomes VCSP
The classical definition of the CSP

Definition

An instance of the constraint satisfaction problem (CSP) is a triple \((V, D, C)\) where:

- \(V\) is a set of variables
- \(D\) is a set of possible domain values
- \(C\) is a set of constraints

Each constraint \(c \in C\) is a pair \((s, R)\) where:

- \(s\) is the scope (list of variables constrained)
- \(R\) is the relation (set of allowed combinations of values)

A solution is a mapping \(\sigma : V \rightarrow D\) which satisfies all the constraints in \(C\): \(\forall (s, R) \in C, \sigma(s) \in R\)
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**Splitting a CSP into two parts**

Any CSP instance can be split into two parts:

- The structure of the constraints
- The nature of the constraints
Splitting a CSP into two parts

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The nature of the constraints
Another definition of a **CSP**

The structure of the constraints
- this is a *relational structure*

\[ S_1 = (V, E_1, E_2, \ldots, E_m) \]

The nature of the constraints
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\[ S_2 = (D, R_1, R_2, \ldots, R_m) \]

**Definition**

An *instance* of the constraint satisfaction problem (CSP) is a pair of similar relational structures \((S_1, S_2)\).

A solution is a homomorphism \(h : S_1 \rightarrow S_2\).
Another definition of a CSP

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An example: Graph colouring as a CSP

The problem of finding a proper $k$-colouring for the vertices of a graph $(N, E)$ can be expressed as a CSP...

$(V, D, C) = (N, \{1, \ldots, k\}, \{(v_1, v_2), \neq\} \mid \{v_1, v_2\} \in E)$

$(N, E) \rightarrow$
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$$(N, E) \xrightarrow{?} (\{1, \ldots, k\}, \neq)$$
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$(N, E) \overset{?}{\rightarrow} K_k$
An example: Satisfiability as a CSP

The problem of finding a satisfying assignment for a set of Boolean clauses $c_1, \ldots, c_m$ can be expressed as a CSP...

\[(V, D, C) = (\text{Vars}(c_1, \ldots, c_m), \{0, 1\}, \{(\text{Vars}(c_i), \text{Models}(c_i)) \mid i = 1, \ldots, m\})\]

\[(\text{Vars}(c_1, \ldots, c_m), E_1, E_2, \ldots, E_8) \rightarrow (\{0, 1\}, R_1, R_2, \ldots, R_8)\]
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\(\text{Vars}(c_1, \ldots, c_m), E_1, E_2, \ldots, E_8 \rightarrow (\{0, 1\}, R_1, R_2, \ldots, R_8)\)
Combining the two parts of a CSP

\[ S_1 = (V, E_1, E_2, \ldots, E_m) \]

\[ S_2 = (D, R_1, R_2, \ldots, R_m) \]

To allow a wider range of descriptive possibilities we combine these two structures to form their product: \( S_1 \times S_2 \)

If we extend both structures with the equality relation, and complement the relations of \( S_2 \), then their product is a structure called the microstructure complement.
The microstructure complement of a CSP

**Definition**

The *microstructure complement* of a CSP \((V, D, C)\) is the relational structure \((V \times D, E)\) where 
\(((v_1, d_1), (v_2, d_2), \ldots, (v_k, d_k)) \in E\) if and only if:

- \((v_1, v_2, \ldots, v_k)\) is the scope of some constraint \(c \in C\) which *disallows* the assignment \((d_1, d_2, \ldots d_k)\); or
- \(k = 2, v_1 = v_2\) and \(d_1 \neq d_2\).
An example: Graph 2-colouring

\[ F = (x \neq y) \land (y \neq z) \land (x \neq z) \]

The microstructure complement of this CSP instance is:
An example: Linear equations modulo 2

\[ F = (x + y + z \equiv 0) \land (w + y \equiv 1) \land (w + z \equiv 0) \pmod{2} \]

The microstructure complement of this CSP instance is:
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4. CSP Becomes VCSP
The CSP is NP-Hard

Testing is easy

We can test any putative homomorphism (variable assignment) to see it meets all constraints in linear time.
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We can test any putative homomorphism (variable assignment) to see it meets all constraints in linear time.

CSP restricted to domain size 2 is NP-hard
3-SAT: The problem of finding a satisfying assignment for a set of Boolean 3-clauses can be expressed as a CSP…
The CSP is NP-Hard

Testing is easy

We can test any putative homomorphism (variable assignment) to see it meets all constraints in linear time.

CSP restricted to arity 2 constraints is NP-hard

3-Colouring: The problem of finding a proper 3-colouring for the vertices of a graph can be expressed as a CSP...
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CSP Languages

Restricting the Right Hand Side

- A set $\Gamma$ of *allowed relations* is called a *constraint language*.

- Some (quite small) languages are sufficient to express interesting problems (e.g. Linear Equations, 3-SAT, 3-Colouring, Scheduling, Line drawing interpretation, Temporal Logics, Interval Logics...).

- For some languages we find effective algorithms.

- For each such algorithm we characterise the (maximal) language that it works on, to build an “algorithm chooser”.
CSP Languages

RHS == Language == Clones == Algebra

- We observed (some 15 years ago) that several maximal tractable languages... are characterised by very simple algebraic properties (clones of polymorphisms).
- We proved that every tractable language has this property: So we characterise tractable algebras instead.
- This is a simpler problem because so much is known about algebras!
Languages are Done

We are so close to finishing... just one polymorphism left

The following result shows that it remains to prove that $\Gamma$ is tractable when it has a cyclic (or a Siggers’) polymorphism.

Theorem

(Siggers, 2009) A finite algebra has a WNU term operation iff it has a 4-ary idempotent term operation $f$ satisfying

$$f(y, x, y, z) = f(x, y, z, x).$$
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CSP Structure

Restricting the Left Hand Side

- We can limit the hypergraph (or relational) structures that are allowed.
- Such structural restrictions are natural and reasonable in practical applications.
- All tractable structural classes are based upon some notion of the width of a relational structure.
Structures are Done

All interesting structural classes are known

The following results show us everything we might want to know:

**treewidth** For classes characterised by *Primal graphs* (Grohe, 2003)

**hypertree width** For *tree decompositions* (Gottlob, 2007)

**treewidth of core** For *bounded arity* (Grohe, 2007)

**submodular width** For *FPT* (parameter: no. vars) (Marx, 2010)
The Structural Gap

Between hypertree and submodular Width

The most general known tractable structural property is bounded \textit{fractional hypertree width} (Grohe, Marx, 2007). This is better than hypertree width and worse than submodular width.

Where exactly are the \textit{boundaries of structural tractability}?
The Structural Gap

Between hypertree and submodular Width

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Key Motivation/Timeliness

We believe the following:

- **Languages** (RHS) are mostly done and its gotten too hard/esoteric.
- **Structure** (LHS) is mostly done and its gotten too hard/esoteric.
- Every tractable class has a definition in terms of its microstructure.
- These are (very like) graph structures.
- We can bring in a whole lot of new theory.
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Perfect Graphs, Salamon 2008

Motivation

- Solving CSPs is equivalent to finding independent sets (cliques) in their microstructure (complement).
- The independent set and clique problems are tractable for triangulated graphs.
- This explains why CSP instances whose microstructure (complement) is triangulated are tractable.

Definition

A class of CSP instances whose microstructure complements have a tractable independent set problem are tractable.

This is a generalised explanation for the tractability of instances of the AllDiff+Unary class.
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Introducing the CSP

CSP Complexity

Hybrid Tractability

CSP Becomes VCSP

Forbidding certain embeddings in the microstructure complement

Negative Patterns

Can we capture tractable classes by specifying forbidden homomorphisms from certain microstructure complements. Here we insist that distinct variables map to distinct variables.

Example

Of course we can just forbid the single edge from embedding, or a pair of linked edges.
Pivots: Characterising Tractability

**Theorem**

A set $\mathcal{X}$ of connected patterns is tractable only if there is some $\chi \in \mathcal{X}$ that embeds into some $\text{PIVOT}(r)$.

**Theorem**

$\text{PIVOT}(1)$ is tractable and any disjoint union of $\text{SEPPIVOT}(r)$ patterns is tractable.

**Conjecture**

A finite set $\mathcal{X}$ of connected patterns is tractable if and only if there is some $\chi \in \mathcal{X}$ that embeds into some $\text{PIVOT}(r)$. 
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Formally: A CSP becomes a VCSP

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Each constraint is a pair \( \langle \sigma, \rho \rangle \) where

- \( \sigma \in V^* \) is the scope
- \( \rho \subseteq D^{\mid \sigma \mid} \) is the relation
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**Definition**

An **assignment** is a mapping \( s : V \mapsto D \).
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A **solution** is an assignment that satisfies all constraints. Formally:

$$\forall \langle \sigma, \rho \rangle \in C, s(\sigma) \in \rho.$$
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  - $\rho$ is a mapping from $D^{\mid \sigma \mid}$ to $\{0, \infty\}$ is the **relation**

**Definition**

A **solution** is an assignment that satisfies all constraints. Formally:

$$\forall \langle \sigma, \rho \rangle \in C, \rho(s(\sigma)) = 0.$$
Formally: A CSP becomes a VCSP

Definition
A VCSP is a four tuple $\langle V, D, C, \Omega \rangle$ where

- $V$ is a finite set of variables
- $D$ is a finite domain
- $C$ is a finite set of constraints.

Each constraint is a pair $\langle \sigma, \phi \rangle$ where

- $\sigma \in V^*$ is the scope
- $\phi$ is a mapping from $D^{\mid \sigma \mid}$ to $\Omega$ called the cost function
Introducing the CSP

CSP Complexity

Hybrid Tractability

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Definition

The valuation structure $\Omega$, is ordered, with a 0 and $\infty$ and an AC aggregation $\oplus$, where $\alpha \geq \beta, \gamma \in \Omega$ and $\alpha \oplus \gamma \geq \beta \oplus \gamma$. 
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  - $\phi$ is a mapping from $D^{\mid \sigma \mid}$ to $\Omega$ called the cost function

Definition
An assignment is a mapping $s : V \mapsto D$. Its cost is given by

\[
Cost_p(s) = \bigoplus_{\langle \sigma, \phi \rangle \in C} \phi(s(\sigma)).
\]
Formally: A CSP becomes a VCSP

**Definition**

A VCSP is a four tuple $\langle V, D, C, \Omega \rangle$ where

- $V$ is a finite set of **variables**
- $D$ is a finite **domain**
- $C$ is a finite set of constraints.

Each constraint is a pair $\langle \sigma, \phi \rangle$ where

- $\sigma \in V^*$ is the **scope**
- $\phi$ is a mapping from $D^{|\sigma|}$ to $\Omega$ called the **cost function**

**Definition**

A **solution** is an assignment with minimal cost.
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# Complexity, Expressibility, Galois, Submodularity

**We have begun**

There is a close association between the **complexity/expressibility** of VCSP languages and a so called **fractional clone**. This **association** is via a well defined **Galois connection** - just as the **lattice of clones** is associated with the complexity/expressibility of CSP languages. We are trying to use this theory to understand the **structure/expressibility** of **submodular optimisation** problems.
Complexity, Expressibility, Galois, Submodularity

Good Work

- Takhanov 2009: A dichotomy theorem for the general minimum cost homomorphism problem.
- Cohen et al: Binary submodular languages are cubic.
- Cohen et al: A dichotomy for Boolean VCSP languages.
- Cooper et al: Arc-consistency for VCSPs
- etc.,